- The purpose of this note is to classify groups of order 2015. This was a qualifying exam problem, so consider carefully whether or not you want to read a solution of it.

- Let G be a group of order $2015 = 5 \cdot 13 \cdot 31$. By the first sylow theorem, G has sylow p-groups for each prime dividing 2015. Let P, Q, R be the respective $5, 13, 31$-sylow subgroups.

- Since n_{31} divides $5 \cdot 13$ and is congruent to 1 mod 31, $n_{31} = 1$. Therefore, R is a normal subgroup of G.

- Since n_{13} divides $5 \cdot 31$ and is congruent to 1 mod 13, $n_{13} = 1$. Therefore, Q is a normal subgroup of G.

- Since n_5 divides $13 \cdot 31$ and is congruent to 1 mod 5, $n_5 = 1$ or $n_5 = 31$.

- If $n_5 = 1$, then P is also a normal subgroup of G. Since $5, 13, 31$ are coprime, Lagrange's theorem implies that the pairwise intersections of P, Q, R are trivial. Therefore G is the direct product of P, Q, and R. Since they are cyclic of coprime orders, G is the cyclic group of order 2015.

- Suppose $n_5 = 31$ and that G is non-abelian. Since Q and R are normal subgroups of G, QR is a normal subgroup of G. Since $Q \cap R = 1$, QR is the (internal) direct product of Q and R. Since Q and R are cyclic of coprime order, QR is a cyclic group of order $13 \cdot 31 = 403$. Furthermore, QR is a normal subgroup of G, $P \cap QR = 1$ (again, by Lagrange's theorem), and $G = PQR$ (since QR has index 5 and the elements of P belong to different cosets of G/QR), we know that G is a semi-direct product of P and QR. So $G \cong QR \rtimes_\psi P$ for some group homomorphism $\psi : P \to \mathrm{Aut}(QR)$. Since QR is cyclic of order 403, $\mathrm{Aut}(QR)$ is cyclic of order $\phi(403) = (13 - 1)(31 - 1) = 360$. (The relationship is that $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ acts on $\mathbb{Z}/n\mathbb{Z}$ as multiplication.)

- Let x be a generator of P and let y be a generator of QR. Because G is non-abelian, ψ is not the trivial map. Since P is simple, this implies that ψ is injective. Therefore, $\psi(x) = \sigma$ is an element of $\mathrm{Aut}(QR)$ of order 5, and we have $xyx^{-1} = \sigma(y)$.

- I claim that the choices of x, y, and σ do not affect the structure of the group. If $\tau \in \mathrm{Aut}(QR)$ is another element of order 5, then $\tau = \sigma^n$ for some n (coprime to 5) because a cyclic group of order 360 has a unique (cyclic) subgroup of order 5. Therefore, $x^n y x^{-n} = \sigma^n(y) = \tau(y)$. Since P has order 5 and n is coprime to 5, the nth power map $P \to P$ is an automorphism. Therefore, changing which element $\psi(x)$ is equivalent to picking another generator of P. Since y is an arbitrary generator of QR, the relation $xyx^{-1} = \sigma(y)$ holds for all of them.

- Now we need only give an example of a specific choice of σ to make this explicit. By the Chinese remainder theorem, $(\mathbb{Z}/403\mathbb{Z})^\times \cong (\mathbb{Z}/13\mathbb{Z})^\times \times (\mathbb{Z}/31\mathbb{Z})^\times$. We want an element of order 5. The relation $n^5 \equiv 1 \mod 13$ implies that $n \equiv 1 \mod 13$ because 5 does not divide $12 = \phi(13)$ (this is another application of Lagrange's theorem). On the other hand, 2 has multiplicative order 5 modulo 31. So we want a number n such that $n \equiv 1 \mod 13$ and $n \equiv 2 \mod 31$. The Euclidean algorithm shows us that

  $$1 = 12 \cdot 13 - 5 \cdot 31.$$

  So we want a number equivalent to $2 \cdot (12 \cdot 13) + 1 \cdot (-5 \cdot 31) \mod 403$, and in fact 157 does the trick. Thus a presentation for our group is

  $$\left\langle x, y : x^5 = y^{403} = 1, xyx^{-1} = y^{157} \right\rangle.$$