Solutions to the Problems in *Introduction to Commutative Algebra* by M. F. Atiyah and I. G. MacDonald

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Contents

1 Rings and Ideals

1.1

Let A be a ring with nilpotent element x and unit element u. We will show that $u+x$ is a unit. Take $u=1$ to see that $1+x$ is a unit.

Constructive Proof: Let $n>0$ be the least integer such that $x^n=0$ and define

$$y := u^{-1}\sum_{i=0}^{n-1}(-u^{-1}x)^i.$$

Note that

$$(-u^{-1}x)^n = (-u^{-1})^n x^n = 0.$$

As a consequence, we see that

$$y(u+x) = \left(\sum_{i=0}^{n-1}(-u^{-1}x)^i\right)(1+u^{-1}x) = 1-(-u^{-1}x)^n = 1.$$

So $u+x$ is a unit with inverse y.

Non-Constructive Proof: By Proposition 1.8, the nilradical of A is the intersection of all prime ideals of A. Since every maximal ideal of A is prime, the nilradical of A is contained in the Jacobson radical. Therefore, x is contained in the Jacobson radical of A. Apply Proposition 1.9 with $y=-u^{-1}$ to seee that, $1+u^{-1}x$ is a unit. Since the units of A form a group,

$$u(1+u^{-1}x) = u+x$$

is a unit as well.

1.2

Let $f = a_0 + a_1x + \ldots + a_nx^n$ and $g = b_0 + b_1x + \ldots + b_mx^m$ be elements of $A[x]$.

(i) Suppose that f is a unit with inverse g. Let $\mathfrak{p}$ be a prime ideal of A. Since $A/\mathfrak{p}$ is a domain and $f\cdot g = 1$ in $(A/\mathfrak{p})[x]$, f must have degree 0 when reduced mod $\mathfrak{p}$. Thus, $a_1,\ldots,a_n$ are contained in every prime ideal of A. On the other hand, Proposition 1.8 tells us that the nilradical of A is the intersection of all the prime ideals of A. So $a_1,\ldots,a_n$ are nilpotent. As a consequence, $a_0 = f + (a_0 - f)$ is the sum of a unit and a nilpotent element, hence a unit by Exercise 1.1.

If a_0 is a unit and $a_1,\ldots,a_n$ are nilpotent, then $f-a_0$ is nilpotent and $f = a_0 + (f-a_0)$ is the sum of a unit and a nilpotent element, hence a unit by Exercise 1.1.

(ii) Let $\mathfrak{p}$ be a prime ideal of A. If $f^k=0$, then $f^k=0$ in $(A/\mathfrak{p})[x]$. Since $A/\mathfrak{p}$ is a domain, $f=0$ in $(A/\mathfrak{p})[x]$. Thus, $a_0,\ldots,a_n$ are contained in every prime ideal of A, and are nilpotent as a consequence of Proposition 1.8.

If $a_0,\ldots,a_n$ are in the nilradical of $A[x]$, then the $A[x]$-linear combination f is also in the nilradical because the nilradical is an ideal by Proposition 1.7.

(iii) If f is a zero-divisor, let g be a non-zero annihilator of f in $A[x]$ of least degree. If g has positive degree, then a_ng is an annihilator of f of degree less than g, hence $a_ng=0$. If k is a positive integer such that $a_jg=0$ for $j\geq k$, then $(f-(a_kx^k+\ldots+a_nx^n))g=0$ implies that $a_{k-1}g=0$. Therefore, the least such k is 0. That is, $a_0g=0$. Furthermore, we have shown that $a_kb_m=0$ for all k. Therefore, $b_mf=0$. (In fact, we have implicitly shown that g has degree 0)

(iv) Let $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ be the ideals generated by the coefficients of f, g, and fg, respectively. The inclusion $\mathfrak{c}\subseteq\mathfrak{ab}$ shows that f and g are primitive if fg is. On the other hand, if fg is not primitive, then $\mathfrak{c}$ is contained in some maximal ideal, $\mathfrak{m}$, of A. Therefore $fg=0$ in $(A/\mathfrak{m})[x]$. Since $A/\mathfrak{m}$ is a field, $f=0$ or $g=0$ in $(A/\mathfrak{m})[x]$. This in turn implies that $\mathfrak{a}\subseteq\mathfrak{m}$ or $\mathfrak{b}\subseteq\mathfrak{m}$. In either case, at least one of f or g is not primitive.

1.3

Define $A_r := A[x_1, \dots, x_r]$ for all positive integers r. Let $f = a_0 + \dots + a_n x_r^n$ be an element of $A_r = A_{r-1}[x_r]$ with $a_0, \dots, a_n$ elements of A_{r-1}.

Suppose that every unit of A_{r-1}, when viewed as a polynomial with coefficients in A, has unit constant term and all other coefficients nilpotent. Furthermore, suppose that every nilpotent element of A_{r-1}, when viewed as a polynomial with coefficients in A, has nilpotent coefficients.

If f is a unit, then by Exercise 1.2 (i), a_0 is a unit and $a_1, \dots, a_n$ are nilpotent in A_{r-1}. By our hypotheses, the constant coefficient of f over A is a unit, and all other coefficients of f over A are nilpotent.

If f is nilpotent, then by Exercise 1.2 (ii), $a_0, \dots, a_n$ are nilpotent in A_{r-1}. By our hypothesis, all coefficients of f over A are nilpotent.

Pick a monomial order on the monomials in the indeterminates $x_1, \dots, x_r$. If f is a zero-divisor, let $g \in A_r$ be a non-zero annihilator of f with minumum leading term. Let $a \in A$ be the lead coefficient of f over A. Then ag is an annihilator of f of degree less than g, hence $ag = 0$. By successively subtracting off the leading terms of f, a similar induction on the chosen monomial order as in Exercise 1.2 (iii) shows that g annihilates every coefficient of f over A. Therefore, $bf = 0$, where b is the lead coefficient of g over A. (Remark 1)

Let $f, g \in A_r$ and let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be the ideals generated by the respective coefficients of f, g, and fg over A. The inclusion $\mathfrak{c} \subseteq \mathfrak{ab}$ shows that f and g are primitive if fg is. On the other hand, if fg is not primitive, then $\mathfrak{c}$ is contained in some maximal ideal, $\mathfrak{m}$, of A. Therefore $fg = 0$ in $(A/\mathfrak{m})[x_1, \dots, x_r]$. Since $A/\mathfrak{m}$ is a field, $f = 0$ or $g = 0$ in $(A/\mathfrak{m})[x_1, \dots, x_r]$. This in turn implies that $\mathfrak{a} \subseteq \mathfrak{m}$ or $\mathfrak{b} \subseteq \mathfrak{m}$. In either case, at least one of f or g is not primitive.

1.4

Since every maximal ideal is prime, the nilradical of any ring is contained in the Jacobson radical.

If f is in the Jacobson radical of $A[x]$, then $1 - fx$ is a unit by Proposition 1.9. By Exercise 1.2 (i), the coefficients of f are nilpotent. Therefore f is in the nilradical of $A[x]$ by Exercise 1.2 (ii).

1.5

Let $f = \sum_{n \geq 0} a_n x^n$ be an element of $A[\![x]\!]$.

(i) If f is a unit with inverse $g = \sum_{n \geq 0} b_n x^n$, then $fg = 1$. On the other hand, fg has constant term $a_0 b_0$. Therefore a_0 is a unit.

Suppose a_0 is a unit. Define $h := 1 - a_0^{-1} f$. Define $g := a_0^{-1} \sum_{n \geq 0} h^n$. Since h has positive order, each of the coefficients in the definition of g is a finite polynomial in the coefficients of h. Thus g is well defined. Furthermore,

$$fg = a_0(1-h)a_0^{-1} \sum_{n \geq 0} h^n = \sum_{n \geq 0} h^n - \sum_{n \geq 0} h^{n+1} = h^0 + \sum_{n \geq 1} h^n - \sum_{n \geq 1} h^n = 1.$$

Therefore, f is a unit.

(ii) Let k be the order of f as a power series and suppose that $f^N = 0$ for some positive integer N. The least degree term of f^N has coefficient $a_k^N = 0$. Thus a_k is nilpotent. Since the nilradical is an ideal, $f - a_k x^k$ is also nilpotent. By induction on k, we see that every coefficient of f is nilpotent.

The converse is false: Define

$$A := \mathbb{F}_p[T, T^{p^{-1}}, T^{p^{-2}}, T^{p^{-3}}, \dots]/(T^p).$$

Claim: For every $n \geq 0$, the element $T^{p^{-n}}$ has nilpotency order p^{n+1} in A.

Proof: Clearly, $\left(T^{p^{-n}}\right)^{p^n} = T$. On the other hand, suppose

$$\left(T^{p^{-n}}\right)^a = T^p \cdot g\left(T^{p^{-m}}\right)$$

for some integers $a, m \geq 0$. Let $N := \max\{m, n\}$. Take p^Nth powers to get

$$T^{ap^{N-n}} = T^{p^{N+1}} \cdot g\left(T^{p^{N-m}}\right)$$

in $\mathbb{F}_p[T]$. By comparing degrees we see that

$$ap^{N-n} \geq p^{N+1}.$$

In particular, $a \geq p^{n+1}$ as desired.

Claim: The element $f := \sum_{n\geq 0} T^{p^{-n}} x^n$ of $A[\![x]\!]$ is not nilpotent.

Proof: It is easy to see that

$$f(x)^p = \left(\sum_{n\geq 0} T^{p^{-n}} x^n\right)^p = \sum_{n\geq 0} T^{p^{1-n}} x^{pn} = \sum_{n\geq 0} T^{p^{-n}} x^{p(n+1)} = x^p f(x^p).$$

By repeated application of this identity, we see that

$$f(x)^{p^n} = x^{p^n} f\left(x^{p^n}\right)$$

for all $n \geq 0$. Since the right hand side has the same coefficients as the left (albeit with new indices), no power of $f(x)$ is zero.

(iii) By Proposition 1.9, f is in the Jacobson radical of $A[\![x]\!]$ if and only if $1 - fg$ is a unit of $A[\![x]\!]$ for all g in $A[\![x]\!]$. By (i), this latter condition is equivalent to requiring that $1 - a_0 b$ is a unit in A for all b in A. Again, Proposition 1.9 shows that this is equivalent to letting a_0 be an element of the Jacobson radical of A.

(iv) Let $\mathfrak{m}$ be a maximal ideal of $A[\![x]\!]$. Since 0 is in the Jacobson radical of A, x is in the Jacobson radical of $A[\![x]\!]$ by (iii). Thus $\mathfrak{m}^c$ consists of the constant terms of elements of $\mathfrak{m}$ and $\mathfrak{m} = \mathfrak{m}^c + xA[\![x]\!]$. Furthermore,

$$A[\![x]\!]/\mathfrak{m} \cong (A[\![x]\!]/xA[\![x]\!])/(\mathfrak{m}/xA[\![x]\!]) \cong A/\mathfrak{m}^c$$

shows that $A/\mathfrak{m}^c$ is a field. As a consequence, $\mathfrak{m}^c$ is maximal.

(v) Let $\mathfrak{p}$ be a prime ideal of A, and define

$$\phi : A[\![x]\!] \to A \to A/\mathfrak{p}$$

by following the evaluation at 0 map with reduction modulo $\mathfrak{p}$. This map is obviously surjective. Furthermore, the kernel of ϕ consists of power series with constant term in $\mathfrak{p}$. That is, $\ker(\phi) = \mathfrak{p} + xA[\![x]\!]$. By the First Isomorphism Theorem, $A[\![x]\!]/(\mathfrak{p} + xA[\![x]\!]) \cong A/\mathfrak{p}$. Since $A/\mathfrak{p}$ is a domain, $\mathfrak{p} + xA[\![x]\!]$ is a prime ideal. On the other hand, $\mathfrak{p} = (\mathfrak{p} + xA[\![x]\!])^c$ shows that $\mathfrak{p}$ is the contraction of a prime ideal of $A[\![x]\!]$.

1.6

Let A be a ring such that every ideal not contained in the nilradical contains a non-zero idempotent. Let e be an idempotent in the Jacobson radical. By Proposition 1.9, $1 - e$ is a unit. Since e is idempotent,

$$e(1 - e) = e - e^2 = 0.$$

Thus $e = 0$. Since, the Jacobson radical contains no non-zero idempotents, it must be contained in the nilradical.

1.7

Let A be a ring such that every element is a root of a polynomial of the form $x^n - x$ with n a positive integer. Let $\mathfrak{p}$ be a prime ideal and x an element of A not in $\mathfrak{p}$. For some positive n, $x^n - x = 0$ is in $\mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal not containing x, we have the equality $x^{n-1} = 1$ in $A/\mathfrak{p}$. We have now shown that the non-zero elements of $A/\mathfrak{p}$ have multiplicative inverses. Therefore, $A/\mathfrak{p}$ is a field and $\mathfrak{p}$ is a maximal ideal.

1.8

Let Σ be the set of prime ideals of A. By Theorem 1.3, Σ is not empty. Order Σ by reverse inclusion and let $\{\mathfrak{p}_n\}_{n=1}^{\infty}$ be a chain. Define $\mathfrak{p} := \bigcap_{n=1}^{\infty} \mathfrak{p}_n$. If x, y are not elements of $\mathfrak{p}$, then there are i, j such that x is not in $\mathfrak{p}_i$ and y is not in $\mathfrak{p}_j$. Let k be the greater of i, j. Then x and y are both not in $\mathfrak{p}_k$. Since $\mathfrak{p}_k$ is a prime ideal, xy is not in $\mathfrak{p}_k$. Thus xy is not in $\mathfrak{p}$. In particular, $\mathfrak{p}$ is a prime ideal. Since $\mathfrak{p}$ is an upper bound for the chain $\{\mathfrak{p}_n\}_{n=1}^{\infty}$, Zorn's Lemma implies that Σ has maximal elements. Therefore, A has minimal prime ideals.

1.9

Let $\mathfrak{a}$ be a proper ideal of A. If $\mathfrak{a} = \sqrt{\mathfrak{a}}$, then $\mathfrak{a}$ is the intersection of the prime ideals that contain it by Proposition 1.14. On the other hand, let $\mathfrak{a}$ be the intersection of some collection of prime ideals Σ and let Σ' be the set of all prime ideals containing $\mathfrak{a}$. Then the following inclusions

$$\mathfrak{a} = \bigcap_{\mathfrak{p} \in \Sigma} \supseteq \bigcap_{\mathfrak{p} \in \Sigma'} = \sqrt{\mathfrak{a}} \supseteq \mathfrak{a}$$

show that $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

1.10

Let A be a ring with nilradical $\mathfrak{R}$.

(i)⇒(ii) Let $\mathfrak{p}$ be the unique prime ideal of A. By Proposition 1.8, $\mathfrak{p} = \mathfrak{R}$. Furthermore, $\mathfrak{p}$ is a maximal ideal by Corollary 1.4. If x is a non-unit, then it is contained in $\mathfrak{p}$ and is nilpotent as a consequence.

(i)⇒(iii) Let $\mathfrak{p}$ be the unique prime ideal of A. By Proposition 1.8, $\mathfrak{p} = \mathfrak{R}$. Furthermore, $\mathfrak{p}$ is a maximal ideal by Corollary 1.4. Thus $A/\mathfrak{R}$ is a field.

(ii)⇒(i) Since every element of A not in $\mathfrak{R}$ is a unit, Proposition 1.6 (i) asserts that A is a local ring with unique maximal ideal $\mathfrak{R}$. On the other hand, Proposition 1.8 states that $\mathfrak{R}$ is contained in every prime ideal of A. Therefore, $\mathfrak{R}$ is the only prime ideal of A.

(ii)⇒(iii) Since every element of A not in $\mathfrak{R}$ is a unit, Proposition 1.6 (i) asserts that A is a local ring with unique maximal ideal $\mathfrak{R}$. In particular, $A/\mathfrak{R}$ is a field.

(iii)⇒(i) Note that every prime ideal of A contains $\mathfrak{R}$. By the Lattice Isomorphism Theorem, $\mathfrak{p}$ is a prime ideal of A if and only if $\mathfrak{p}/\mathfrak{R}$ is a prime ideal of $A/\mathfrak{R}$. Since $A/\mathfrak{R}$ is a field, every prime ideal of A is equal to $\mathfrak{R}$. Thus A has exactly one prime ideal.

(iii)⇒(ii) Let x be a non-nilpotent element of A. Let y be an element of A such that $\overline{xy} = 1$ in $A/\mathfrak{R}$. In other words, $xy - 1$ is nilpotent. By Exercise 1.1, $xy = 1 + (xy - 1)$ is a unit. So x is a unit with inverse $y(xy)^{-1}$.

1.11

Let A be a Boolean ring.

(i) Given x in A,

$$0 = (x+1)^2 - (x+1) = x^2 + 2x + 1 - x - 1 = 2x.$$

(ii) If x is not an element of a prime ideal $\mathfrak{p}$, then $x(1-x)=0$ implies that $x=1$ in $A/\mathfrak{p}$. Thus $A/\mathfrak{p}$ is the field with two elements and $\mathfrak{p}$ is a maximal ideal.

(iii) Let x,y be elements of an ideal $\mathfrak{a}$. Define $z := x+y+xy$ and note that

$$xz = x(x+y+xy) = x^2 + 2xy = x$$

and

$$yz = y(x+y+xy) = y^2 + 2xy = y.$$

Therefore, $(x,y)=(z)$. If $\mathfrak{a}$ is a finitely generated ideal with more than one generator, we may apply this process to reduce the number of generators by one. By repeated application, we can find a principal generator for$\mathfrak{a}$.

