

I will omit proofs and focus on the gist of the topic. Use wikipedia for undefined terms (e.g. limit ordinal).

1. A σ -algebra Ω on a set X is a nonempty subset of the power set of X that is closed under countable unions and complements. A set $E \subset X$ is called measurable iff $E \in \Omega$, and (X, Ω) is called a measurable space.

Consequently, Ω is also closed under countable intersections, and \emptyset, X are measurable (the first is the null union and the second is the null intersection).

2. If X is a topological space, then the Borel σ -algebra \mathcal{B} of X is the smallest σ -algebra of X such that open sets are measurable.

In other words, let $\mathcal{T} \subset \mathcal{P}(X)$ be the topology of X (the set of open sets). Since $\mathcal{P}(X)$ is a σ -algebra, we can take the intersection of all σ -algebras of X containing \mathcal{T} . This intersection is \mathcal{B} , the set of Borel sets.

An inductive construction is as follows: Let \mathcal{T}^{n+1} be the set of all complements and countable unions of elements of \mathcal{T}^n for non-limit ordinals n ($\mathcal{T}^0 = \mathcal{T}$), and $\mathcal{T}^\alpha = \bigcup_{\beta < \alpha} \mathcal{T}^\beta$ for limit ordinals. Then $\mathcal{B} = \mathcal{T}^{\omega_1}$, where ω_1 is the least uncountable ordinal.

A consequence is that sets that occur naturally (meaning you are not trying *very hard* for the contrary) are Borel sets.

3. A (positive) measure μ on a measurable space (X, Ω) is a function $\mu : \Omega \rightarrow [0, \infty]$ such that the following hold:

- $\mu(\emptyset) = 0$
- σ -additive: $\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sum_{n \in \mathbb{N}} \mu(E_n)$, if $E_n \in \Omega$ are disjoint.

A measure is

- Monotone: $A \subseteq B$ implies $\mu(A) \leq \mu(B)$.
- Subadditive: $\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) \leq \sum_{n \in \mathbb{N}} \mu(E_n)$.
- If E_n is increasing, then $\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n)$.
- If E_n is decreasing and $\mu(E_0) < \infty$, then $\mu\left(\bigcap_{n \in \mathbb{N}} E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n)$.

Given μ , (X, Ω, μ) is called a measure space. A probability space is the case that $\mu(X) = 1$.

4. Lebesgue measure (pronounced "luh-bayg" emphasis on second syllable): Define $m([a, b]) = b - a$ for reals $a \leq b$. We can extend the domain of m to the Borel σ -algebra of \mathbb{R} so that it is σ -additive, hence a measure. This is the Lebesgue measure. Classical sets have exactly the measure they should. Multiplication lets us define the measure on rectangular boxes in \mathbb{R}^k , and again we extend to the Borel algebra. All your calculus intuition of approximating sets with unions and intersections of boxes works verbatim. The axiom of choice implies that it is impossible to extend the domain of the Lebesgue measure to the power set of \mathbb{R} in a consistent manner.

5. It is convenient to extend the domain of a Borel measure by adjoining subsets of sets of measure zero (this actually implies that a lot more are added). This is called the completion of the measure. If two sets differ by a set of measure zero, then they have the same measure. In general, there are non-Borel sets of measure zero. In \mathbb{R} there are uncountably many.

6. A (positive) function $f : X \rightarrow [0, \infty]$ is measurable iff the inverse image of every open set is measurable. Let $f : X \rightarrow \overline{\mathbb{R}}$ (where $\overline{\mathbb{R}}$ is the extended reals), and define $f^+ = \max(f, 0)$, $f^- = \max(0, -f)$ (these are called the positive and negative parts of f). Then $f = f^+ - f^-$ and we say that f is measurable iff f^+ and f^- are. We define $f : X \rightarrow \mathbb{C}$ as measurable iff the positive and negative parts of the real and imaginary parts of f are measurable. Just as with continuity, we rarely prove a function is measurable from the definition, but rather break it into pieces.
- continuous functions are measurable
 - compositions of measurable functions are measurable
 - the identity on \mathbb{C} is measurable
 - characteristic functions of measurable sets are measurable (is 1 on set and 0 off set)
 - thus polynomials, trig functions, exponentials, logarithms, power series, infinite sums of non-negative functions, and say the function that maps a real number to its k th decimal digit are all measurable
7. A simple measurable function is a finite sum $s(x) = \sum_{i=1}^n a_i \chi_{A_i}(x)$ where $a_i \in [0, \infty]$, A_i are measurable, and $\chi_{A_i}(x) = 1$ iff $x \in A_i$ and is zero otherwise. If f is a positive measurable function, there is a sequence of simple measurable functions s_n such that $s_n \leq s_{n+1} \leq f$ (pointwise) and $s_n(x) \rightarrow f(x)$ for all x as $n \rightarrow \infty$. An example of such a sequence is the truncated binary expansion of f (this turns out to be simple to construct).
8. The Integral: Define $\int s d\mu = \sum_{i=1}^n a_i \mu(A_i)$ for every simple measurable function. If f is a positive measurable function, define $\int f d\mu = \sup \int s d\mu$ where the supreme is over all simple measurable functions pointwise bounded above by f . For complex and real valued functions extend by positive and negative, real and imaginary parts. To integrate over a set, multiply f by a characteristic function and then integrate. If μ is the Lebesgue measure and f is Riemann integrable, then this definition agrees with the Riemann integral of f .
9. Using counting measures on \mathbb{N} (every point has measure 1), every infinite sum is an integral in this sense. The Lebesgue monotone convergence theorem and dominated convergence theorem allow commuting limits with integrals in a manner much simpler than is required for Riemann integrals. In general, the Lebesgue integral is much easier to compute than the Riemann. Furthermore, the Lebesgue theory allows one to think of functions as defined on all but a set of measure zero, hence in terms of their distributional content regardless of their pointwise values (changing the values of a function on the rational numbers has no effect on its integral). On the other hand, there are cases where the Riemann integral is useful and the Lebesgue integral is undefined (particularly in some integral transforms on the real line and applications to analytic number theory).