Some things about algebraic geometry

J. David Taylor

March 13, 2019

Contents

1 Flatness

Theorem 2.16 (page 11) of Qing Liu tells you what flatness is in a large number of cases. Here are some examples:

1. A finitely generated flat module is free on stalks.

2. If $\mathcal{F}$ is a finitely generated flat $\mathcal{O}_X$-module for some scheme X (definition 1.1, page 157, a very broad generalization of many constructions from differential geometry or complex analysis), then the stalk of $\mathcal{F}$ is free at each point of X.

3. An example of the previous, if X is a k-variety (in the sense of Qing Liu: a k-scheme of finite type, this includes classical varieties), then every point of X has an open neighborhood U such that $\mathcal{O}_X(U)$ is a k-vector space.

4. Another example of (2): If $f : X \to Y$ is a flat morphism of k-varieties and $x \in X$, then there are affine open neighborhoods $x \in U \subseteq X$ and $f(x) \in V \subseteq Y$ such that $f(U) \subseteq V$, $V = \operatorname{Spec} A$, $U = \operatorname{Spec} B$, and B is a free A-module. That is, X is an affine space bundle (i.e. a vector bundle without vector space structure) over Y (even in the sense of differential geometry). The right analogue of vector bundles in commutative algebra is projective modules, and it turns out that finitely generated flat modules over a commutative noetherian ring are projective.

5. Many results in scheme theory are proven by reducing to the finitely generated case, and this is fiberwise flat over a field, so the above situation is not idiosyncratic. To elaborate on the fiberwise bit: if $f : X \to Y$ is any morphism of schemes, let $y \in Y$ have residue field κ. Then $Z := f^{-1}(y) = X \times_Y \operatorname{Spec} y$ is a κ-scheme. Any point $z \in Z$ has an open affine neighborhood $U = \operatorname{Spec} A$ and A is a κ-algebra. Any arithmetic statement in A involving finitely many elements and equations takes place in some subring $B := \kappa[a_1, a_2, \dots, a_e]/(r_1, r_2, \dots, r_d)$ where a_i's are elements of A and r_j's are polynomials of the a_i's that vanish in A. Now, $\operatorname{Spec} B$ is a variety over κ (in the sense of Qing Liu) and many results about $f : X \to Y$ can be deduced from proving them for B for appropriately chosen B. To my knowledge this was originally Grothendieck's idea. He uses it throughout EGA.

6. Stacks tag 00HK is called the "equational criterion" for flatness. Roughly, a sheaf/morphism is flat if it doesn't cause any additional collapse/identification of elements that isn't already present in the base ring.

2 Functor of Points Examples from Qing Liu

Here are some "functor of points interpretations" of some results in Qing Liu:

1. Propostion 3.25 (page 48) tells you how to compute the functor of points of any affine scheme (or how to go in reverse). Here are some examples:

2. $\mathbb{A}^1_S$: Let X be an S-scheme. Using prop 3.25,

$$\mathbb{A}^1_S(X) = \operatorname{Hom}_{S-sch}(X, \mathbb{A}^1_S) \cong \operatorname{Hom}_{\mathcal{O}_S(S)-algebras}(\mathcal{O}_S(S)[t], \mathcal{O}_X(X)) = \mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X), \qquad (1)$$

   where the second to last identification on the right comes from the universal property of polynomial rings: a ring homomorphism from a polynomial ring is entirely determined by where the coefficients go ($\mathcal{O}_S(S)$-algebra structure) and where the variable goes (some element of $\mathcal{O}_X(X)$.

3. (going in reverse, same logic) If you assume that $\mathbb{G}_{\mathrm{m}}$ is the functor that takes a scheme X to $\Gamma(X, \mathcal{O}_X)^\times$, and a morphism $f : X \to Y$ to the group homomorphism $f^\# : \Gamma(Y, \mathcal{O}_Y)^\times \to \Gamma(X, \mathcal{O}_X)^\times$, then which affine scheme works for $\mathbb{G}_{\mathrm{m}}$?