1.12

Let $A,\mathfrak{m}$ be a local ring with an idempotent element x. If x is in $\mathfrak{m}$, then $1-x$ is a unit because $\mathfrak{m}$ is the Jacobson radical. Thus $x(1-x)=0$ implies that $x=0$. If x is a unit, then $x(1-x)=0$ implies that $x=1$. Therefore the only idempotents of A are 0 and 1.

1.13

Let $K_0 = K$ be a field. Given a non-negative integer n for which the field, K_n, is defined, let Σ_n be the set of monic irreducible elements of $K_n[x]$ and let A_n be the polynomial ring over K_n generated by the set of indeterminants $\{x_f | f \in \Sigma_n\}$. Define $\mathfrak{a}_n$ as the ideal of A_n generated by the set $\{f(x_f) \in A | f \in \Sigma_n\}$. Since K_n is a field, A_n is a domain. Thus every element of $\mathfrak{a}_n$ has positive degree and $\mathfrak{a}_n$ does not contain 1. Let $\mathfrak{m}_n$ be a maximal ideal of A_n containing $\mathfrak{a}_n$ and define $K_{n+1} = A_n/\mathfrak{m}_n$. The map

$$K_n \to A_n \to A_n/\mathfrak{m}_n = K_{n+1},$$

given by the composition of the canonical inclusion and quotient maps, is a field homomorphism. Thus it is injective and we may identify K_n with a subfield of K_{n+1}.

Let $\overline{K} = \bigcup_{n\geq 0} K_n$ (really, $\overline{K} = \varinjlim K_n$). If x,y are in $\overline{K}$, then they are contained in some subfields K_n, K_m. Letting k be the greater of m and n, x and y are in K_k. Therefore the sum, difference, and product of x,y are in $\overline{K}$. If y is non-zero, then the quotient x/y is in $\overline{K}$ as well. Since $0,1$ are in $\overline{K}$, and any field arithmetic of $\overline{K}$ can be performed in a subfield, $\overline{K}$ is a field.

Let f be an irreducible monic polynomial in $\overline{K}[x]$. Since f has only finitely many coefficients, there is some n such that f is an irreducible monic polynomial in $K_n[x]$. By construction, f has a root in K_{n+1}, hence in $\overline{K}$. By the Euclidean division, f must have degree 1. Therefore, $\overline{K}$ is algebraically closed.

By construction, the field extension K_{n+1}/K_n is algebraic for every n. Since the class algebraic extensions is a distinguished class of field extensions (Lang's *Algebra*), K_n/K is an algebraic extension for every $n \geq 0$. Therefore, every element of $\overline{K}$ is algebraic over K, and $\overline{K}$ is the algebraic closure of K up to isomorphism.

1.14

Let Σ be the set of ideals of the ring A which consist of solely zero-divisors. This set is non-empty because it contains (0). Order Σ by inclusion and suppose that $\{\mathfrak{a}_n\}_{n\geq 1}$ is a chain. Since $\mathfrak{a} = \bigcup_{n\geq 1} \mathfrak{a}_n$ is an ideal of A consisting solely of zero divisors, it is an upper bound for $\{\mathfrak{a}_n\}_{n\geq 1}$ in Σ. By Zorn's Lemma, Σ has a maximal element $\mathfrak{b}$.

If x,y are not elements of $\mathfrak{b}$, then $(x)+\mathfrak{b}, (y)+\mathfrak{b}$ each contain non-zero-divisors, because $\mathfrak{b}$ is a maximal element of Σ. Since the product of two non-zero-divisors is a non-zero-divisor and $((x)+\mathfrak{b})((y)+\mathfrak{b}) \subseteq (xy)+\mathfrak{b}$, some element of $(xy)+\mathfrak{b}$ is not a zero-divisor. Thus xy is not an element of $\mathfrak{b}$. Therefore, $\mathfrak{b}$ is a prime ideal of A.

If d is a zero-divisor, then (d) is an element of Σ. Thus (d) is contained in a maximal element of Σ, which in turn is a prime ideal of A. Therefore, the union of the ideals that are maximal in Σ is the set of zero-divisors, and each of these is prime ideal.

1.15

Let X be the set of all prime ideals of the ring A. For any subset E of A, define $V(E)$ as the set of prime ideals containing E.

(i) If $\mathfrak{a}$ is the ideal of A generated by E, then $V(\mathfrak{a}) \subseteq V(E)$ by the definition of V. If $\mathfrak{p}$ is a prime ideal containing E, then $\mathfrak{p}$ contains $\mathfrak{a}$ because $\mathfrak{a}$ is the smallest ideal containing E. Thus $V(E) \subseteq V(\mathfrak{a})$ as well.

Given that every ideal is contained in its radical, $V\left(\sqrt{\mathfrak{a}}\right) \subseteq V(\mathfrak{a})$. Since $\sqrt{\mathfrak{a}}$ is the intersection of all prime ideals containing $\mathfrak{a}$, $V(\mathfrak{a}) \subseteq V\left(\sqrt{\mathfrak{a}}\right)$.

(ii) Since (0) is contained in every ideal of A, $V(0) = X$. On the other hand, no proper ideals of A contain 1, hence $V(1)$ is empty.

(iii) Let $\{E_i\}_{i\in I}$ be a family of subsets of A and let $\mathfrak{p}$ be a prime ideal. If $\mathfrak{p}$ contains $\bigcup_{i\in I} E_i$, then $\mathfrak{p}$ contains each E_i. Thus $V\left(\bigcup_{i\in I} E_i\right) \subseteq \bigcap_{i\in I} V(E_i)$. If $\mathfrak{p}$ contains each E_i, then clearly $\mathfrak{p}$ contains their union. Thus $\bigcap_{i\in I} V(E_i) \subseteq V\left(\bigcup_{i\in I} E_i\right)$ as well.

(iv) If $\mathfrak{a}, \mathfrak{b}$ are ideals of A, then

$$V(\mathfrak{a}\mathfrak{b}) = V\left(\sqrt{\mathfrak{a}\mathfrak{b}}\right) = V\left(\sqrt{\mathfrak{a}\cap\mathfrak{b}}\right) = V(\mathfrak{a}\cap\mathfrak{b})$$

by Exercise 1.13 of the text.

Given that $\mathfrak{a}\mathfrak{b}$ is contained in $\mathfrak{a}$ and $\mathfrak{b}$, $V(\mathfrak{a})\cup V(\mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$. If $\mathfrak{p}$ is a prime ideal containing $\mathfrak{a}\mathfrak{b}$, then $\mathfrak{p}$ contains $\mathfrak{a}$ or $\mathfrak{b}$, by Proposition 1.11. Thus $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.

1.16

- Because $\mathbb{Z}$ is a principal ideal domain, the prime ideals of $\mathbb{Z}$ are of the form (p), where p is a prime number, and (0). Therefore, $\operatorname{Spec}\mathbb{Z}$ is a curve through all the principal ideals of $\mathbb{Z}$ of the form (p), with p a prime ideal number, and the point (0). For geometric reasons that will be clear later, draw (0)) off to the side of the curve.

  Let $U \subseteq \operatorname{Spec}\mathbb{Z}$ be an open set. By definition, $U = V(\mathfrak{a})^c$ for some ideal $\mathfrak{a}$ of $\mathbb{Z}$. Since $\mathbb{Z}$ is a principal ideal domain, $\mathfrak{a} = (a)$ for some integer a. If $a = 0$, then $V(\mathfrak{a}) = V(0) = \operatorname{Spec}\mathbb{Z}$ and $U = \varnothing$. If U is non-empty, then $a \neq 0$. By the Fundamental Theorem of Arithmetic, $a = \pm\prod_{i=1}^k p_i^{\alpha_i}$ for some prime numbers $p_1, \ldots, p_k$ and positive integers $\alpha_1, \ldots, \alpha_k$. Define $\sqrt{a}$ as $\prod_{i=1}^k p_i$. Since non-zero prime ideals of $\mathbb{Z}$ are maximal, we have

  $$V(\mathfrak{a}) = V(a) = V(\sqrt{a}) = \bigcup_{i=1}^k V(p_i) = \{(p_1), \ldots, (p_k)\}.$$

  Therefore, $U = \{(p_1), \ldots, (p_k)\}^c$. In summary, non-empty open subsets of $\operatorname{Spec}\mathbb{Z}$ are cofinite sets containing (0).

- Since $\mathbb{R}$ is a field, $\operatorname{Spec}\mathbb{R}$ is the one point space containing (0) under the trivial topology.

- As before, $\mathbb{C}[x]$ is a principal ideal domain. Thus its prime ideals are of the form (f), where f is a monic irreducible or $f = 0$. Since $\mathbb{C}$ is algebraically closed, the monic irreducible elements of $\mathbb{C}[x]$ are of the form $x - a$, where a is a complex number. Therefore, $\operatorname{Spec}\mathbb{C}[x]$ can be drawn as the complex plane, where the point $(x - a)$ is identified with the complex number a, with an additional point corresponding to the prime ideal (0).

Let $U \subseteq \operatorname{Spec}\mathbb{C}[x]$ be an open set. By definition, $U = V(\mathfrak{a})^c$ for some ideal $\mathfrak{a}$ of $\mathbb{C}[x]$. Since $\mathbb{C}[x]$ is a principal ideal domain, $\mathfrak{a} = (f)$ for some monic polynomial $f \in \mathbb{C}[x]$. If $f = 0$, then $V(\mathfrak{a}) = V(0) = \operatorname{Spec}\mathbb{C}[x]$ and $U = \varnothing$. If U is non-empty, then $f \neq 0$. By the Fundamental Theorem of Algebra, $f = \prod_{i=1}^{k}(x - a_i)^{\alpha_i}$ for some complex numbers $a_1, \dots, a_k$ and positive integers $\alpha_1, \dots, \alpha_k$. Define $\sqrt{f}$ as $\prod_{i=1}^{k}(x - a_i)$. Since non-zero prime ideals of $\mathbb{C}[x]$ are maximal, we have

$$V(\mathfrak{a}) = V(f) = V\left(\sqrt{f}\right) = \bigcup_{i=1}^{k} V(x - a_i) = \{(x - a_1), \dots, (x - a_k)\}.$$

Therefore, $U = \{(x - a_1), \dots, (x - a_k)\}^c$. In summary, non-empty open subsets of $\operatorname{Spec}\mathbb{C}[x]$ are cofinite sets containing (0).

- As before, $\mathbb{R}[x]$ is a principal ideal domain. Therefore, its prime ideals are of the form (f), where f is a monic irreducible polynomial or $f = 0$. Since $\mathbb{C}/\mathbb{R}$ is a degree 2 extension and $\mathbb{C}$ is algebraically closed, the irreducible polynomials in $\mathbb{R}[x]$ are degree 1 or 2. Let (f) be a non-zero prime ideal of $\mathbb{R}[x]$ and let $\iota : \mathbb{R}[x] \to \mathbb{C}[x]$ be the inclusion map. If $\deg(f) = 1$, then $\iota(f) = x - r$ for some real number r, and $\iota^*((x - r)) = (f)$. If $\deg(f) = 2$, then $\iota(f) = (x - a)(x - \overline{a})$ for some non-real complex number a, and $\iota^*((x - a)) = \iota^*((x - \overline{a})) = (f)$. By Exercise 1.21 (i), $\iota^* : \operatorname{Spec}\mathbb{C}[x] \to \operatorname{Spec}\mathbb{R}[x]$ is a continuous map. We have just shown that it is surjective and that $\operatorname{Spec}\mathbb{R}[x]$ is a quotient of $\operatorname{Spec}\mathbb{C}[x]$ by identifying conjugate points. So $\operatorname{Spec}\mathbb{R}[x]$ can be drawn as a copy of the complex plane, folded over the real axis, with an additional point corresponding to the prime ideal (0).

  Let $U \subseteq \operatorname{Spec}\mathbb{R}[x]$ be a non-empty open set. Then $\iota^{*-1}(U)$ is a non-empty open subset of $\operatorname{Spec}\mathbb{C}[x]$. By our previous arguments, $\iota^{*-1}(U) = \{(x - a_1), \dots, (x - a_k)\}^c$ for some complex numbers $a_1, \dots, a_k$. Since ι^* is surjective, we know that

  $$U = \iota^*(\iota^{*-1}(U)) = \{\iota^*((x - a_1)), \dots, \iota^*((x - a_k))\}^c.$$

  In summary, non-empty open subsets of $\operatorname{Spec}\mathbb{R}[x]$ are cofinite sets containing (0).

- Let $\iota : \mathbb{Z} \to \mathbb{Z}[x]$ be the inclusion map. By Exercise 1.21 (i), $\iota^* : \operatorname{Spec}\mathbb{Z}[x] \to \operatorname{Spec}\mathbb{Z}$ is continuous.

  Claim: The map ι^* is surjective.

  Proof: Let p be a prime number. The identification, $\mathbb{Z}[x]/p\mathbb{Z}[x] \cong (\mathbb{Z}/p\mathbb{Z})[x] = \mathbb{F}_p[x]$ shows that $p\mathbb{Z}[x]$ is a prime ideal of $\mathbb{Z}[x]$. Clearly, $\iota^*(p\mathbb{Z}[x]) = (p)$. On the other hand, $\iota^*((0)) = (0)$. So ι^* is surjective.

  Therefore, we can partition $\operatorname{Spec}\mathbb{Z}[x]$ as a union of fibers of the map ι^* via

  $$\operatorname{Spec}\mathbb{Z}[x] = \iota^{*-1}((0)) \cup \left(\bigcup_{p \text{ prime number}} \iota^{*-1}((p)) \right).$$

  Let $\mathfrak{p}$ be a non-zero prime ideal of $\mathbb{Z}[x]$. Then the map $\mathbb{Z}/\iota^{-1}(\mathfrak{p}) \to \mathbb{Z}[x]/\mathfrak{p}$ obtained from $\iota : \mathbb{Z} \to \mathbb{Z}[x]$ by passing to quotients is injective. In particular, if $\iota^*(\mathfrak{p}) = (c)$ with $c \geq 0$, then c is the characteristic of $\mathbb{Z}[x]/\mathfrak{p}$.

  Claim: The fiber $\iota^{*-1}((0))$ consists of prime ideals of the form (f) where f is either an irreducible polynomial or 0.

  Proof: If $\iota^*(\mathfrak{p}) = (0)$, then every non-zero element of $\mathfrak{p}$ has positive degree. Suppose that $\mathfrak{p} \neq (0)$. Let $f \in \mathfrak{p}$ be a irreducible polynomial with minimum degree. Let g be any element of $\mathfrak{p}$. Since $\mathbb{Q}[x]$ is a Euclidean Domain, there are $q, r \in \mathbb{Q}[x]$ such that $g = qf + r$ and r has degree less than f. Since f, g are in $\mathfrak{p}$, r is in $\mathfrak{p}$ as well. By our hypothesis on f, r is zero. By Gauss' Lemma, q is in $\mathbb{Z}[x]$. Therefore, $\mathfrak{p} = (f)$.

  Claim: Let p be a prime number. The fiber $\iota^{*-1}((p))$ consists of prime ideals of the form (p, f) where f is a polynomial that is irreducible mod p or f is 0.

  Proof: If $\iota^*(\mathfrak{p}) = (p)$, then the quotient map $\mathbb{Z}[x] \to \mathbb{Z}[x]/\mathfrak{p}$ factors through the quotient $\mathbb{Z}[x] \to \mathbb{F}_p[x]$. So $\mathbb{Z}[x]/\mathfrak{p}$ is a quotient of $\mathbb{F}_p[x]$ by some prime ideal $\mathfrak{q}$. Since $\mathbb{F}_p[x]$ is a Principal Ideal Domain, $\mathfrak{q} = (\overline{f})$ for some element $\overline{f}$ of $\mathbb{F}_p[x]$ which is irreducible or 0. By the Lattice Isomorphism Theorem, $\mathfrak{q}$ lifts to the ideal (p, f) of $\mathbb{Z}[x]$, where f is any lift of $\overline{f}$.

In summary, the prime ideals of $\mathbb{Z}[x]$ break into four types:

(f), where $f \in \mathbb{Z}[x]$ is an irreducible polynomial of positive degree,

(p), where $p \in \mathbb{Z}$ is a prime number,

(p, f), where $p \in \mathbb{Z}$ is a prime number and $f \in \mathbb{Z}[x]$ is irreducible modulo p, and

(0).

Therefore, $\operatorname{Spec}\mathbb{Z}[x]$ can be drawn as a grid. The bottom of the grid consists of ideals (c) where c is either a prime number or 0. To maintain consistency with the description of $\operatorname{Spec}\mathbb{Z}$ given above, draw (0) off to the side from the other ideals of this form.

Above each ideal on the bottom row, place the ideals (c, f), where f is irreducible mod c. Each such f is irreducible, so you may as well put the ideal (p, f) in the same row as the ideal f. (Though f won't necessarily be irreducible mod every prime. For instance, $x^2 + x + 1$ is irreducible mod 2 but not mod 3.)

Claim: Every proper closed subset of $\operatorname{Spec}\mathbb{Z}[x]$ is a finite union of sets of the form $V(g_1, \ldots, g_n)$, where $g_1, \ldots, g_n$ are irreducible elements of $\mathbb{Z}[x]$.

