- The purpose of this note is to classify groups of order 12. Let G be a group of order 12.
- If G is abelian, then the fundamental theorem of finite abelian groups tells us that G is isomorphic to one of $C_{12}, C_6 \times C_2$ (these are invariant factor decompositions). In the following we shall assume that G is non-abelian.
- By the first sylow theorem, G has a sylow 2-subgroup, P, and a sylow 3-subgroup Q. Since 3 is prime, Q is cyclic. Let q be a generator. Since $P \cap Q$ is a subgroup of P and Q, its order divides the gcd of 4 and 3 by Lagrange's theorem. Thus $P \cap Q = 1$. We will index our cases by the size of n_3. Typically, indexing with n_p, where p is the largest prime dividing $|G|$ leads to a cleaner proof than breaking into cases along a smaller prime.
- Since n_3 is congruent to 1 mod 3 and n_3 divides 12, we have $n_3 = 1$ or $n_3 = 4$.
- Suppose $n_3 = 1$. Then Q is a normal subgroup of G of index 4. Therefore, $G = PQ$, $P \cap Q = 1$, and $P \leq N_G(Q)$. This tells us that G is isomorphic to some semi-direct product $Q \rtimes_\psi P$ for some $\psi : P \to \mathrm{Aut}(Q)$ (and the isomorphism is just the inclusion when restricted to P or Q). Since Q is cyclic of order 3, $\mathrm{Aut}(Q)$ has order $\phi(3) = 2$ (where ϕ is Euler's totient function). Since we are assuming G is non-abelian, ψ is not the trivial map.
- If $P = V_4 = \langle x, y \rangle$, then at least one of $\psi(x), \psi(y), \psi(xy)$ is the non-identity element of $\mathrm{Aut}(Q)$ (the map that transposes q and q^2 and fixes the identity). Without loss of generality, it is $\psi(x)$. Therefore, $xqx^{-1} = q^2$. If $\psi(y) = \psi(xy) = 1$, then $\psi(x) = \psi(xy)\psi(y) = 1$, contrary to assumption. Therefore, one of $\psi(y), \psi(xy)$ is also non-trivial. Let it be $\psi(xy)$. Therefore, $yqy^{-1} = q$. So $(yq)^2 = yqyq = (yqy^{-1})q = q^2$ (where we are using $|y| = 2$), and we see that yq is an element of order 6. Since $(yq)^3 = y$, we have that x and yq generate the group. The former has order 2 and the latter has order 6. Furthermore, $x(yq) = y(xqx^{-1})x = yq^2x = (yq)^5x$. So G is the dihedral group of order 12.
- If $P = C_4 = \langle x \rangle$, then $\psi(x)$ must be the non-trivial element of $\mathrm{Aut}(Q)$. Therefore, G is generated by x and q, $|x| = 4, |q| = 3$, and $xqx^{-1} = q^2$. These relations determine the group G completely because they allow us to write any element in the form $q^m x^n$ for some m, n with $0 \leq m < 3, 0 \leq n < 4$. Since the sylow 2-subgroup is cyclic, this group is not the dihedral group of order 12.
- Now, suppose that $n_3 = 4$. By the second sylow theorem, G acts transitively by conjugation on the set of sylow 3-subgroups. This action gives us a group homomorphism, $\rho : G \to S_4$. Since $Q \leq N_G(Q)$ and $4 = n_3 = |G : N_G(Q)|$, we must have $N_G(Q) = Q$. Since the normalizer of Q is its stabilizer under conjugation, and the kernel of ρ is contained in every stabilizer, we know that $\ker(\rho)$ is a normal subgroup of G contained in Q. Since Q is simple and non-normal, $\ker(\rho) = 1$ and ρ is injective. Thus G may be identified with a subgroup of S_4. Since $|G| = 12$, G has index 2 in S_4. Therefore, G is a normal subgroup of S_4. Since the proper normal subgroups of S_n are contained in the alternating group (for any n), $G = A_4$ by cardinality.