4. Similarly, this will let you reverse-engineer what μ_n should be, as well as any matrix group as a scheme – they are all affine, and the fact that the real matrix groups sit in $\mathbb{R}^{n^2}$ should clue you in to that.

5. Similarly, given an affine scheme $X = \operatorname{Spec} A$ and $f \in A$, the distinguished open subscheme $D(f)$ has functor of points: Let R be a ring. $D(f)(R)$ consists of ring homs $A \to R$ that send f to a unit. If k is a field, then $D(f)(k)$ consists of pairs $(\mathfrak{p}, \sigma)$ where $\mathfrak{p} \in \operatorname{Spec} A$, f doesn't vanish at $\mathfrak{p}$, and σ is an embedding of the residue field of $\mathfrak{p}$ into k. In the classical setting that X is an affine variety (in characteristic 0) and $k = \mathbb{C}$, $D(f)(\mathbb{C})$ is the set of points of $X(\mathbb{C})$ where f doesn't vanish. (Note that X doesn't have to be a $\mathbb{C}$-scheme in this last case, it's sufficient to be finite type over some number field. The interesting bit is that you can still take $\mathbb{C}$-points and you still get what you should. This remark applies as well to the next example.)

6. Again, let $X = \operatorname{Spec} A$ be an affine scheme and let $f \in A$. The closed subscheme $V(f)$ has functor of points: Let R be a ring, $V(f)(R)$ consists of ring homs $A \to R$ killing f. If k is a field, then $V(f)(k)$ consists of pairs $(\mathfrak{p}, \sigma)$ where $\mathfrak{p} \in \operatorname{Spec} A$, and f vanishes at $\mathfrak{p}$ (i.e. $f \in \mathfrak{p}$), and σ is an embedding of the residue field at $\mathfrak{p}$ into k. In the classical setting that X is an affine variety (in characteristic 0) and $k = \mathbb{C}$, $V(f)(\mathbb{C})$ is the set of points of $X(\mathbb{C})$ where f vanishes.

7. A note on the sets of pairs above: The reason for a set of pairs $(\mathfrak{p}, \sigma)$ for points of $X(k)$ is that k is often not algebraically closed. When k is extended to its algebraic closure, the base-change of X has (typically) more points – the prime $\mathfrak{p}$ splits. The different embeddings are actually keeping track of the different "geometric" points that are collapsed to $\mathfrak{p}$ when working over k instead of its algebraic closure. Here's a simple example:

8. Let $X = \operatorname{Spec} \mathbb{Q}[x, y]/(x^2 + y^2 - 1)$. Let's examine the functor of points of X. A point of $X(\mathbb{Q})$ is a point $\mathfrak{p}$ of X and an embedding of $\kappa(\mathfrak{p})$ (the residue field of $\mathfrak{p}$) into $\mathbb{Q}$. Since $\mathbb{Q}$ is a prime field (no subfields), the embedding is only the requirement that $\kappa(\mathfrak{p}) = \mathbb{Q}$. This is the same as requiring the $\mathfrak{p}$ be a maximal ideal of the form $(x - a, y - b)$ where $a^2 + b^2 = 1$ (by the nullstellensatz). Therefore, $X(\mathbb{Q})$ gives you all the points corresponding to right triangles with (signed) integer sides (as in your talk last semester) – which is precisely what it should be.