Proof: Since $\mathbb{Z}$ is a Euclidean Domain, it is Noetherian. By Theorem 7.5, $\mathbb{Z}[x]$ is Noetherian. By Exercises 6.7 and 6.8, $V(\mathfrak{a})$ is a finite union of irreducible closed subspaces of $\operatorname{Spec}\mathbb{Z}[x]$. If g_1, g_2 are elements of $\mathbb{Z}[x]$, then Exercise 1.15 (iv) implies $V(g_1 g_2) = V(g_1) \cup V(g_2)$. Therefore, if $V(g)$ is irreducible, then g is irreducible. By Exercise 1.18 (ii) and our taxonomy of the prime ideals in $\mathbb{Z}[x]$, the converse holds as well. By Exercise 1.17 (i), the sets X_g form a basis for the topology on $X = \operatorname{Spec}\mathbb{Z}[x]$. By Exercise 6.6, every open subset of $\operatorname{Spec}\mathbb{Z}[x]$ is quasi-compact. If g has factorization into irreducibles $\prod_{i=1}^n g_i$, then $X_g = \bigcap_{i=1}^n X_{g_i}$. By De Morgan's Laws and the preceding facts, every closed subset of $\operatorname{Spec}\mathbb{Z}[x]$ is a finite union of sets of the form $V(g_1, \ldots, g_n)$, where $g_1, \ldots, g_n$ are irreducible elements of $\mathbb{Z}[x]$.

Claim: Every proper closed subset of $\operatorname{Spec}\mathbb{Z}[x]$ is a finite union of sets $V(p), V(f), V(p, f)$ for varying p, f, where p is a prime number and f is either irreducible (for (f)) or irreducible mod p (for (p, f)). Note that *irreducible mod p* implies *irreducible.*

Proof: Let $g_1, \ldots, g_n$ be distinct irreducible elements of $\mathbb{Z}[x]$. Let $\mathfrak{p}$ be a minimal prime ideal containing the ideal $(g_1, \ldots, g_n)$. If $\mathfrak{p} = (f)$ for some irreducible f, then all the g_i's are multiples of f. If this case, $g_1 = f$ and $n = 1$. If $n \neq 1$, then $(g_1, \ldots, g_n)$ has only finitely many minimal primes, and each must be of the form (p, f) for p a prime number and f irreducible mod p.

Note that $V(p)$, for p a prime number, consists of (p) and all prime ideals (p, f) with f irreducible mod p. Furthermore, $V(f)$, for f an irreducible polynomial of positive degree, consists of (f) and all prime ideals of the form (p, f) where p is a prime number such that f is irreducible mod p.

Conclusion: Every non-empty open subset of $\operatorname{Spec}\mathbb{Z}[x]$ can be formed by puncturing finitely many closed points and removing finitely many horizontal and vertical grid lines from the total space, and every non-empty open set contains (0).

1.17

For each f in A, define $X_f = V(f)^c$ in $X = \operatorname{Spec} A$. Let U be an open set containing the point $\mathfrak{p}$ of X. By definition of the Zariski topology, there is an ideal $\mathfrak{a}$ such that $U^c = V(\mathfrak{a})$. Let f be an element of $\mathfrak{a}$ not contained in $\mathfrak{p}$. Then $U^c = V(\mathfrak{a}) \subseteq V(f)$ implies that $X_f \subseteq U$ and $\mathfrak{p}$ is in X_f. Thus the sets X_f form a basis for the Zariski topology.

(i) By Exercise 1.15 (iv),

$$X_f \cap X_g = V(f)^c \cap V(g)^c = (V(f) \cup V(g))^c = V(fg)^c = X_{fg}.$$

(ii) An element of A is nilpotent if and only if it is contained in every prime ideal. Since X_f is the set of prime ideals not containing f, the equivalence follows.

(iii) By Corollary 1.5, every non-unit is contained in a maximal ideal. Conversely, no prime ideal contains a unit. Therefore, f is contained in no prime ideals precisely when f is a unit.

(iv) Every prime ideal containing g contains f if and only if $\sqrt{(g)} \supseteq \sqrt{(f)}$. The first condition is equivalent to $X_f \subseteq X_g$. Swapping the roles of f and g gives the desired equivalence.

(v) Let $\{X_f\}_{f\in E}$ be an open cover of X. Taking complements shows that $V(E)$ is empty. Therefore, 1 is in the ideal generated by E. This in turn implies that there are $f_1, \ldots, f_n$ in E and $a_1, \ldots, a_n$ in A such that $1 = \sum_{i=1}^n a_i f_i$. Thus $V(f_1, \ldots, f_n)$ is empty. Taking complements again shows that $\{X_{f_i}\}_{i=1}^n$ is an open cover of X.

(vi) Let $\{X_g\}_{g\in E}$ be an open cover of X_f. Taking complements shows that $V(f) \supseteq V(E)$. Therefore, f is in the radical ideal generated by E. This in turn implies that there are $g_1, \ldots, g_n$ in E, $a_1, \ldots, a_n$ in A, and a positive integer m such that $f^m = \sum_{i=1}^n a_i g_i$. Thus $V(f) \supseteq V(g_1, \ldots, g_n)$. Taking complements again shows that $\{X_{g_i}\}_{i=1}^n$ is an open cover of X_f.

(vii) Let U be a quasi-compact open set of X. Since U is open, it is the union of a collection of basic open sets $\{X_f\}_{f\in E}$. On the other hand, only finitely many such sets are needed to cover U because it is quasi-compact.

Conversely, finite unions of quasi-compact sets are quasi-compact.

1.18

(i) By (ii), the closure of $\{x\}$ is the set of prime ideals containing $\mathfrak{p}_x$. Thus $\{x\}$ is closed if and only if $\mathfrak{p}_x$ is not properly contained in any other prime ideals. In other words, $\{x\}$ is closed if and only if $\mathfrak{p}_x$ is maximal.

(ii) The closure of $\{x\}$ is the intersection of all closed sets containing x. Furthermore, x is in $V(E)$ if and only if E is a subset of $\mathfrak{p}_x$. Thus every closed set containing x contains $V(\mathfrak{p}_x)$. On the other hand, $V(\mathfrak{p}_x)$ is a closed set containing x.

(iii) By (ii), y is in the closure of $\{x\}$ if and only if $\mathfrak{p}_y \supseteq \mathfrak{p}_x$.

(iv) If x, y are distinct points in X such that y is in the closure of $\{x\}$, then $V(\mathfrak{p}_y) \subseteq V(\mathfrak{p}_x)$ by (iii). Since x and y are distinct, x is not in $V(\mathfrak{p}_y)$. Therefore, x is in the complement of the closure of $\{y\}$.

1.19

If Spec A is irreducible and fg is nilpotent, then $X_f \cap X_g = X_{fg}$ is empty by Exercise 1.17. Thus X_f or X_g is empty, and correspondingly f or g is nilpotent.

Suppose that the nilradical of A is prime ideal. Let X_f, X_g be non-empty open sets. Then f, g are not nilpotent and neither is fg. Thus $X_f \cap X_g = X_{fg}$ is non-empty.

1.20

Let X be a topological space.

(i) Let U, V be open subsets of X such that $U \cap \overline{Y}$ and $V \cap \overline{Y}$ are non-empty. By the definition of the closure, $U \cap Y$ and $V \cap Y$ are non-empty open sets of Y. Since Y is irreducible, $(U \cap Y) \cap (V \cap Y)$ is a non-empty subset of $(U \cap \overline{Y}) \cap (V \cap \overline{Y})$. Thus $\overline{Y}$ is irreducible.

(ii) Note that every point is irreducible. If Y is an irreducible subspace of X, then let Σ be the set of irreducible subspaces of X containing Y, ordered by inclusion. Let $\{Z_n\}_{n\geq 1}$ be a chain in Σ and define $Z = \bigcup_{n\geq 1} Z_n$. Suppose that U, V are open sets intersecting Z. Then there are positive integers i, j such that $U \cap Z_i$ and $V \cap Z_j$ are non-empty. If k is the greater of i, j, then both U and V intersect Z_k. Since Z_k is irreducible, $U \cap V$ contains a point of Z_k. Thus $U \cap V$ contains a point of Z. Therefore, Z is an upper bound for $\{Z_n\}_{n\geq 1}$ in Σ. By Zorn's Lemma, there is a maximal irreducible subspace of X containing Y.

(iii) If Y is a maximal irreducible subspace of X, then it must equal its closure by (i). As in (ii), note that points are irreducible subspaces of X. By (ii), the maximal irreducible subspaces cover X. In a Hausdorff space, any subspace with more than one point has disjoint non-empty open sets, and is thus not irreducible.

(iv) Let $X = \operatorname{Spec} A$ for some ring A. If Y is a closed irreducible subspace of X, then there is a radical ideal $\mathfrak{a}$ such that $Y = V(\mathfrak{a})$. If f, g are not in $\mathfrak{a}$, then X_f, X_g both meet Y, because $\mathfrak{a}$ is its own radical. Since Y is irreducible, $X_f \cap X_g = X_{fg}$ intersects Y. Thus fg is also not in $\mathfrak{a}$, and we see that $\mathfrak{a}$ is a prime ideal.

If $\mathfrak{p}$ is a prime ideal, $Y = V(\mathfrak{p})$, and X_f, X_g intersect Y, then f, g are not in $\mathfrak{p}$. Thus fg is not in $\mathfrak{p}$ and $X_f \cap X_g = X_{fg}$ intersects Y. Therefore, Y is irreducible.

For any prime ideals $\mathfrak{p}, \mathfrak{q}$, $V(\mathfrak{p}) \subseteq V(\mathfrak{q})$ if and only if $\mathfrak{p} \supseteq \mathfrak{q}$. Thus maximal irreducible components of X correspond to minimal prime ideals.

1.21

Let $\phi : A \to B$ be a ring homomorphism, $X = \operatorname{Spec} A$, and $Y = \operatorname{Spec} B$. If $\mathfrak{q}$ is a prime ideal of B, then $A/\phi^{-1}(\mathfrak{q}) \to B/\mathfrak{q}$ is injective by the First Isomorphism Theorem. Thus $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A. Define $\phi^* : Y \to X$ by $\phi^*(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$.

(i) If f is an element of A, then $\phi^{-1}(\mathfrak{q})$ is contained in X_f if and only if $\mathfrak{q}$ is contained in $Y_{\phi(f)}$. Therefore, $\phi^{*-1}\left(X_f\right) = Y_{\phi(f)}$.

(ii) If $\phi^{-1}(\mathfrak{q}) \supseteq \mathfrak{a}$, then $\mathfrak{q} \supseteq \mathfrak{q}^{ce} \supseteq \mathfrak{a}^e$. Conversely, $\mathfrak{q} \supseteq \mathfrak{a}^e$ implies that $\phi^{-1}(\mathfrak{q}) \supseteq \mathfrak{a}^{ec} \supseteq \mathfrak{a}$. Therefore, $\phi^{*-1}\left(V(\mathfrak{a})\right) = V(\mathfrak{a}^e)$.

(iii) Let $\mathfrak{q}$ be a prime ideal containing $\mathfrak{b}$ and define $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$. Then $\mathfrak{b}^c$ is contained in $\mathfrak{p}$. Thus $\phi^*\left(V(\mathfrak{b})\right) \subseteq V(\mathfrak{b}^c)$.

Without loss of generality, $\mathfrak{b} = \sqrt{\mathfrak{b}}$. Let $\mathfrak{p} \in V(\mathfrak{b}^c)$ and suppose $f \notin \mathfrak{p}$. By the transitivity of containment, $\phi(f) \notin \mathfrak{b} = \sqrt{\mathfrak{b}}$. By Proposition 1.14, there exists $\mathfrak{q} \in V(\mathfrak{b})$ such that $\phi(f) \notin \mathfrak{q}$. Therefore,

$$\phi^*(\mathfrak{q}) \in \phi^*\left(Y_{\phi(f)} \cap V(\mathfrak{b})\right) \subseteq X_f \cap \phi^*(V(\mathfrak{b})).$$

Hence, $V(\mathfrak{b}^c) \subseteq \overline{\phi^*(V(\mathfrak{b}))}$.

(iv) If ϕ is surjective, then $\overline{\phi} : A/\ker\phi \to B$ is an isomorphism by the First Isomorphism Theorem. By the Lattice Isomorphism Theorem, $\phi^* : Y \to V(\ker\phi)$ is bijective and continuous. Since ϕ is surjective, every basic open set of Y can be put in the form $Y_{\phi(f)}$ for some f in A. By (i), $\phi^*\left(Y_{\phi(f)}\right) = X_f \cap V(\ker\phi)$. Therefore, ϕ^* is a homeomorphism of Y onto its image.

(v) Suppose that $\ker\phi$ is the in the nilradical of A and let X_f be a non-empty open set of X. Since f is not in the nilradical of A, $\phi(f)$ is not nilpotent. Therefore, $Y_{\phi(f)}$ contains some point $\mathfrak{q}$. By (i), $\phi^*(\mathfrak{q})$ is in X_f. Therefore, $\phi^*(Y)$ is dense in X.

Suppose that $\phi^*(Y)$ is dense in X and let f be an element of $\ker\phi$. Then $Y_{\phi(f)} = \phi^{*-1}(X_f)$ is empty. Thus X_f is empty and f is nilpotent.

(vi) If $\psi : B \to C$ is another ring homomorphism and $\mathfrak{r}$ is a prime ideal of C, then

$$(\psi \circ \phi)^*(\mathfrak{r}) = (\psi \circ \phi)^{-1}(\mathfrak{r}) = \phi^{-1}(\psi^{-1}(\mathfrak{r})) = (\phi^* \circ \psi^*)(\mathfrak{r}).$$

(vii) Let A be a domain with exactly one non-zero prime ideal $\mathfrak{p}$, and let K be the field of fractions of A. Define $B = (A/\mathfrak{p}) \times K$ and $\phi : A \to B$ by $\phi(x) = (\overline{x}, x)$.

The prime ideals of B are $(A/\mathfrak{p}) \times (0)$ and $(0) \times K$. Furthermore, ϕ^* is bijective due to $\phi^*((A/\mathfrak{p}) \times (0)) = (0)$ and $\phi^*((0) \times K) = \mathfrak{p}$. On the other hand, $\operatorname{Spec} B$ has the discrete topology, but the closure of $\{(0)\}$ in $\operatorname{Spec} A$ is the whole space.

1.22

Let $A := \prod_{i=1}^n A_i$ be a direct product of the rings A_i, let $\pi_i : A \to A_i$ be the canonical projections, let $\mathfrak{a}_i := \ker \pi_i$, and define $X_i := V(\mathfrak{a}_i)$. By the Chinese Remainder Theorem, $\psi : A \to \prod_{i=1}^n A/\mathfrak{a}_i$ is an isomorphism that matches a prime ideal $\mathfrak{p}$ to the product $\prod_{i=1}^n (\mathfrak{p} + \mathfrak{a}_i)/\mathfrak{a}_i$. Therefore,

$$A/\mathfrak{p} \cong \prod_{i=1}^n A/(\mathfrak{p} + \mathfrak{a}_i)$$

is a domain. Thus all but one factor ring must be zero. Calling the remainding index j, $A/\mathfrak{p} \cong A/(\mathfrak{p} + \mathfrak{a}_j)$ by the natural projection. So $\mathfrak{p}$ must contain $\mathfrak{a}_j$. We have now shown that $\operatorname{Spec} A$ is the disjoint union of the closed sets X_i. Since the complement of each X_i is a finite union of closed sets, each X_i is open. By Exercise 1.21 (iv), π_i^* gives a homeomorphism from $\operatorname{Spec} A_i$ to X_i.

(i)⇒(ii) If $X = \operatorname{Spec} A$ is disconnected, then there are proper non-zero ideals $\mathfrak{a}, \mathfrak{b}$ such that $X = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$ and $\varnothing = V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} + \mathfrak{b})$. By Exercise 1.15, there are f, g in $\mathfrak{a}, \mathfrak{b}$ and a positive integer n such that $f + g = 1$ and $(fg)^n = 0$. Since $(f, g) \subseteq \sqrt{(f^n, g^n)}$ and $V(f, g)$ is empty, $V(f^n, g^n)$ is also empty. Thus $(f^n) + (g^n) = (1)$ and the Chinese Remainder Theorem implies that $A \to (A/(f^n)) \times (A/(g^n))$ is an isomorphism. Neither of f, g is a unit, because they are elements of the proper ideals $\mathfrak{a}, \mathfrak{b}$.

(i)⇒(iii) Suppose that A is reduced. If $X = \operatorname{Spec} A$ is disconnected, then there are proper non-zero ideals $\mathfrak{a}, \mathfrak{b}$ such that $X = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$ and $\varnothing = V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} + \mathfrak{b})$. By Exercise 1.15, there are f, g in $\mathfrak{a}, \mathfrak{b}$ such that $f + g = 1$ and $fg = 0$. Therefore,

$$f^2 = f(1 - g) = f - fg = f.$$

Furthermore, $f \neq 0, 1$ because $\mathfrak{a}, \mathfrak{b}$ are both proper ideals. (Remark 2)

(ii)⇒(i) This was shown above.

(ii)⇒(iii) Let $\phi : A \to A_1 \times A_2$ be an isomorphism, where neither of A_1, A_2 are the zero ring. Then $\phi^{-1}((1, 0))$ is a non-trivial idempotent of A.

(iii)⇒(i) If e is a non-trivial idempotent of A, then

$$V(e) \cap V(1 - e) = V(1) = \varnothing$$

and

$$V(e) \cup V(1 - e) = V(e - e^2) = V(0) = \operatorname{Spec} A$$

is a separation of $\operatorname{Spec} A$.

(iii)⇒(ii) Let e be a non-trivial idempotent of A. Then $(e) + (1 - e) = (1)$ and $(e)(1 - e) = (0)$. By the Chinese Remainder Theorem, $A \cong A/(e) \times A/(1 - e)$.

