Taking Fixed Points Commutes with Base Change

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The purpose of this note is to show that taking fixed points commutes with base change when working with G-modules. We make no assumptions on the cardinality of G nor the characteristic of the base ring. Additionally, the identification is canonical.

Let G be a group with subgroup H, and let M be a $A[G]$ module for some commutative ring, A, with unit. Furthermore, let B be a flat, commutative, unital, A-algebra. As a specific case, B/A can be a field extension and M can be a vector space over A on which G acts linearly.

Let $\bigoplus_{h\in H} M$ be a G-module, given by the action

$$g \cdot (x_h)_h = (y_h)_h, \text{ where } y_h = g x_{g^{-1}hg}.$$

We show that this is an action as follows. Let $g_1, g_2 \in G$ and $(x_h)_h \in \bigoplus_{h\in H} M$. Let

$$g_1 \cdot (x_h)_h = (y_h)_h \text{ and } g_2 \cdot (y_h)_h = (z_h)_h.$$

Then we have

$$y_h = g_1 x_{g_1^{-1}hg_1} \text{ and } z_h = g_2 y_{g_2^{-1}hg_2} = g_2 \left(g_1 x_{g_1^{-1}(g_2^{-1}hg_2)g_1} \right) = (g_2g_1) x_{(g_2g_1)^{-1}h(g_2g_1)}.$$

Therefore, $(g_2g_1) \cdot (x_h)_h = (z_h)_h$, and this is indeed an action.

Define the map

$$f_M : M \to \bigoplus_{h\in H} M : m \mapsto (m - hm)_{h\in H}.$$

Let $g \in G$ and note that

$$f_M(gm) = (gm - hgm)_h = (g(m - (g^{-1}hg)m))_h = g \cdot f_M(m).$$

It is clear that f_M is A-linear, so it is a homomorphism of $A[G]$-modules.

Note that M^H is the kernel of f_M. So we have the following exact sequence of $A[G]$-modules:

$$0 \to M^H \longrightarrow M \xrightarrow{f_M} \bigoplus_{h\in H} M.$$

Since B is flat, $- \otimes_A B$ is an exact functor. Therefore,

$$0 \to M^H \otimes_A B \longrightarrow M \otimes_A B \xrightarrow{f_M \otimes \mathrm{id}} \left(\bigoplus_{h\in H} M \right) \otimes_A B$$

is an exact sequence of $B[G]$-modules. Since tensor products distribute over direct sums,

$$\left(\bigoplus_{h\in H} M \right) \otimes_A B \cong \bigoplus_{h\in H} (M \otimes_A B)$$

as B-modules. It is easy to see that this homomorphism is G-equivariant. Thus we have an exact sequence of $B[G]$-modules,

$$0 \to M^H \otimes_A B \longrightarrow M \otimes_A B \xrightarrow{f_M \otimes \mathrm{id}} \bigoplus_{h\in H} (M \otimes_A B).$$

Since $f_M \otimes \mathrm{id} = f_{M\otimes_A B}$, we canonically have

$$M^H \otimes_A B = \ker f_M \otimes \mathrm{id} \cong \ker f_{M\otimes_A B} = (M \otimes_A B)^H,$$

as $B[G]$-modules.