   On the other hand, $\mathfrak{p} = (x^2 + 1, y^2 - 2)$ is a point of X and it is completely invisible if we take $\mathbb{Q}$-points. On the other hand, if we take $K = \mathbb{Q}(i, \sqrt{2})$-points then not only does $\mathfrak{p}$ become visible, but it's actually four points! Namely, K is the residue field of $\mathfrak{p}$ and the embeddings of $K \to K$ are necessarily automorphisms (injective maps of 4-dimensional $\mathbb{Q}$-vector spaces are surjective). In fact, $\mathfrak{p}$ will be four points for any field containing K. So $\mathfrak{p}$ is a point of X as a scheme, but it's not visible in the functor of points until we use a field containing K. If we were looking at the solution set of $x^2 + y^2 = 1$ in the affine plane in classical algebraic geometry (point sets, characteristic 0, no schemes), then this is exactly what we'd like – any field containing $i, \sqrt{2}$ picks up the four points $(\pm i, \pm\sqrt{2})$, but they won't be visible otherwise. In summary of this example, the functor of points perspective correctly manages the galois-arithmetic nature of schemes.

9. Let $X = \operatorname{Spec} \mathbb{Q}[x, y]/(x^2)$ be the doubled y-axis. This is a naturally occuring object: continuously deform two overlapping cylinders until they meet tangentially along a line, the subscheme of intersection is like X. If we only look at the points of X over a field (which is what people used to do), then we won't be able to pick of the multiplicity of the intersection. That is, $X(k) = k$ for every field k (and every integral domain, for that matter). So, thinking in terms of point sets over fields, X is indistinguishable from a single line. This is no good. However, let R be a $\mathbb{Q}$-algebra with nilpotent elements, then $X(R)$ can be identified with the set of pairs (r, n) where $r, n \in R$ and $n^2 = 0$ (prop 3.25 Qing Liu, page 48). This is a different set than $\mathbb{A}^1(R)$. In fact, $X(R)$ "contains" the line in its first coordinate, but the nilpotent bit is orthogonal to the line. That is, the n is like a

vector tangent to the point (r, n) pointing off the line. Why is this a tangent vector? In differential geometry you learn that tangent vectors are point derivations. Let $f \in R[x]$ vanish at r. Then $f(r+n) = f(r) + f'(r)n = f'(r)(n)$ by Taylor's theorem. So (r, n) defines a point derivation – a tangent vector. Some authors (notably Vakil) say that X is an infinitessimal thickening of the line. It's thicker because it tells you just a little about how to propogate function values off of it (a derivative is a precise way of measuring/controlling the propogation of function values). In summary of this example, the functor of points perspective keeps track of infinitessimal/non-reduces behavior that is not visible when looking at literal solution sets and is somewhat obscure when looking at the scheme X itself as a space.

10. The functor of points of projective space is given in Proposition 1.31 (page 169). "Invertible sheaf", "line bundle", and "rank 1 vector bundle" are more or less synonyms – even if someone defines them differently, they are all equivalent.

11. Here's a final example: Let $X = \operatorname{Spec} \mathbb{Z}$. We want to make sense of the difference between the closed points and the generic point. Note that if R is a ring, then there is a unique map of (commutative) rings (with 1) $\mathbb{Z} \to R$. Let k be a field. $X(k)$ consists of one point. If k has characteristic p, then that point can be identified with the canonical embedding of the residue field of X at (p) into k – this is true even when $p = 0$. This behavior is true more generally: the non-closed points of a scheme are visible in the points of rings that are not finitely generated over that scheme. e.g. The generic point of $\mathbb{A}^1_{\mathbb{C}}$ is visible over $\mathbb{C}(t)$: $\mathbb{A}^1_{\mathbb{C}}(\mathbb{C}(t))$ picks up every rational function, and they are all located at the generic point. Before Grothendieck came up with schemes, Andre Weil developed his foundations along this principle – he fixed some algrebraically closed field Ω of infinite transcendence degree over the field you'd be working with, and took literal point set solutions to the systems of equations with coordinates in Ω and called these generic points. The scheme theoretic solution is cleaner and the functor of points is cleaner than that. The more clunky Weil perspective does help to clarify what a generic point is though: it's the behavior you'd expect barring special circumstances (i.e. the high transcendence degree says that the coordinates of a generic point won't hold relations to one another outside of those required by the equations).