(iii)⇒(ii) (Alternative) Let e be a non-trivial idempotent of A. The ideal Ae is a ring with identity e. Since

$$(1 - e)^2 = 1 - 2e + e^2 = 1 - e,$$

$1 - e$ is also a non-trivial idempotent. Let $\phi : A \to Ae \times A(1 - e)$ be the ring homomorphism given by

$$\phi(x) = (xe, x(1 - e)).$$

If $x \in \ker \phi$, then $x = xe + x(1 - e) = 0$. So ϕ is injective. On the other hand, let y, z be any elements of A. Then $\phi(ye + z(1 - e)) = (ye, z(1 - e))$. So ϕ is surjective and an isomorphism.

Let A be a local ring. By Exercise 1.12, A contains no non-trivial idempotent. Therefore, $\operatorname{Spec} A$ is connected by (iii).

1.23

Let A be a Boolean ring and $X = \operatorname{Spec} A$.

(i) If f is an element of A, then $X_f \cap X_{(1-f)} = X_0$ is empty and

$$X_f \cup X_{(1-f)} = (V(f) \cap V(1-f))^c = V((f) + (1-f))^c = V(1)^c = X.$$

Therefore, $X_f = X^c_{(1-f)}$ is closed.

(ii) If $f_1, \dots, f_n$ are elements of A, then by (i)

$$\bigcup_{i=1}^{n} X_{f_i} = \bigcup_{i=1}^{n} V(1-f_i) = V\left(\prod_{i=1}^{n}(1-f_i)\right) = X_f,$$

where $f = 1 - \prod_{i=1}^{n}(1-f_i)$.

(iii) Let Y be a closed and open subset of X. Because Y is open, it is a union of basic open sets. Since Y is closed and X is quasi-compact, Y is also quasi-compact. Thus Y is a finite union of basic open sets. By (ii), Y is a basic open set.

(iv) If x, y are distinct points in X, then, without loss of generality, there is some f in A such that X_f contains x but not y, by Exercise 1.18 (iv). By (i), y is in $X_{(1-f)}$ and the two sets are disjoint. X is quasi-compact by Exercise 1.17 (v).

1.24

Let L be a Boolean algebra. Then L has a greatest element 1 and a least element 0, each of $\vee, \wedge$ distribute over each other, and each element a has a unique complement a'.

Let $A(L)$ be the set of L under the operations

$$a + b = (a \wedge b') \vee (a' \wedge b), \;\; a \cdot b = a \wedge b.$$

Let a, b, c be elements of $A(L)$. First, $+, \cdot$ are commutative because $\vee, \wedge$ are. The additive identity is 0 by

$$a + 0 = (a \wedge 1) \vee (a' \wedge 0) = a \vee 0 = a.$$

The multiplicative identity is given by 1. The additive inverse of a is a by

$$a + a = (a \wedge a') \vee (a' \wedge a) = 0 \vee 0 = 0.$$

Addition is associative by

$$\begin{aligned}(a+b)+c &= \big[((a \wedge b') \vee (a' \wedge b)) \wedge c'\big] \vee \big[((a' \vee b) \wedge (a \vee b')) \wedge c\big] \\ &= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \vee (a' \wedge b' \wedge c) \vee (a \wedge b \wedge c) \\ &= a + (b + c),\end{aligned}$$

where the second line follows from the first by distributing c, c' and further shows that the expression is symmetric in a, b, and c. Multiplication is associative because $\wedge$ is. Finally, the distributive law holds:

$$\begin{aligned}a(b+c) &= a \wedge \big((b \wedge c') \vee (b' \wedge c)\big) \\ &= [(a \wedge b) \wedge c'] \vee [b' \wedge (a \wedge c)] \\ &= [(a \wedge b) \wedge (a' \vee c')] \vee [(a' \vee b') \wedge (a \wedge c)] \\ &= ab + ac.\end{aligned}$$

Therefore, $A(L)$ is a Boolean ring.

Let A be a Boolean ring with elements a, b and define $a \vee b = a + b + ab$, $a \wedge b = ab$, and $a \leq b$ if and only if $ab = a$.

First, we show that $\leq$ is a partial order. If $a \leq b$ and $b \leq a$, then $a = ab = ba = b$. So $\leq$ is anti-symmetric. If $a \leq b \leq c$, then $ac = abc = ab = a$, and $a \leq c$. So $\leq$ is transitive.

Second, we show that A is a lattice under this ordering.

$$a(a \vee b) = a(a + b + ab) = a \text{ and } a(a \wedge b) = aab = a \wedge b$$

shows that $a \wedge b \leq a \leq a \vee b$. By symmetry, $a \wedge b \leq b \leq a \vee b$ as well. If $c \leq a, b$, then

$$c(a \wedge b) = cab = c$$

shows that $c \leq a \wedge b$. Thus $a \wedge b$ is the infimum of a and b. If $a, b \leq c$, then

$$c(a \vee b) = c(a + b + ab) = a \vee b$$

shows that $a \vee b \leq c$. Thus $a \vee b$ is the supremum of a and b.

Third, we show that this is a Boolean lattice. In general,

$$a \vee 1 = a + 1 + a \cdot 1 = 1$$

and

$$a \wedge 0 = a \cdot 0 = 0.$$

For distributivity,

$$(a \wedge b) \vee c = ab + c + abc = (a + c + ac)(b + c + bc) = (a \wedge c) \vee (b \wedge c)$$

and

$$(a \vee b) \wedge c = ac + bc + abc = ac + bc + acbc = (a \wedge c) \vee (b \wedge c).$$

Finally, if we define $a' = 1 + a$, then

$$a \wedge a' = a(1 + a) = a + a = 0$$

and

$$a \vee a' = a + (1 + a) + a(1 + a) = 1.$$

Furthermore, if x is a complement of a, then

$$0 = a \wedge x = ax$$

and

$$1 = a \vee x = a + x + ax.$$

These equations imply that $x = 1 + a = a'$. So complements are unique. Thus $A, \leq$ is a Boolean lattice.

1.25

Given a Boolean lattice L, define

$$\phi : L \to \mathcal{P}(\operatorname{Spec} A(L)) : f \mapsto X_f.$$

By Exercise 1.24, $f \leq g$ in L if and only if $fg = f$. This in turn implies that $X_f \cap X_g = X_{fg} = X_f$, which yields $X_f \subseteq X_g$.

If $X_f = X_g$, then

$$X_f = X_g = X^c_{(1+g)}$$

by the solution to Exercise 1.23 (i). So $X_f \cap X_{(1+g)} = X_{f(1+g)}$ is empty. Therefore, $f(1 + g)$ is nilpotent. It is easy to see that every Boolean ring is reduced. So $f(1 + g) = 0$. In particular, $f = fg$. So $f \leq g$. Swap the roles of f and g to see that $g \leq f$. Since the ordering on L is reflexive, $f = g$ and ϕ is injective.

On the other hand, the image of ϕ is precisely the class of open-and-closed subspaces of the compact Hausdorff space $\operatorname{Spec} A(L)$ by Exercise 1.23.

1.26

We will use the notation of the problem.

(i) Shown in the text.

(ii) Shown in the text.

(iii) Since $\mu : X \to \tilde{X}$ is bijective,

$$\begin{aligned}\mu(U_f) &= \mu(\{x \in X : f(x) \neq 0\}) \\ &= \{\mu(x) : x \in X \text{ and } f(x) \neq 0\} \\ &= \{\mathfrak{m}_x : x \in X \text{ and } f \notin \mathfrak{m}_x\} \\ &= \{\mathfrak{m} \in \widetilde{X} : f \notin \mathfrak{m}\} \\ &= \widetilde{U}_f.\end{aligned}$$

Let U be an open subset of X. If $x \in U$, then $\{x\}$ and U^c are disjoint closed sets. By Urysohn's Lemma there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \notin U$. Therefore, $x \in U_f \subseteq U$. So the sets U_f form a basis for the topology on X.

Let $\tilde{U}$ be an open subset of $\tilde{X}$. Since $\tilde{X} \subseteq \operatorname{Spec} C(X)$ has the subspace topology, there is an open set $\tilde{W}$ of $\operatorname{Spec} C(X)$ such that $\tilde{U} = \tilde{X} \cap \tilde{W}$. Let $\mathfrak{m} \in \tilde{U} \subseteq \tilde{W}$. By Exercise 1.17, there is $f \in C(X)$ such that $\mathfrak{m} \in (\operatorname{Spec} C(X))_f \subseteq \tilde{W}$. Note that $\tilde{X} \cap (\operatorname{Spec} C(X))_f = \tilde{U}_f$. Therefore,

$$\mathfrak{m} \in \tilde{U}_f = \tilde{X} \cap (\operatorname{Spec} C(X))_f \subseteq \tilde{X} \cap \tilde{W} = \tilde{U}.$$

So the sets $\tilde{U}_f$ form a basis for the topology on $\tilde{X}$.

1.27

We will use the notation of the problem.

We will prove that the map $\mu : X \to \widetilde{X}$ is surjective. If $\mathfrak{m}$ is a maximal ideal of $P(X)$, then $P(X)/\mathfrak{m}$ is a finitely generated k-algebra that is also a field. By Corollary 5.24, $P(X)/\mathfrak{m}$ is a finite field extension of k. Since k is algebraically closed, $k \to P(X)/\mathfrak{m}$ is an isomorphism of k-algebras. For each i, let x_i be the image of ξ_i in k. Then the elements $\xi_1 - x_1, \ldots, \xi_n - x_n$ each map to 0 in k. Therefore, $\mathfrak{m} = (\xi_1 - x_1, \ldots, \xi_n - x_n) = \mathfrak{m}_x$, and μ is surjective.

1.28

Let X, Y be affine algebraic varieties in k^n, k^m respectively, let $\phi : X \to Y$ be a regular map, let $P(X) = k[t_1, \ldots, t_n]/I(X)$ and $P(Y) = k[s_1, \ldots, s_m]/I(Y)$. Furthermore, let $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_m$ be the respective images of $t_1, \ldots, t_n$ and $s_1, \ldots, s_m$ in $P(X)$ and $P(Y)$.

(Inverse Functor) Given a k-algebra homomorphism $\phi^* : P(Y) \to P(X)$, the elements $\phi^*\eta_1, \ldots, \phi^*\eta_m$ lift to polynomials $g_1, \ldots, g_m$ in $k[t_1, \ldots, t_n]$. Let $\psi : X \to k^m$ be the regular morphism with coordinate functions $g_1, \ldots, g_m$. Since all lifts of $g_1, \ldots, g_m$ are congruent mod $I(X)$, the regular morphism ψ only depends on ϕ^* and not on the choice of $g_1, \ldots, g_m$.

Suppose that $h \in I(Y)$. Then

$$0 = \phi^*(h(\eta_1, \ldots, \eta_m)) = h(g_1, \ldots, g_m) \mod I(X) = (h \circ \psi)(\xi_1, \ldots, \xi_n).$$

Therefore, $\operatorname{im} \psi \subseteq Y$. By abuse of notation, we call the corestricted morphism $\psi : X \to Y$.

($\phi \mapsto \phi^*$ is injective) If $\phi^* : P(Y) \to P(X)$ is induced by some regular morphism $\phi : X \to Y$, then

$$f_j \mod I(X) = \phi^*\eta_j = g_j \mod I(X)$$

for all j. So $\psi = \phi$ as functions from X to Y.

($\phi \mapsto \phi^*$ is surjective) Furthermore, $\psi^* \eta_j = g_j \mod I(X)$. So $\psi^* = \phi^*$ as k-algebra homomorphisms.

2 Modules

2.1

Direct Proof: If m, n are coprime, then there are integers x, y such that $mx + ny = 1$. For any simple tensor, $f \otimes g$, we get

$$f \otimes g = (mx + ny)(f \otimes g) = x(mf \otimes g) + y(f \otimes ng) = 0.$$

Thus $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$.

Proof via Localization: (Remark 3) If p is a prime not dividing m, then the Euclidean algorithm provides integers q, r such that $p = qm + r$. Since any common factor of m and r divides p, r is coprime to m. Therefore, p is a unit in $\mathbb{Z}/m\mathbb{Z}$. In particular, $(\mathbb{Z}/m\mathbb{Z})_{(p)} = 0$.

Let $X = \operatorname{Spec}\mathbb{Z}$. If m, n are coprime, then $(m) + (n) = (1)$. Therefore,

$$X_m \cup X_n = V(m, n)^c = V(1)^c = X.$$

Let p be a prime. If $(p) \in X_m$, then

$$(\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z})_{(p)} = (\mathbb{Z}/m\mathbb{Z})_{(p)} \otimes (\mathbb{Z}/n\mathbb{Z})_{(p)} = 0 \otimes (\mathbb{Z}/n\mathbb{Z})_{(p)} = 0.$$

Swap the roles of m and n to see that $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$ localizes to 0 at every closed point of X. By Proposition 3.8, $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = 0$.

Proof via Deferment: Set $A = \mathbb{Z}$, $\mathfrak{a} = (m)$, and $M = \mathbb{Z}/n\mathbb{Z}$. Apply Exercise 2.2 and note that $\mathfrak{a}M = M$ in this case.

2.2

If $\mathfrak{a}$ is an ideal of A and M is an A-module, then

$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$$

is a short exact sequence of A-modules. Tensoring with M yields

$$\mathfrak{a} \otimes_A M \to A \otimes_A M \to (A/\mathfrak{a}) \otimes_A M \to 0,$$

another exact sequence of A-modules, since the tensor product is right exact. If we define $f : \mathfrak{a} \times M \to \mathfrak{a}M$ by $f(a, m) = am$, then f is obviously A-bilinear and surjective. By the universal property of the tensor product, there is a unique A-linear map $f^{\otimes} : \mathfrak{a} \otimes_A M \to \mathfrak{a}M$ that satisfies $f^{\otimes}(a \otimes m) = am$. Furthermore, define $g : (A/\mathfrak{a}) \times M \to M/\mathfrak{a}M$ by $g(\overline{a}, m) = \overline{am}$. This map is obviously A-bilinear and surjective as well. By the universal property of the tensor product, there is a unique A-linear map $g^{\otimes} : (A/\mathfrak{a}) \otimes_A M \to M/\mathfrak{a}M$ such that $g^{\otimes}(\overline{a} \otimes m) = \overline{am}$.

On the other hand,

$$0 \to \mathfrak{a}M \to M \to M/\mathfrak{a}M \to 0$$

is a short exact sequence of A-modules. Putting this together gives the following commutative diagram with exact rows, where central vertical arrow is the canonical isomorphism.

$$\begin{array}{ccccccccc}
 & & \mathfrak{a} \otimes_A M & \longrightarrow & A \otimes_A M & \longrightarrow & (A/\mathfrak{a}) \otimes_A M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f^{\otimes}} & & \downarrow & & \downarrow{\scriptstyle g^{\otimes}} & & \\
0 & \longrightarrow & \mathfrak{a}M & \longrightarrow & M & \longrightarrow & M/\mathfrak{a}M & \longrightarrow & 0
\end{array}$$

Applying the Snake Lemma to this diagram yields another exact sequence,

$$0 \to \ker g^{\otimes} \to \operatorname{coker} f^{\otimes} = 0.$$

Therefore, $g^{\otimes}$ is injective. Given that $g^{\otimes}$ is surjective as well,

$$g^{\otimes} : (A/\mathfrak{a}) \otimes_A M \to M/\mathfrak{a}M$$

is an isomorphism of A-modules.

2.3

Let A be a local ring with maximal ideal $\mathfrak{m}$ and finitely generated A-modules M, N such that $M \otimes_A N = 0$. Additionally, let $\kappa = A/\mathfrak{m}$ be the residue field of A. By Proposition 2.14 and Exercise 2.15,

$$\begin{aligned} M_\kappa \otimes_\kappa N_\kappa &= (\kappa \otimes_A M) \otimes_\kappa (\kappa \otimes_A N) \\ &= [(\kappa \otimes_A M) \otimes_\kappa \kappa] \otimes_A N \\ &= (\kappa \otimes_A M) \otimes_A N \\ &= \kappa \otimes_A (M \otimes_A N) \\ &= (M \otimes_A N)_\kappa = 0, \end{aligned}$$

where the equalities are the canonical identifications. Since M, N are finitely generated A-modules, M_κ, N_κ are finitely generated κ-vector spaces by Proposition 2.17. If d, e are the respective dimensions of M_κ, N_κ, then $M_\kappa \otimes_\kappa N_\kappa$ is isomorphic to $\kappa^d \otimes_\kappa \kappa^e$, which can further be identified with κ^{de} by Proposition 2.14. This shows that d or e must by zero, because $M_\kappa \otimes_\kappa N_\kappa = 0$. Without loss of generality, $d = 0$ and $M_\kappa = 0$. On the other hand, $M_\kappa = M/\mathfrak{m}M$ by Exercise 2.2. Finally, Nakayama's Lemma implies that $M = 0$.

2.4

Let $\{M_i\}_{i \in I}$ be a family of A-modules with direct sum M.

Claim: The module $M \otimes_A N$ is canonically isomorphic to $\bigoplus_{i \in I}(M_i \otimes_A N)$ for regardless of the A-module N.

Proof 1: (Construct Maps) By the universal property of the tensor product, the inclusions $\iota_i : M_i \to M$ induce maps $M_i \otimes_A N \to M \otimes_A N$. These in turn induce a unique A-linear map $f : \bigoplus_{i \in I}(M_i \otimes_A N) \to M \otimes_A N$ such that $f(m_j \otimes n) = m_j \otimes n$. On the other hand, the map $M \times N \to \bigoplus_{i \in I}(M_i \otimes_A N)$ given by $(\sum_i m_i, n) \mapsto \sum_i m_i \otimes n$ is A-bilinear. By the universal property of the tensor product, it induces a unique A-linear map $g : M \otimes_A N \to \bigoplus_{i \in I}(M_i \otimes_A N)$ such that $g(\sum_i m_i \otimes n) = \sum_i m_i \otimes n$. By construction, $f \circ g = \mathrm{id}_{M \otimes_A N}$ and $g \circ f = \mathrm{id}_{\bigoplus_{i \in I}(M_i \otimes_A N)}$ and $M \otimes_A N$ is canonically isomorphic to $\bigoplus_{i \in I}(M_i \otimes_A N)$.

Proof 2: (Yoneda) We will make use of the adjunction preceding Proposition 2.18 and the following characterization of the direct sum:

> "There are morphisms $\iota_i : M_i \to M$ such that for any A-module N and any family of A-linear maps $\{f_i : M_i \to N\}_{i \in I}$, there is a unique A-linear map $f : M \to N$ such that $f \circ \iota_i = f_i$ for all $i \in I$."

In particular, the map

$$\hom_A(M, N) \to \prod_{i \in I} \hom_A(M_i, N) : f \mapsto (f \circ \iota_i)_{i \in I}$$

is a bijection, natural in N.

Let N, P be any A-modules. We have the following natural isomorphisms,

$$\begin{aligned}\hom_A(M \otimes_A N, P) &= \hom_A(M, \hom_A(N, P)) \\ &= \prod_{i \in I} \hom_A(M_i, \hom_A(N, P)) \\ &= \prod_{i \in I} \hom_A(M_i \otimes_A N, P) \\ &= \hom_A\left(\bigoplus_{i \in I} (M_i \otimes_A N), P\right)\end{aligned}$$

By Yoneda's Lemma, $M \otimes_A N$ is naturally isomorphic to $\bigoplus_{i \in I}(M_i \otimes_A N)$.

(Remark 4)

Solution of Problem 1: (Element Chase) Let $\phi : N' \to N$ be injective, let M be a flat A-module, and let η be an element of $\ker(\mathrm{id}_{M_i} \otimes \phi)$. Then

$$0 = (\iota_i \otimes \mathrm{id}_N)(\mathrm{id}_{M_i} \otimes \phi)(\eta) = (\mathrm{id}_M \otimes \phi)(\iota_i \otimes \mathrm{id}_{N'})(\eta).$$

Since M is flat, $\mathrm{id}_M \otimes \phi$ is injective. Furthermore, for each $i \in I$, the map $\iota_i \otimes \mathrm{id}_{N'}$ is injective because it is inclusion into the ith coordinate of the direct sum $M \otimes_A N'$. Therefore, $\eta = 0$ and $\mathrm{id}_{M_i} \otimes \phi$ is injective. By Proposition 2.19, the A-modules M_i are all flat.

Now suppose that every M_i is a flat A-module and let $\sum_i \theta_i$ be an element of $\ker(\mathrm{id}_M \otimes \phi)$. Then

$$(\iota_i \otimes \mathrm{id}_N)(\mathrm{id}_{M_i} \otimes \phi)(\theta_i) = (\mathrm{id}_M \otimes \phi)(\iota_i \otimes \mathrm{id}_{N'})(\theta_i) = 0$$

for every i. Since $\mathrm{id}_{M_i} \otimes \phi$ and $\iota_i \otimes \mathrm{id}_N$ are injective, $\theta_i = 0$. Thus $\sum_i \theta_i = 0$, $\mathrm{id}_M \otimes \phi$ is injective, and M is a flat A-module.

Solution of Problem 2: (Exactness) Let $f : N' \to N$ be injective. The map

$$\mathrm{id}_M \otimes f : M \otimes_A N' \to M \otimes_A N$$

has the property that for each $i \in I$, this map takes the ith coordinate into the ith coordinate. In particular, it is injective if and only if it is injective in each coordinate. By Proposition 2.19, the equivalence follows.

2.5

Define $f : \bigoplus_{n \geq 0} A \to A[x]$ by $f(\sum_{n \geq 0} a_n e_n) = \sum_{n \geq 0} a_n x^n$. The map f is obviously an A-linear isomorphism. By Exercise 2.4, $\bigoplus_{n \geq 0} A$ is a flat A-module due to each A being a flat A-module. Therefore, $A[x]$ is a flat A-algebra because it is A-linearly isomorphic to a flat A-module.

2.6

Let M be an A module and define $M[x]$ as the space of polynomials with coefficients in M. Let E be the endomorphism ring of M and let E' be the endomorphism ring of $M[x]$ (as abelian groups). Letting E act on $M[x]$ diagonally gives an injective ring homomorphism $E \to E'$. Since M is an A-module, we already have a ring homomorphism $A \to E$. Therefore, $M[x]$ is an A-module via the composition $A \to E \to E'$. Extend $A \to E'$ to $A[x] \to E'$ be letting $x \cdot mx^n := mx^{n+1}$ and extending by linearity. By the Universal Property of Polynomial Rings, this map is uniquely determined – and it makes $M[x]$ into an $A[x]$-module.

We have the following identifications

$$A[x] \otimes_A M = \left(\bigoplus_{n \geq 0} Ax^n\right) \otimes_A M = \bigoplus_{n \geq 0} (Ax^n \otimes_A M) = \bigoplus_{n \geq 0} Mx^n = M[x].$$

2.7

If $\mathfrak{p}$ is a prime ideal of A, then $(A/\mathfrak{p})[x]$ is a domain. By Exercise 2.6, this is isomorphic to $A[x] \otimes_A (A/\mathfrak{p})$ as an $A[x]$-module. Furthermore, Exercise 2.2 shows that this is isomorphic to $A[x]/\mathfrak{p}[x]$ as an A-module. The composite map $f : (A/\mathfrak{p})[x] \to A[x]/\mathfrak{p}[x]$ is an A-linear map such that $f(\sum_{n\geq 0} \overline{a}_n x^n) = \overline{\sum_{n\geq 0} a_n x^n}$. Thus f is an $A[x]$-algebra isomorphism, $A[x]/\mathfrak{p}[x]$ is a domain, and $\mathfrak{p}[x]$ is a prime ideal of $A[x]$.

On the other hand, (2) is a maximal ideal of $\mathbb{Z}$, but

$$\mathbb{Z}[x]/(2)[x] = (\mathbb{Z}/2\mathbb{Z})[x] = \mathbb{F}_2[x]$$

is not a field (x is not a unit).

2.8

(i) Let $0 \to P' \to P$ be exact, and let M, N be flat A-modules. Tensoring with M shows that

$$0 \to P' \otimes_A M \to P \otimes_A M$$

is exact. Now tensor with N to show that

$$0 \to (P' \otimes_A M) \otimes_A N \to (P \otimes_A M) \otimes_A N$$

is exact. By the Proposition 2.14 (ii),

$$0 \to P' \otimes_A (M \otimes_A N) \to P \otimes_A (M \otimes_A N)$$

is exact. Thus $M \otimes_A N$ is a flat A-module.

(ii) Let $0 \to P' \to P$ be an exact sequence of A-modules, let B be a flat A-algebra, and let N be a flat B-module. Extend scalars by B to get the exact sequence of B-modules

$$0 \to B \otimes_A P' \to B \otimes_A P.$$

Tensoring with N yields another exact sequence of B-modules,

$$0 \to N \otimes_B (B \otimes_A P') \to N \otimes_B (B \otimes_A P).$$

By Exercise 2.15,

$$0 \to (N \otimes_B B) \otimes_A P' \to (N \otimes_B B) \otimes_A P$$

is an exact sequence of A-modules. Thus $N = N \otimes_B B$ is a flat A-module.

2.9

Let

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be an exact sequence of A-modules with M', M'' finitely generated.

Constructive Proof: Let $x_1, \ldots, x_n$ be generators for M' and $g(y_1), \ldots, g(y_m)$ be generators for M''. Given z in M, there are $b_1, \ldots, b_m$ in A such that $g(z) = \sum_{i=1}^m b_i g(y_i)$. Thus $z - \sum_{i=1}^m b_i y_i$ is in the kernel of g. By exactness of the sequence, there are $a_1, \ldots, a_n$ in A such that $z = \sum_{j=1}^n a_j f(x_j) + \sum_{i=1}^m b_i y_i$. Therefore, $f(x_1), \ldots, f(x_n), y_1, \ldots, y_m$ is a finite generating set for M.

Non-Constructive Proof: Let $\alpha : A^{\oplus n} \to M'$ and $\beta : A^{\oplus m} \to M''$ be surjective A-linear maps. Since $A^{\oplus m}$ is a free A-module, it is projective. So there is an A-linear map $\gamma : A^{\oplus m} \to M$ such that $g \circ \gamma = \beta$. Define

$$\delta = f \circ \alpha \oplus \gamma : A^{\oplus(n+m)} = A^{\oplus n} \oplus A^{\oplus m} \to M$$

via the universal property of the direct limit. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{\oplus n} & \longrightarrow & A^{\oplus(n+m)} & \longrightarrow & A^{\oplus m} & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \delta & & \downarrow \beta & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

Apply the Snake Lemma to get the exact sequence

$$0 = \operatorname{coker} \alpha \to \operatorname{coker} \delta \to \operatorname{coker} \beta = 0.$$

Therefore, δ is surjective.

2.10

Let A be a ring, let $\mathfrak{a}$ be an ideal in its Jacobson radical, and let $u : M \to N$ be a homomorphism of A-modules such that N is finitely generated. Let v, w be the maps induced from u by restriction and passing to the quotient. Then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a}M & \longrightarrow & M & \longrightarrow & M/\mathfrak{a}M & \longrightarrow & 0 \\ & & \downarrow v & & \downarrow u & & \downarrow w & & \\ 0 & \longrightarrow & \mathfrak{a}N & \longrightarrow & N & \longrightarrow & N/\mathfrak{a}N & \longrightarrow & 0 \end{array}$$

is commutative and has exact rows. Applying the Snake Lemma yields an exact sequence

$$\operatorname{coker} v \xrightarrow{f} \operatorname{coker} u \to \operatorname{coker} w.$$

Note that $\operatorname{im} f = \mathfrak{a} \operatorname{coker} u$.

Suppose that w is surjective. Then f is surjective,

$$\operatorname{coker} u = \operatorname{im} f = \mathfrak{a} \operatorname{coker} u,$$

and $\operatorname{coker} u$ is finitely generated because it is a quotient of N. By Nakayama's Lemma, $\operatorname{coker} u = 0$. Therefore, u is surjective.

2.11

Let A be a non-zero ring and let $\phi : A^{\oplus m} \to A^{\oplus n}$ be a homomorphism of free A-modules.

If ϕ is surjective, then the sequence of A-modules,

$$A^{\oplus m} \xrightarrow{\phi} A^{\oplus n} \to 0$$

is exact. Tensor this sequence with $\kappa = A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of A to get an exact sequence

$$\kappa^{\oplus m} \to \kappa^{\oplus n} \to 0$$

of κ-vector spaces. Since m, n are the respective dimensions of $\kappa^{\oplus m}$ and $\kappa^{\oplus n}$, $m \geq n$.

On the other hand, suppose that $m > n$ and embed $A^{\oplus n}$ as the submodule of the first n coordinates of $A^{\oplus m}$. The containment,

$$\phi(A^{\oplus m}) \subseteq A^{\oplus n} \subsetneq A^{\oplus m},$$

yields the polynomial equation,

$$\phi^d + a_1 \phi^{d-1} + \ldots + a_{d-1}\phi + a_d = 0,$$

by the Cayley-Hamilton Theorem. Let d be the minimal degree for such an equation and let $\pi : A^{\oplus m} \to A^{\oplus(m-n)}$ be the projection onto the last $m-n$ coordinates. Then

$$0 = \pi(\phi^d + a_1\phi^{d-1} + \ldots + a_{d-1}\phi + a_d) = a_d\pi$$

implies that $a_d = 0$. Since d is minimal, there is an x in $A^{\oplus m}$ such that

$$y = (\phi^{d-1} + a_1\phi^{d-2} + \ldots + a_{d-1})(x)$$

is non-zero, but $\phi(y) = 0$. Therefore, if ϕ is injective, then $m \leq n$.

Putting the above results together shows that if ϕ is an isomorphism, then $m = n$.

2.12

Let M be a finitely generated A-module, $\phi : M \to A^n$ a surjective A-module homomorphism, and $e_1, \ldots, e_n$ be the standard basis of A^n. Let $f_1, \ldots, f_n$ be lifts of $e_1, \ldots, e_n$ to M via ϕ, and define the A-linear map $\psi : A^n \to M : e_i \mapsto f_i$. Since $\phi \circ \psi = \mathrm{id}_{A^n}$, M is isomorphic to $\mathrm{im}(\psi \circ \phi) \oplus \ker(\phi)$ by the map

$$m \mapsto ((\psi \circ \phi)(m), m - (\psi \circ \phi)(m)).$$

Therefore, $\ker \phi$ is a quotient of the finitely generate A-module M, and is itself finitely generated as a consequence. (Remark 5)

2.13

Let $f : A \to B$ be a ring homomorphism and let N be a B-module. Give N an A-module structure by restriction of scalars, define

$$g : N \to N_B : n \mapsto 1 \otimes n,$$

and define

$$p : N_B \to N : b \otimes y \mapsto by.$$

Since $p \circ g = \mathrm{id}_N$, g is injective and N_B is isomorphic to $\mathrm{im}(g) \oplus \ker(p)$ by the map

$$b \otimes y \mapsto ((g \circ p)(b \otimes y), b \otimes y - (g \circ p)(b \otimes y)).$$

(Remark 5)

2.14

We need only show that $\mu_i = \mu_j \circ \mu_{ij}$. If $x_i \in M_i$, then

$$(\mu_j \circ \mu_{ij})(x_i) = \mu_{ij}(x_i) + D = \mu_{ij}(x_i) + (x_i - \mu_{ij}(x_i)) + D = x_i + D = \mu_i(x_i).$$

(Remark 6)

2.15

Let $x \in M$. By the construction of M, $x = \sum_{i \in I} x_i + D$ for some $x_i \in M_i$, and I is finite. Since I is finite, its elements have a common upper bound, k. Then

$$x = \sum_{i \in I} x_i + D = \sum_{i \in I} x_i - \sum_{i \in I}(x_i - \mu_{ik}(x_i)) + D = \sum_{i \in I} \mu_{ik}(x_i) + D = \mu_k\left(\sum_{i \in I} \mu_{ik}(x_i)\right).$$

If $\mu_i(x_i) = 0$, then $x_i \in D$. Let $x_i = \sum_{l=1}^n (x_{j_l} - \mu_{j_l k_l}(x_{j_l}))$ be a representation of x_i as a sum of elements of D with $j_l < k_l$ for each l. Without loss of generality, j_1 is minimal. If $j_1 = j_2$, then

$$(x_{j_1} - \mu_{j_1 k_1}(x_{j_1})) + (x_{j_2} - \mu_{j_2 k_2}(x_{j_2})) = ((x_{j_1} + x_{j_2}) - \mu_{j_1 k_1}(x_{j_1} + x_{j_2})) + \mu_{j_1 k_1}(x_{j_2}) - \mu_{j_2 k_2}(x_{j_2}).$$

That is, $\mu_{j_1 k_1}(x_{j_2}) - \mu_{j_2 k_2}(x_{j_2}) \in D$ and is 0 in coordinate j_1. So we may assume that $j_1 < j_l$ for all $l > 1$. If $j_1 \neq i$, then $x_{j_1} = 0$ by the independence of coordinates in the direct sum.

Without loss generality, j_2 is minimal in $I \setminus \{i\}$. If $j_2 \neq k_1$, then $x_{j_2} = 0$ by the independence of coordinates. Therefore, $x_{j_2} = \mu_{ik_1}(x_i)$, and

$$\begin{aligned}
x_i &= \sum_{l=1}^n (x_{j_l} - \mu_{j_l k_l}(x_{j_l})) \\
&= x_i - \mu_{ik_1}(x_i) + x_{j_2} - \mu_{j_2 k_2}(x_{j_2}) + \sum_{l=3}^n (x_{j_l} - \mu_{j_l k_l}(x_{j_l})) \\
&= x_i - \mu_{k_1 k_2}(\mu_{ik_1}(x_i)) + \sum_{l=3}^n (x_{j_l} - \mu_{j_l k_l}(x_{j_l})) \\
&= x_i - \mu_{ik_2}(x_i) + \sum_{l=3}^n (x_{j_l} - \mu_{j_l k_l}(x_{j_l})).
\end{aligned}$$

By repeated application of the above argument we arrive at

$$x_i = x_i - \mu_{ik_n}(x_i).$$

Therefore, $\mu_{ik_n}(x_i) = 0$ in M_{k_n}. (Remark 7)

Alternative Proof: We will use the construction of the direct limit in Remark 6. The first statement follows from the construction of M as a quotient of the disjoint union $\coprod_{i \in I} M_i$. The second statement follows directly from the definition of $\sim$.

2.16

Let $\alpha_i : M_i \to N$ be a family of A-module homomorphisms such that

$$\alpha_j \circ \mu_{ij} = \alpha_i$$

for $i \leq j$, and let $M = \varinjlim_{i \in I} M_i$. If $i, j \leq k$ such that $\mu_i(x_i) = \mu_j(x_j)$, then

$$0 = \mu_i(x_i) - \mu_j(x_j) = \mu_k(\mu_{ik}(x_i)) - \mu_k(\mu_{jk}(x_j)) = \mu_k(\mu_{ik}(x_i) - \mu_{jk}(x_j)).$$

By Exercise 2.15 there is $l \geq k$ such that $\mu_{il}(x_i) = \mu_{jl}(x_j)$. Thus

$$\alpha_i(x_i) = \alpha_l(\mu_{il}(x_i)) = \alpha_l(\mu_{jl}(x_j)) = \alpha_j(x_j).$$

Therefore, we may define $\alpha : M \to N$ by $\alpha(\mu_i(x_i)) = \alpha_i(x_i)$. The preceding calculations have show that this is well defined and satisfies the requirement that $\alpha \circ \mu_i = \alpha_i$. Furthermore, this requirement forces the map to be unique. (Remark 8)

Alternative Proof: We will use the construction of the direct limit in Remark 6. Define

$$\alpha : M \to N : [(x, i)] \mapsto \alpha_i(x).$$

Suppose that $\mu_{ik}(x) = \mu_{jk}(y)$. Then

$$\alpha_i(x) = \alpha_k(\mu_{ik}(x)) = \alpha_k(\mu_{jk}(y)) = \alpha_j(y).$$

So α is well-defined. It is easy to verify that α is A-linear. Additionally, $\alpha(\mu_i(x)) = \alpha_i(x)$ by construction. Therefore, this condition uniquely determines α.

2.17

Let $\{M_i\}_{i\in I}$ be a family of submodules of an A-module, such that for each i, j there exists k such that $M_i + M_j \subseteq M_k$. Define $i \leq j$ to mean $M_i \subseteq M_j$ and define $\mu_{ij} : M_i \to M_j$ as the inclusion map.

Since $M_j \subseteq \bigcup_{i\in I} M_i$ for every j, $\sum M_i \subseteq \bigcup M_i$. On the other hand, $\bigcup M_i$ is a union of subsets of $\sum M_i$. Thus $\bigcup M_i \subseteq \sum M_i$.

For each j, let $\alpha_j : M_j \to \bigcup M_i$ be the inclusion map, and let $\alpha : \varinjlim M_i \to \bigcup M_i$ be the unique induced A-module homomorphism. If $x \in \bigcup M_i$, then there is j such that $x \in M_j$. Thus, $\alpha(\mu_j(x)) = \alpha_j(x) = x$, and α is surjective. On the other hand, $\alpha(\mu_j(x_j)) = \alpha_j(x_j) = x_j$ shows that α is injective. Therefore, α is an isomorphism.

Take $\{M_i\}_{i\in I}$ to be the family of finitely generated submodules of M. Order I by $i \leq j$, if $M_i \subseteq M_j$. Then

$$M = \bigcup_{i\in I} M_i = \varinjlim M_i.$$

2.18

Let $\mathbf{M} = (M_i, \mu_{ij})$ and $\mathbf{N} = (N_i, \nu_{ij})$ be direct systems of A modules over the same directed set, I. Let M, N be the respective direct limits and $\mu_i : M_i \to M, \nu_i : N_i \to N$ be associated homomorphisms. A homomorphism $\Phi : \mathbf{M} \to \mathbf{N}$ of direct systems is defined to be a collection of A-module homomorphisms $\phi_i : M_i \to N_i$ such that

$$\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$$

for all $i \leq j$.

Let $\Phi : \mathbf{M} \to \mathbf{N}$ be a homomorphism of direct systems. For each i, define the map $\alpha_i = \nu_i \circ \phi_i : M_i \to N$. We check that the collection of maps α_i are compatible by this derivation:

$$\alpha_j \circ \mu_{ij} = \nu_j \circ \phi_j \circ \mu_{ij} = \nu_j \circ \nu_{ij} \circ \phi_i = \nu_i \circ \phi_i = \alpha_i.$$

By the universal property of the direct limit there is a unique map $\phi : M \to N$ such that $\phi \circ \mu_i = \alpha_i = \nu_i \circ \phi_i$ for every i.

2.19

Let $\mathbf{M} \xrightarrow{\mathbf{f}} \mathbf{N} \xrightarrow{\mathbf{g}} \mathbf{P}$ be an exact sequence of direct systems over a common index set I. Let M, N, P be their respective direct limits and let $f : M \to N, g : N \to P$ be the unique maps obtained from $\mathbf{f}, \mathbf{g}$ as in Exercise 2.18. Then we have

$$g \circ f \circ \mu_i = g \circ \nu_i \circ f_i = \rho_i \circ g_i \circ f_i = 0$$

for every i. By Exercise 2.15, the images of the maps μ_i cover M. Therefore $f \circ g = 0$.

Let $n \in \ker(g)$. By Exercise 2.15 there is $n_i \in N_i$ such that $\nu_i(n_i) = n$. Thus, by Exercise 2.18,

$$0 = g(n) = (g \circ \nu_i)(n_i) = (\rho_i \circ g_i)(n_i).$$

We again cite Exercise 2.15 to obtain $j \geq i$ such that

$$0 = (\rho_{ij} \circ g_i)(n_i) = (g_j \circ \nu_{ij})(n_i).$$

Since $\nu_{ij}(n_i) \in \ker(g_j)$ and $\mathbf{M} \xrightarrow{\mathbf{f}} \mathbf{N} \xrightarrow{\mathbf{g}} \mathbf{P}$ is exact, there is $m_j \in M_j$ such that $f_j(m_j) = \nu_{ij}(n_i)$. Let $m = \mu_j(m_j)$. We have obtained the following:

$$f(m) = (f \circ \mu_j)(m_j) = (\nu_j \circ f_j)(m_j) = (\nu_j \circ \nu_{ij})(n_i) = \nu_i(n_i) = n.$$

Therefore, $\ker(g) \leq \operatorname{im}(f)$.

2.20

Let M, D be as in the statement of Exercise 2.14, and let M', D' be their analogues for the system $(M_i \otimes_A N, \mu_{ij} \otimes \mathrm{id}_N)$. So we have two short exact sequences of A-modules

$$0 \to D \to M \to \varinjlim M_i \to 0 \tag{1}$$

and

$$0 \to D' \to M' \to P \to 0. \tag{2}$$

Note that

$$M \otimes_A N = \left(\bigoplus_{i \in I} M_i \right) \otimes_A N = \bigoplus_{i \in I} (M_i \otimes_A N) = M',$$

and

$$D \otimes_A N \to D' : (x_i - \mu_{ij}(x_i)) \otimes n \mapsto x_i \otimes n - (\mu_{ij} \otimes \mathrm{id}_N)(x_i \otimes n)$$

is clearly surjective.

Tensor (1) with N to obtain the following commutative diagram with exact rows (The right vertical arrow is induced from the middle vertical arrow by passing to the quotient):

$$\begin{array}{ccccccccc}
 & & D \otimes_A N & \longrightarrow & M \otimes_A N & \longrightarrow & (\varinjlim M_i) \otimes_A N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D' & \longrightarrow & M' & \longrightarrow & P & \longrightarrow & 0
\end{array}$$

Apply the Snake Lemma and recall that the middle arrow is an isomorphism and the left arrow is surjective to see that the rightmost vertical arrow has trivial kernel and cokernel.

2.21

Let $i \in I$. For each $j \geq i$, ring homomorphism $\alpha_{ij} : A_i \to A_j$ makes A_j into an A_i-algebra. Therefore (Remark 9),

$$\varinjlim_{j \in J_i} A_j = \varinjlim_{j \in I} A_j = A$$

has a canonical A_i-module structure, and this is true for every $i \in I$. Let $\alpha_i(x), \alpha_j(y) \in A$. Define

$$\alpha_i(x) \cdot \alpha_j(y) := x \cdot \alpha_j(y),$$

where the operation on the right is scalar multiplication of A as an A_i-module. The associative and distributive properties are inherited from the A_i-module axioms. Let us check that this is well-defined and commutative. Let $k \geq i, j$. Then $\alpha_i(x) = \alpha_k(\alpha_{ik}(x))$ and $\alpha_j(y) = \alpha_k(\alpha_{jk}(y))$. Furthermore, the definition of scalar multiplication yields

$$x \cdot \alpha_j(y) = x \cdot \alpha_k(\alpha_{jk}(y)) = \alpha_k(\alpha_{ik}(x)\alpha_{jk}(y)).$$

The expression on the far right is obviously commutative and independent of the choice of x, y as representatives. By construction, the maps $A_i \to A$ are ring homomorphisms.

Suppose that $A = 0$. Let $i \in I$. Then $\alpha_i(1) = 0$. By Exercise 2.15, there is $j \geq i$ such that $1 = \mu_{ij}(1) = 0$. Therefore, $A_j = 0$.

2.22

Let "ρ" be the symbol for the structure maps on the system of $\mathfrak{N}_i$'s. Let $i, j \in I$ with $i \leq j$. Since $\mathfrak{N}_j$ is an A_j-module and A_j is an A_i-algebra, we can view $\mathfrak{N}_j$ as an A_i-module via restriction of scalars. Therefore,

$$\varinjlim_{j \in J_i} \mathfrak{N}_j = \varinjlim_{i \in I} \mathfrak{N}_i =: \mathfrak{N}$$

is an A_i-module. Furthermore, $\iota_j : \mathfrak{N}_j \to A_j$ is A_i-linear for all $j \geq i$. By Exercise 2.18, there is a unique A_i-linear map $\iota : \mathfrak{N} \to A$ such that $\iota \circ \rho_i = \iota_i$ for all $i \in I$. These A_i-linear maps are compatible for varying $i \in I$. Therefore, $\mathfrak{N} \to A$ is A-linear. By Exercise 2.19, ι is injective. So $\mathfrak{N}$ may be canonically identified with an ideal of A.

By Exercise 2.15, every element of $\mathfrak{N}$ is nilpotent, so $\mathfrak{N}$ is contained in the nilradical of A. Suppose that $x \in A$ and $x^n = 0$ for some $n \geq 0$. Let $x = \alpha_i(\xi)$ for some $i \in I$. Let $j \geq i$ such that $0 = \alpha_{ij}(\xi^n) = \alpha_{ij}(\xi)^n$. Then $\alpha_{ij}(\xi) = \iota_j(y)$ for some $y \in \mathfrak{N}_j$. Then we have

$$x = \alpha_j(\alpha_{ij}(\xi)) = \alpha_j(\iota_j(y)) = \iota(\rho_j(y)).$$

So every element of the nilradical of A is in the image of ι.

Suppose that A_i is an integral domain for each $i \in I$. Let $x, y \in A$ such that $xy = 0$. By Exercise 2.16, x, y lift to a common index k such that $xy = 0$ in A_k. If $x \neq 0$ in A, then $x \neq 0$ in A_k. Since A_k is an integral domain, $y = 0$ in A_k; and $y = 0$ in A as a consequence. Therefore, A is an integral domain.

2.23

This Exercise asks no questions. I'm suppose the reader is expected to verify the assertions. These follow from Exercises 2.14, 2.15, 2.16, 2.18, 2.19, and 2.21.

2.24

(Remark 10)

(i)⇒(ii) Let N be an A-module. Let F_* be a free resolution of N. Since M is flat, the functor $M \otimes_A$ is exact. So the complex $M \otimes_A F_*$ is exact. Therefore, $\mathrm{Tor}_n^A(M, N) = H_n(M \otimes_A F_*) = 0$.

(ii)⇒(iii) This is immediate.

(iii)⇒(i) Let

$$0 \to N' \to N \to N'' \to 0$$

be a short exact sequence of A-modules. Apply the functor $\mathrm{Tor}_*^A(M, \cdot)$ to get the exact sequence

$$0 = \mathrm{Tor}_1^A(M, N'') \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0.$$

By Proposition 2.19, M is flat.

2.25

Let

$$0 \to N' \to N \to N'' \to 0$$

be a short exact sequence of A-modules, let N'' be flat, and let $f : M' \to M$ be an injective A-linear map. The Tor long exact sequence yields the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 = \mathrm{Tor}_1^A(M', N'') & \longrightarrow & M' \otimes_A N' & \longrightarrow & M' \otimes_A N & \longrightarrow & M' \otimes_A N'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f \otimes \mathrm{id}_{N'}} & & \downarrow{\scriptstyle f \otimes \mathrm{id}_N} & & \downarrow{\scriptstyle f \otimes \mathrm{id}_{N''}} & & \\
0 = \mathrm{Tor}_1^A(M, N'') & \longrightarrow & M \otimes_A N' & \longrightarrow & M \otimes_A N & \longrightarrow & M \otimes_A N'' & \longrightarrow & 0
\end{array}$$

By the Snake Lemma, the sequence

$$0 \to \ker(f \otimes \mathrm{id}_{N'}) \to \ker(f \otimes \mathrm{id}_N) \to \ker(f \otimes \mathrm{id}_{N''}) = 0$$

is exact. Therefore, the map $\ker(f \otimes \mathrm{id}_{N'}) \to \ker(f \otimes \mathrm{id}_N)$ is an isomorphism and N is flat if and only if N' is flat.

2.26

(⇒) This follows from Exercise 2.24.

(⇐) Let $(M_i)_{i \in I}$ be a directed system of A-modules with direct limit M and let N be an A-modules. Let $F_* \to N$ be a flat resolution of N. By Exercises 2.19 and 2.20,

$$\mathrm{Tor}_n^A(M, N) = H_n(M \otimes_A F_*) = H_n\left(\varinjlim (M_i \otimes_A F_*)\right) = \varinjlim H_n(M_i \otimes_A F_*) = \varinjlim \mathrm{Tor}_n^A(M_i, N). \tag{3}$$

Suppose that N is an A-module such that $\mathrm{Tor}_1^A(A/\mathfrak{a}, N) = 0$ for all finitely generated ideals $\mathfrak{a}$ of A. Let $\mathfrak{b}$ be an ideal of A and let Σ be the directed family of all finitely generated subideals of $\mathfrak{b}$ (as in Exercise 2.17). By the exactness of direct limits and (3),

$$\mathrm{Tor}_1^A(A/\mathfrak{b}, N) = \mathrm{Tor}_1^A\left(\varinjlim_{\mathfrak{a} \in \Sigma} A/\mathfrak{a}, N\right) = \varinjlim_{\mathfrak{a} \in \Sigma} \mathrm{Tor}_1^A(A/\mathfrak{a}, N) = 0.$$

So $\mathrm{Tor}_1^A(\cdot, N)$ vanishes on cyclic A-modules.

Let $M = \sum_{i=1}^n Am_i$ be a finitely generated A-module, and define $M_k := \sum_{i=1}^k Am_i$ for $1 \le k \le n$. Note that $A \to M_k/M_{k-1} : 1 \mapsto m_k$ is surjective, so M_k/M_{k-1} is cyclic. The Tor functor yields exact sequences

$$\mathrm{Tor}_1^A(M_{k-1}, N) \to \mathrm{Tor}_1^A(M_k, N) \to \mathrm{Tor}_1^A(M_k/M_{k-1}, N) = 0,$$

where the equality is due to M_k/M_{k-1} being a cyclic A-module. By induction on k, there is a surjective map from $\mathrm{Tor}_1^A(M_1, N)$ onto $\mathrm{Tor}_1^A(M_n, N) = \mathrm{Tor}_1^A(M, N)$. Since M_1 is a cyclic A-module, $\mathrm{Tor}_1^A(M, N) = 0$.

Let M be any A-module and let $(M_i)_{i \in I}$ be the directed family of its finitely generated submodules. By (3), $\mathrm{Tor}_1^A(M, N) = 0$. Since M was arbitrary, Exercise 2.24 implies that N is flat. (Remark 2.26)

2.27 Incomplete

(i)⇒(ii) Let A be absolutely flat with element a. Since the inclusion $(a) \to A$ is injective and $A/(a)$ is flat, the tensored map,

$$(a)/(a)^2 = (a) \otimes_A A/(a) \to A \otimes_A A/(a) = A/(a)$$

is also injective. However, $xa \mapsto 0$ for all $x \in A$. Therefore, $(a)/(a)^2 = 0$.

(ii)⇒(iii) If a is an element of A, then $(a) = (a)^2$ implies that there is an element b such that $a = a^2 b$. Furthermore, $(ab)^2 = a^2 b \cdot b = ab$ shows that ab is idempotent, and $a \cdot ab = a^2 b = a$ shows that $(a) = (ab)$. Thus every principal ideal is generated by an idempotent element of A.

If e, f are idempotent elements of A, then

$$e(e + f - ef) = e + ef - ef = e$$

and

$$(e + f - ef)f = ef + f - ef = f$$

show that $(e, f) = (e + f - ef)$. Additionally,

$$(e + f - ef)^2 = e(e + f - ef) + f(e + f - ef) - ef(e + f - ef) = e + f - ef.$$

By induction, every finitely generated ideal is principal, generated by an idempotent element. Thus, if $\mathfrak{a}$ is a finitely generated ideal, there is an idempotent g such that $\mathfrak{a} = (g)$. Then $A = (g) \oplus (1-g)$ shows that $\mathfrak{a}$ is a direct summand of A.

(iii)⇒(i) Let N be an A-module, $\mathfrak{a}$ a finitely generated ideal, and $A = \mathfrak{a} \oplus \mathfrak{b}$. Additionally, let $\iota : \mathfrak{a} \to A$ be the natural inclusion and let $\pi : A \to \mathfrak{a}$ be the natural projection. Then $\pi \otimes \mathrm{id}_N$ is a left inverse for $\iota \otimes \mathrm{id}_N$, and $\iota \otimes \mathrm{id}_N$ is injective. Thus $\mathrm{Tor}_1^A(A/\mathfrak{a}, N) = 0$. By Exercise 2.26, N is flat and A is absolutely flat.

(Remark 12)

2.28

If a is an element in a boolean ring A, then $a^2 = a$ implies that $(a) = (a)^2$. Thus A is absolutely flat by Exercise 2.27.

Let A be a ring with a map $n : A \to \mathbb{Z}$ such that $n(x) \geq 2$ and $x^{n(x)} = x$ for every x. Then $x^{n(x)-2} \cdot x^2 = x$ implies that $(x) \subseteq (x)^2 \subseteq (x)$. Thus A is absolutely flat.

Let $f : A \to B$ be a ring homomorphism with A absolutely flat. If y is in the image of f, then there are a, x in A such that $x = ax^2$ and

$$y = f(x) = f(ax^2) = f(a)y^2.$$

Thus $(y) = (y)^2$ and the image of f is absolutely flat.

Let A be an absolutely flat local ring with maximal ideal $\mathfrak{m}$. Let x be an element of $\mathfrak{m}$ and a an element of A such that $x = ax^2$. Then ax is a non-unit idempotent in a local ring. By Exercise 1.12, ax is zero. Therefore, $x = ax \cdot x = 0$. As a consequence, $\mathfrak{m}$ is the zero ideal and A is a field.

Let A be absolutely flat with $x = ax^2$. Then $x(1 - ax) = 0$. If x is not a zero-divisor, then $1 = ax$ and x is a unit.

3 Rings and Modules of Fractions

(Remark 13)

3.1 Current

Let S be a multiplicatively closed subset of the ring A and let $M = \sum_{i=1}^k Ax_i$ be an A-module. By Corollary 3.4, $S^{-1}M = 0$ if and only if $S^{-1}Ax_j = 0$ for every j. This is equivalent to the requirement that $S \cap \mathrm{Ann}_A(Ax_j)$ is non-empty for each j. On the other hand, the existence of s in S such that $sM = 0$ is equivalent to the condition that $S \cap \mathrm{Ann}_A(M)$ is non-empty.

The following inclusions show that $S \cap \mathrm{Ann}_A(M)$ is non-empty precisely when every $S \cap \mathrm{Ann}_A(Ax_j)$ is non-empty.

$$\begin{aligned}\prod_{i=1}^k (S \cap \mathrm{Ann}_A(Ax_i)) &\subseteq \bigcap_{j=1}^k (S \cap \mathrm{Ann}_A(Ax_j)) \\ &= S \cap \left(\bigcap_{j=1}^k \mathrm{Ann}_A(Ax_j)\right) \\ &= S \cap \mathrm{Ann}_A(M) \\ &\subseteq S \cap \mathrm{Ann}_A(Ax_j),\end{aligned}$$

where the step from the second to third line uses Exercise 2.2 (i).

3.2

Let $\mathfrak{a}$ be an ideal of A, and define $S = 1 + \mathfrak{a}$. If $\frac{a}{1+b}$ is an element of $S^{-1}\mathfrak{a}$, then

$$1 - \frac{a}{1+b} = \frac{1+b-a}{1+b}$$

is a unit in $S^{-1}A$. Thus $S^{-1}\mathfrak{a}$ is in the Jacobson radical of $S^{-1}A$.

Let M be a finitely generated A-module such that $\mathfrak{a}M = M$. Then $(S^{-1}\mathfrak{a})(S^{-1}M) = S^{-1}M$ and $S^{-1}\mathfrak{a}$ is in the Jacobson radical of $S^{-1}A$. By Nakayama's Lemma, $S^{-1}M = 0$. By Exercise 3.1 there is some s in $S = 1 + \mathfrak{a}$ such that $sM = 0$.

3.3

Let S, T be multiplicatively closed subsets of A, let $f : A \to S^{-1}A$ be the natural map, and let $U = f(T)$. Note that ST is a multiplicatively closed subset of A and let $g : A \to (ST)^{-1}A$ be the natural map. Since $g(s) = \frac{s}{1}$ is a unit in $(ST)^{-1}A$ for each s in S, g has a unique lift $h : S^{-1}A \to (ST)^{-1}A$ such that $h \circ f = g$ by Proposition 3.1.

If $u = f(t)$ is in $U = f(T)$, then

$$h(u) = h(f(t)) = g(t) = \frac{t}{1}$$

is a unit in $(ST)^{-1}A$. If $0 = h(a/s) = g(a)g(s)^{-1} = a/s$ in $(ST)^{-1}A$, then there are elements s_0, t_0 in S, T such that $s_0t_0a = 0$ in A. Thus

$$0 = \frac{1}{s_0} \cdot f(s_0t_0a) = f(t_0)\frac{a}{1}$$

in $S^{-1}A$. Finally, for every element $\frac{a}{st}$ of $(ST)^{-1}A$,

$$\frac{a}{st} = g(a)g(st)^{-1} = (g(a)g(s)^{-1})g(t)^{-1} = h(a/s)h(f(t))^{-1}.$$

By Corollary 3.2, h lifts to an isomorphism from $U^{-1}(S^{-1}A)$ to $(ST)^{-1}A$.

3.4

Let $f : A \to B$ be a ring homomorphism, S a multiplicatively closed subset of A, and $T = f(S)$. Consider B as an A-module by restriction of scalars, and form the $S^{-1}A$-module $S^{-1}B$. Let $\frac{b}{s} = \frac{b'}{s'}$ in $S^{-1}B$. Then

$$f(s'')(f(s')b - f(s)b') = s'' \cdot (s' \cdot b - s \cdot b') = 0$$

in B. Thus, the map $g : S^{-1}B \to T^{-1}B : \frac{b}{s} \mapsto \frac{b}{f(s)}$ is well defined and bijective. We show that it is additive and $S^{-1}A$-homogeneous of degree 1.

The calculation,

$$\begin{aligned} g\left(\frac{b}{s} + \frac{b'}{s'}\right) = g\left(\frac{s' \cdot b + s \cdot b'}{ss'}\right) &= \frac{s' \cdot b + s \cdot b'}{f(ss')} = \frac{f(s')b + f(s)b'}{f(s)f(s')} \\ &= \frac{b}{f(s)} + \frac{b'}{f(s')} = g\left(\frac{b}{s}\right) + g\left(\frac{b'}{s'}\right), \end{aligned}$$

shows that g is additive.

On the other hand,

$$g\left(\frac{a}{s} \cdot \frac{b}{s'}\right) = g\left(\frac{a \cdot b}{ss'}\right) = \frac{a \cdot b}{f(ss')} = \frac{f(a)}{f(s)} \frac{b}{f(s')} = (S^{-1}f)\left(\frac{a}{s}\right) g\left(\frac{b}{s'}\right) = \frac{a}{s} \cdot g\left(\frac{b}{s'}\right)$$

shows that g is $S^{-1}A$-homogeneous of degree 1. Therefore, g is an isomorphism of $S^{-1}A$-modules.

3.5

By Corollary 3.12, localization commutes with taking nilradicals. If A is a ring such that $(\mathfrak{N}(A))_\mathfrak{p} = \mathfrak{N}(A_\mathfrak{p}) = 0$ for every prime ideal $\mathfrak{p}$, then $\mathfrak{N}(A) = 0$ by Proposition 3.8.

Let $A = \mathbb{Z}/6\mathbb{Z}$. Then $(2), (3)$ are maximal ideals, and $2 \cdot 3 = 0$ shows that every prime ideal contains 2 or 3. Thus $\operatorname{Spec} A = \{(2), (3)\}$. Furthermore, $2 \cdot 3 = 0$ implies that $A_{(2)} = \mathbb{F}_2$ and $A_{(3)} = \mathbb{F}_3$. Yet, A is not a domain.

3.6

Let A be a non-zero ring and let Σ be the set of multiplicatively closed subsets of A that do not contain 0. Since A is not the zero ring, $\{1\}$ is an element of Σ. Let $\{S_i\}_{i \in I}$ be a chain in Σ and define $S = \bigcup_{i \in I} S_i$. Then $1 \in S$ and if $x, y \in S$, then there is some $j \in I$ such that $x, y \in S_j$. Thus $xy \in S_j \subseteq S$. Furthermore, if 0 were in S, then it would have to be in some S_i. Since S is a multiplicatively closed subset of A not containing 0, it is an element of Σ. By Zorn's Lemma, Σ has maximal elements.

Let S be a maximal element of Σ and let x be an element of A. Since S is maximal in Σ, x is not in S if and only if the multiplicative sub-monoid of A generated by x and S contains 0. This in turn is equivalent to the condition that x is a nilpotent element of $S^{-1}A$. Therefore, $A \setminus S$ is the contraction of the nilradical of $S^{-1}A$, hence an ideal. Since S is multiplicatively closed, $\mathfrak{p} = A \setminus S$ is a prime ideal not meeting S. By Proposition 3.11 (iv), $S^{-1}\mathfrak{p} = \mathfrak{N}(A_\mathfrak{p})$ is a minimal prime ideal, and thus $\mathfrak{p}$ is a minimal prime ideal.

Let $\mathfrak{p}$ be a minimal prime ideal of A, x an element, and define $S = A \setminus \mathfrak{p}$. Then $S^{-1}\mathfrak{p} = \mathfrak{N}(A_\mathfrak{p})$. Thus x is not in S if and only if x is nilpotent in $A_\mathfrak{p}$. This in turn is equivalent to the condition that the multiplicative sub-monoid of A generated by x and S contains 0. Thus S is a maximal element of Σ.

3.7

Let S be a multiplicatively closed subset of the ring A,

(i) If S is saturated, then $A^\times \subseteq S$ because 1 is in S. If 0 is in S then $S = A$, and $A \setminus S$ is the empty union of prime ideals. Suppose that 0 is not in S, and let $\{\mathfrak{m}_i\}_{i\in I}$ be the set of maximal ideals of $S^{-1}A$. If a/s is a unit in $S^{-1}A$, then there is some s' in S such that $s'(a-s) = 0$ in A. In other words, $s'a = s's$ is in S. Thus, a is in S, because S is saturated. Therefore, $A \setminus S \subseteq \bigcup_{i\in I} \mathfrak{m_i}^c$. Conversely, $\bigcup_{i\in I} \mathfrak{m_i}^c \subseteq A \setminus S$ by Proposition 3.11 (iv).

Now, suppose that $A \setminus S = \bigcup_{i\in I} \mathfrak{p}_i$ where the $\mathfrak{p}_i$ are prime ideals. If xy is in S, then xy is in the complement of each prime ideal $\mathfrak{p}_i$. By primality, x, y are each in the complement of each prime ideal $\mathfrak{p}_i$. Thus x, y are both in S, and S is saturated.

(ii) Let Σ be the set of X such that $S \subseteq X \subseteq A$ and X is saturated. Since A is in Σ, it is non-empty. Order Σ by reverse inclusion, let $\{X_i\}_{i\in I}$ be a chain, and let $\widehat{X} = \bigcap_{i\in I} X_i$. Then $\widehat{X}$ is obviously a multiplicatively closed set containing S. Furthermore, $A \setminus \widehat{X} = \bigcup_{i\in I} A \setminus X_i$ is a union of prime ideals by (i). Thus $\widehat{X}$ is saturated and an element of Σ. By Zorn's Lemma, Σ has minimal elements. If S_1, S_2 are elements of Σ, then $S_1 \cap S_2$ is in Σ. Thus there is a unique minimal element, $\overline{S}$ in Σ.

Let $\{\mathfrak{m}_i\}_{i\in I}$ be the set of maximal ideals in $S^{-1}A$. Then $A \setminus \left(\bigcup_{i\in I} \mathfrak{m}_i^c\right)$ is a saturated subset of A, by (i), that contains S. Thus $A \setminus \left(\bigcup_{i\in I} \mathfrak{m}_i^c\right)$ is in Σ and $\overline{S} \subseteq A \setminus \left(\bigcup_{i\in I} \mathfrak{m}_i^c\right)$. Since $\overline{S}$ is saturated, $A \setminus \overline{S} = \bigcup_{j\in J} \mathfrak{q}_j$ where the $\mathfrak{q}_j$ are prime ideals of A not meeting $\overline{S}$. Therefore, $\bigcup_{j\in J} \mathfrak{q}_j \supseteq \bigcup_{i\in I} \mathfrak{m}_i^c$.

On the other hand, if $\mathfrak{p}$ is a prime ideal of A not meeting $\overline{S}$, then $\mathfrak{p}$ does not meet S, and $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}A$. Thus $\bigcup_{j\in J} \mathfrak{q}_j \subseteq \bigcup_{i\in I} \mathfrak{m}_i^c$ holds as well.

Let $\mathfrak{a}$ be an ideal of A and let $S = 1 + \mathfrak{a}$. Note that every factor of an element of S is in $\overline{S}$, and that the collection of these factors is a multiplicatively closed subset of A. Thus $\overline{S}$ is the set of factors of elements of S. We make this more concrete as follows:

If x is in $\overline{S}$, then there is y in $\overline{S}$ such that $xy \equiv 1 \mod \mathfrak{a}$. Conversely, if $xy \equiv 1 \mod \mathfrak{a}$, then there is a in $\mathfrak{a}$ such that $xy = 1 + a$, and x, y are in $\overline{S}$. Thus

$$\overline{S} = \pi^{-1}\left((A/\mathfrak{a})^\times\right),$$

where $\pi : A \to A/\mathfrak{a}$ is the natural projection.

3.8

Let $S \subset T$ be multiplicatively closed subsets of the ring A, and let

$$\phi : S^{-1}A \to T^{-1}A$$

be the lift of $f : A \to T^{-1}A$ as given by Proposition 3.1.

(i)⇒(ii) If t be an element of T, then $\phi^{-1}\left(\frac{1}{t}\right) \cdot \frac{t}{1} = \phi^{-1}\left(\frac{1}{t} \cdot \frac{t}{1}\right) = 1$ in $S^{-1}A$.

(ii)⇒(iii) Let $\frac{x}{s}$ be the inverse of $\frac{t}{1}$ in $S^{-1}A$. The equation $\frac{xt}{s} = \frac{1}{1}$ in $S^{-1}A$ implies that there is s' in S such that $s'(xt - s) = 0$ in A. Hence, $(s'x)t = ss'$ is in S.

(iii)⇒(iv) For each t in T there is x in A such that xt is in S. Thus t is in $\overline{S}$.

(iv)⇒(v) Since $T \subseteq \overline{S}$, $A \setminus T$ contains every prime ideal not meeting S, by Exercise 3.7. Thus every prime ideal meeting T also meets S.

(v)⇒(i) Note that $\phi : S^{-1}A \to T^{-1}A$ is a homomorphism of $S^{-1}A$-modules. If $\mathfrak{p}$ is a prime ideal of $S^{-1}A$, then $S \subseteq T \subseteq A \setminus \mathfrak{p}^c$. By Exercise 3.3, $(S^{-1}A)_\mathfrak{p} = A_{\mathfrak{p}^c} = (T^{-1}A)_\mathfrak{p}$, and $\phi_\mathfrak{p}$ is the identity on $A_\mathfrak{p}$, hence bijective. By Proposition 3.9, ϕ is bijective.

3.9

Let S_0 be the set of all non-zero-divisors in the non-zero ring A. Then S_0 is obviously a saturated multiplicatively closed subset and its complement D is a union of prime ideals by Exercise 3.7. If $\mathfrak{p}$ is a minimal prime ideal, then $S = A \setminus \mathfrak{p}$ is a saturated, maximal multiplicatively closed set not containing 0, by Problems 3.6 and 3.7. If x is not in S, then 0 is in the multiplicative monoid generated by S and x. Therefore x is a zero divisor, and $\mathfrak{p} = A \setminus S \subseteq D$.

(i) Let S be a multiplicatively closed subset of A containing a non-zero, zero-divisor, d, annihiliated by c. Then

$$\frac{c}{1} = \frac{cd}{d} = \frac{0}{d} = 0$$

shows that $A \to S^{-1}A$ is not injective. On the other hand, if $a/1 = 0$ in $S_0^{-1}A$, then there is s in S_0 such that $sa = 0$ in A. Since s is not a zero-divisor, $a = 0$. Thus, the canonical map $A \to S_0^{-1}A$ is injective.

(ii) If d is a zero-divisor of A, annihilated by c, then $\frac{d}{s} \cdot \frac{c}{1} = \frac{cd}{s} = 0$ for every s in S_0. Therefore, if a/s is not a zero-divisor, then a is in $S_0 = A \setminus D$, and a/s is a unit with inverse s/a.

(iii) If $A = A^\times \cup D$, then $S_0 = A^\times$. Let $f : A \to S_0^{-1}A$ be the canonical map. The f is surjective, since

$$\frac{a}{s} = \frac{as^{-1}}{ss^{-1}} = \frac{as^{-1}}{1} = f(as^{-1}),$$

and f is injective, since $0 = f(a) = \frac{a}{1}$ implies that there is s in S such that $as = 0$ in A. Thus $0 = ass^{-1} = a$. Therefore, f is an isomorphism.

3.10

Let A be a ring.

(i) Suppose A is absolutely flat, S is a multiplicatively closed subset, and N is an $S^{-1}A$-module. Then N is a flat A-module by restriction of scalars. Now extend scalars to get $N_{S^{-1}A} = S^{-1}A \otimes_A N$, which is a flat $S^{-1}A$-module by Exercise 2.20. Furthermore, $N_{S^{-1}A} = S^{-1}N$ by Proposition 3.5. Finally, $S^{-1}N = N$ because N is an $S^{-1}A$-module. Thus N is a flat $S^{-1}A$-module, and $S^{-1}A$ is absolutely flat.

(ii) If A is absolutely flat, then $A_\mathfrak{m}$ is an absolutely flat local ring for each maximal ideal $\mathfrak{m}$, hence a field.

On the other hand, let A be a ring that localizes to a field at each maximal ideal, $\mathfrak{m}$, and let N be an A-module. Then $N_\mathfrak{m}$ is an $A_\mathfrak{m}$-vector space, hence a free $A_\mathfrak{m}$-module. Therefore, $N_\mathfrak{m}$ is a flat $A_\mathfrak{m}$-module. Since $\mathfrak{m}$ was arbitrary, N is a flat A-module by Proposition 3.10.

3.11

Let A be a ring.

(iv)⇒(iii) Every Hausdorff space is T_1.

(iii)⇒(ii) Closed points of Spec(A) correspond to maximal ideals by Exercise 1.18 (i).

(ii)⇒(i) By the Lattice Isomorphism Theorem, we may identify the maximal ideals of A and $A/\mathfrak{R}(A)$. Let $\mathfrak{m}$ be such an ideal. Then

$$(A/\mathfrak{R}(A))_\mathfrak{m} = A_\mathfrak{m}/\mathfrak{R}(A_\mathfrak{m}) = A_\mathfrak{m}/\mathfrak{m}$$

is a field, and $A/\mathfrak{R}(A)$ is absolutely flat.

(i)⇒(iv) By Exercise 1.21 (iv), Spec(A) and Spec$(A/\mathfrak{R}(A))$ are naturally homeomorphic. Without loss of generality, $\mathfrak{R}(A) = 0$ and A is absolutely flat. Let $\mathfrak{p}$ be a prime ideal of A contained in a maximal ideal $\mathfrak{m}$. Since $A_\mathfrak{m}$ is a field, it has only one prime ideal, the zero ideal. Thus $\mathfrak{p} = \mathfrak{m}$ by Proposition 3.11 (iv), and every prime ideal of A is maximal.

Suppose that $\mathfrak{m}, \mathfrak{n}$ are distinct maximal ideals. Then there is $f \in \mathfrak{m}$ and $g \in \mathfrak{n}$ such that $f + g = 1$. On the other hand, the elements of $\mathfrak{m}$ vanish in $A_\mathfrak{m}$. So there is s not in $\mathfrak{m}$ such that $sf = 0$. Together, we have $s = sf + sg = sg$ is an element of $\mathfrak{n}$ and $f = 1 - g$ is not. Therefore, $\mathfrak{m} \subseteq X_s$, $\mathfrak{n} \subseteq X_f$ and $X_s \cap X_f = X_{sf} = \varnothing$. So Spec$(A)$ is Hausdorff.

In general, Spec(A) is quasi-compact by Exercise 1.17 (v). If it is Hausdorff, then it is compact. Let X be a subset of Spec(A) containing distinct points x, y, and let X_f contain x but not y. Since X_f is a quasi-compact subset of a Hausdorff space, it is closed. Thus $X_f \cap X, X \setminus X_f$ is a disjoint cover of X by non-empty, relatively open sets, and X is disconnected. Therefore, the only connected subsets of Spec(A) are singletons.

3.12

Let A be a domain, M an A-module, and $T(M)$ the set of torsion elements of M. If $x, y \in T(M), a \in A$, and b, c are respectively non-zero elements of $\operatorname{Ann}_A(x), \operatorname{Ann}_A(y)$, then

$$bc(x + ay) = c(bx) + ab(cy) = 0,$$

and $bc \neq 0$ because A is a domain. Thus $T(M)$ is a submodule of M.

(i) Let $x \in M, 0 \neq a \in A$ such that $ax \in T(M)$. Then there is some non-zero $b \in \operatorname{Ann}_A(ax)$, and $(ba)x = b(ax) = 0$. Since A is a domain, ba is a non-zero element of $\operatorname{Ann}_A(x)$, and $x \in T(M)$. Thus $T\left(M/T(M)\right) = 0$, and $M/T(M)$ is torsion-free.

(ii) Let $f : M \to N$ be an A-module homomorphism, $x \in T(M)$, and a a non-zero element of $\operatorname{Ann}_A(x)$. Then $af(x) = f(ax) = f(0) = 0$. Thus $f(T(M)) \subseteq T(N)$.

(iii) Suppose

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M''$$

is an exact sequence of A-modules, and consider the induced sequence

$$0 \to T(M') \xrightarrow{T(f)} T(M) \xrightarrow{T(g)} T(M'').$$

Since it is the restriction of an injective function, $T(f)$ is injective. Similarly, $T(g) \circ T(f) = 0$, because $g \circ f = 0$. If $x \in T(M)$ such that $T(g)(x) = 0$, then $g(x) = 0$. Since the first sequence is exact, and $T(M)$ is a submodule of M, there is $y \in M'$ such that $f(y) = x$. If a is a non-zero element of $\operatorname{Ann}_A(x)$, then $f(ay) = af(y) = ax = 0$. Since f is injective, $ay = 0$. Thus $y \in T(M')$ and $T(f)(y) = x$. Therefore the induced sequence on torsion is exact.

(iv) Let K be the field of fractions of A and define $\phi : M \to K \otimes_A M$ by $f(x) = 1 \otimes x$. If x is in $T(M)$, then it is annihilated by some a in A. Thus $\phi(x) = 1 \otimes x = a^{-1} \otimes ax = 0$, and x is in the kernel of ϕ. The following demonstrates the converse.

The family $\{A\xi\}_{\xi \in K}$ is a family of A-submodules of K. Given $\xi, \eta \in K$, there are non-zero a, b in A such that $a\xi, b\eta$ are in A. Thus $A\xi + A\eta \subseteq A\frac{1}{ab}$. By Exercise 2.17,

$$K = \bigcup_{\xi \in K} A\xi = \varinjlim_{\xi \in K} A\xi.$$

By Exercise 2.20, we may identify $K \otimes_A M$ with $\varinjlim_{\xi \in K}(A\xi \otimes_A M)$.

If $\phi(x) = 0$, then there is some ξ in K such that $1 \otimes x = 0$ in $A\xi \otimes_A M$, by Exercise 2.15. Since 1 is in $A\xi$, ξ^{-1} is in A. Furthermore,

$$A \to A\xi : a \mapsto a\xi$$

is an isomorphism of A-modules because K is a field. Therefore,

$$\psi : A\xi \otimes_A M \to A \otimes_A M \to M : a\xi \otimes x \mapsto ax$$

is an A-module isomorphism. Finally, we look at the composition

$$0 = \psi(0) = (\psi \circ \phi)(x) = \psi(1 \otimes x) = \xi^{-1}x.$$

Thus x is in $T(M)$.

3.13

Let S be a multiplicatively closed subset of the domain A. For any x in M and s in S,

$$\operatorname{Ann}(x/s) = \operatorname{Ann}(S^{-1}Ax) = S^{-1}\operatorname{Ann}(Ax) = S^{-1}\operatorname{Ann}(x)$$

by Proposition 3.14. Thus $\frac{x}{s}$ is in $T(S^{-1}M)$ if and only if it is in $S^{-1}T(M)$.

In particular, $T(M_{\mathfrak{p}}) = T(M)_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of A. By Proposition 3.8, (i), (ii), and (iii) are equivalent.

3.14

Let $\mathfrak{a}$ be an ideal of A and let M be an A-module such that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ containing $\mathfrak{a}$. Since localization is exact, $M/\mathfrak{a}M$ is an $A/\mathfrak{a}$-module such that $(M/\mathfrak{a}M)_{\mathfrak{m}} = M_{\mathfrak{m}}/(\mathfrak{a}M)_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of $A/\mathfrak{a}A$. By Proposition 3.8, $M/\mathfrak{a}M = 0$. Thus $M = \mathfrak{a}M$.

3.15

Let $F = A^n$ for some ring A. Let $x_1, \ldots, x_n$ be a set of generators for F, let $e_1, \ldots, e_n$ be the canonical basis, and let $\phi : F \to F$ be the map given by $\phi(e_i) = x_i$. The map ϕ is surjective because $x_1, \ldots, x_n$ generate F. The following demonstrates that ϕ is injective.

Without loss of generality, we may assume A is a local ring by Proposition 3.9. Let $\mathfrak{m}$ be the maximal ideal of A, let k be its residue field, and let N be the kernel of ϕ. Then the sequence

$$0 \to N \to F \xrightarrow{\phi} F \to 0$$

is exact. Since F is a flat A-module,

$$0 \to k \otimes_A N \to k \otimes_A F \xrightarrow{\widehat{\phi}} k \otimes_A F \to 0$$

is exact. Furthermore, $k \otimes_A F$ is naturally isomorphic to k^n. Thus $\widehat{\phi}$ is a surjective k-linear map of n-dimensional vector spaces, and bijective as a consequence. By Exercise 2.12, N is a finitely generated A-module. Since $k \otimes_A N = 0$, Exercise 2.3 implies that $N = 0$.

3.16

Let $f : A \to B$ be a flat A-algebra.

(i)⇒(ii) Let $\mathfrak{p}$ be a point in $\operatorname{Spec}(A)$. By Proposition 3.16 there is $\mathfrak{q}$ in $\operatorname{Spec}(B)$ such that $f^*(\mathfrak{q}) = \mathfrak{q}^c = \mathfrak{p}$.

(ii)⇒(iii) Let $\mathfrak{m}$ be a maximal ideal of A and let $\mathfrak{q}$ be a prime ideal of B such that $f^*(\mathfrak{q}) = \mathfrak{m}$. Then $\mathfrak{m}^e = \mathfrak{q}^{ce} \subseteq \mathfrak{q} \neq (1)$.

(iii)⇒(iv) Let x be a non-zero element of M and let $\mathfrak{a} = \operatorname{Ann}(x)$. Since x is non-zero, $\mathfrak{a}$ is a proper ideal of A. By hypothesis, $\mathfrak{a}^e$ is then a proper ideal of B. By Exercise 2.2 and the First Isomorphism Theorem, we have the following isomorphisms of B-modules

$$B/\mathfrak{a}^e \cong B \otimes_A (A/\mathfrak{a}) \cong B \otimes_A Ax.$$

Since B is flat, the map $B \otimes_A Ax \to B \otimes_A M$ induced by the inclusion $Ax \to M$ is injective. Thus M_B is a non-zero B-module.

(iv)⇒(v) Let M' be the kernel of the natural map $M \to B \otimes_A M$ so that we have a short exact sequence of A-modules

$$0 \to M' \to M \to M_B \to 0.$$

Since B is flat, we may tensor with $B \otimes_A -$ to get a short exact sequence of B-modules

$$0 \to M'_B \to M_B \to (M_B)_B \to 0.$$

By Exercise 2.13, the map on the right is injective. Therefore $M'_B = 0$. By hypothesis, this implies that $M' = 0$. Thus $M \to B \otimes_A M$ is injective.

(v)⇒(i) By hypothesis, the natural map

$$A/\mathfrak{a} \to (A/\mathfrak{a})_B \cong B/\mathfrak{a}^e$$

is injective. Thus $\mathfrak{a} \subseteq \mathfrak{a}^{ec} \subseteq \mathfrak{a}$.

3.17

Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a pair of ring homomorphisms such that $g \circ f$ is flat and g is faithfully flat, and let $h : M \to N$ be an injective homomorphism of A-modules. Let K be the kernel of the extension $h_B : M_B \to N_B$. Now tensor with C to get the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K \otimes_B C & \longrightarrow & (M \otimes_A B) \otimes_B C & \longrightarrow & (N \otimes_A B) \otimes_B C \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & M \otimes_A C & \longrightarrow & N \otimes_A C, \end{array}$$

where the right and center vertical arrows are the canonical isomorphisms. The bottom row is exact because $g \circ f$ is flat, and the top row is exact because g is flat. Thus $K \otimes_B C = 0$. Since g is faithfully flat, $K = 0$ by Exercise 3.16 (iv). Therefore, $f : A \to B$ is a flat A-algebra.

3.18

Let $f : A \to B$ be a flat homomorphism of rings, let $\mathfrak{q}$ be a prime ideal of B, and let $\mathfrak{p} = \mathfrak{q}^c$. By Proposition 3.10, $B_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ and $B_\mathfrak{q}$ is flat over $B_\mathfrak{p}$. By Exercise 2.8 (ii), $B_\mathfrak{q}$ is flat over $A_\mathfrak{p}$. Since $\mathfrak{p}^e = \mathfrak{q}^{ce} \subseteq \mathfrak{q}$, $\widehat{f} : A_\mathfrak{p} \to B_\mathfrak{q}$ satisfies the conditions of Exercise 3.16 (iii). Therefore, $B_\mathfrak{q}$ is a faithfully flat $A_\mathfrak{p}$-algebra, and $f^* : \operatorname{Spec}(B_\mathfrak{q}) \to \operatorname{Spec}(A_\mathfrak{p})$ is surjective.

3.19

Let M be an A-module and define the support of M by

$$\operatorname{Supp}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A) | M_\mathfrak{p} \neq 0\}.$$

We will use the result that $S^{-1}A = 0$ if and only if $0 \in S$. In particular, $S^{-1}(A/\mathfrak{a}) = 0$ if and only if $S \cap \mathfrak{a} \neq \varnothing$.

(i) My Proposition 3.8, $M = 0$ if and only if $M_\mathfrak{p} = 0$ for all prime ideals $\mathfrak{p}$. This is equivalent to $\operatorname{Supp}(M) = \varnothing$.

(ii) A given prime ideal $\mathfrak{p}$ contains $\mathfrak{a}$ if and only if $(A \setminus \mathfrak{p}) \cap \mathfrak{a} = \varnothing$. Thus $\mathfrak{p}$ is in $V(\mathfrak{a})$ if and only if $(A/\mathfrak{a})_\mathfrak{p} \neq \varnothing$.

(iii) Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of A-modules. By Proposition 3.3,

$$0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$$

is an exact sequence of $A_\mathfrak{p}$-modules for every prime ideal $\mathfrak{p}$ in A. Thus $M_\mathfrak{p} = 0$ if and only if $M'_\mathfrak{p} = M''_\mathfrak{p} = 0$.

(iv) By (ii), $\operatorname{Supp}(M_i) \subseteq \operatorname{Supp}(M)$ for each i. Conversely, suppose that x/s is a non-zero element of $M_\mathfrak{p}$. Then

$$x/s \in \left(\sum_{j=1}^{k} M_{i_j}\right)_\mathfrak{p} = \sum_{j=1}^{k} (M_{i_j})_\mathfrak{p}$$

for some indices $i_1, \ldots, i_k$. Therefore, $(M_i)_\mathfrak{p} \neq 0$ for some i, and $\operatorname{Supp}(M) \subseteq \bigcup_{i\in I} \operatorname{Supp}(M_i)$ as a consequence.

(v) A prime ideal $\mathfrak{p}$ contains $\operatorname{Ann}(M)$ if and only if $(A \setminus \mathfrak{p}) \cap \operatorname{Ann}(M) = \varnothing$. This is equivalent to the condition that $(A/\operatorname{Ann}(M))_\mathfrak{p} = A_\mathfrak{p}/\operatorname{Ann}(M)_\mathfrak{p}$ is non-zero. Since $M_\mathfrak{p} = 0$ if and only if $A_\mathfrak{p} = \operatorname{Ann}(M_\mathfrak{p})$, we are done.

(vi) By Proposition 3.7, $(M \otimes_A N)_\mathfrak{p} = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p}$ as $A_\mathfrak{p}$-modules. If $M_\mathfrak{p} = 0$ or $N_\mathfrak{p} = 0$, then $M_\mathfrak{p} \otimes_{a_\mathfrak{p}} N_\mathfrak{p} = 0$. Thus

$$\operatorname{Supp}(M \otimes_A N) \subseteq \operatorname{Supp}(M) \cap \operatorname{Supp}(N).$$

By Corollary 3.4, $M_\mathfrak{p}, N_\mathfrak{p}$ are finitely generated $A_\mathfrak{p}$-modules. Since $A_\mathfrak{p}$ is a local ring, if $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} = 0$, then $M_\mathfrak{p} = 0$ or $N_\mathfrak{p} = 0$ by Exercise 2.3. Therefore,

$$\operatorname{Supp}(M \otimes_A N) \supseteq \operatorname{Supp}(M) \cap \operatorname{Supp}(N).$$

(vii) By Exercise 2.2, $M/\mathfrak{a} = (A/\mathfrak{a}) \otimes_A M$. Thus, parts (ii), (v), and (vi) imply that

$$\begin{aligned} \operatorname{Supp}(M/\mathfrak{a}M) &= \operatorname{Supp}((A/\mathfrak{a}) \otimes_A M) \\ &= \operatorname{Supp}(A/\mathfrak{a}) \cap \operatorname{Supp}(M) \\ &= V(\mathfrak{a}) \cap V(\operatorname{Ann}(M)) \\ &= V(\mathfrak{a} + \operatorname{Ann}(M)). \end{aligned}$$

By Proposition 3.3 and Exercise 2.2,

$$(M/\mathfrak{a}M)_\mathfrak{p} = M_\mathfrak{p}/\mathfrak{a}_\mathfrak{p} M_\mathfrak{p} = (A_\mathfrak{p}/\mathfrak{a}_\mathfrak{p}) \otimes_{A_\mathfrak{p}} M_\mathfrak{p} = (A/\mathfrak{a})_\mathfrak{p} \otimes_{A_\mathfrak{p}} M_\mathfrak{p}.$$

By Corollary 3.4, $M_\mathfrak{p}$ is a finitely generated $A_\mathfrak{p}$-module. Therefore, $(M/\mathfrak{a}M)_\mathfrak{p} = 0$ if and only if $(A/\mathfrak{a})_\mathfrak{p} = 0$ or $M_\mathfrak{p} = 0$.

(viii) Let $\mathfrak{q}$ be a point in $\operatorname{Spec}(B)$ and let $\mathfrak{p} = f^*(\mathfrak{q})$. By Proposition 3.5,

$$\begin{aligned} (B \otimes_A M)_\mathfrak{q} &= B_\mathfrak{q} \otimes_B (B \otimes_A M) \\ &= B_\mathfrak{q} \otimes_A M \\ &= (B_\mathfrak{q} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}) \otimes_A M \\ &= B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p}. \end{aligned}$$

Therefore, $\operatorname{Supp}(B \otimes_A M) \subseteq f^{*-1}(\operatorname{Supp}(M))$.

Let $\kappa_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ and $\kappa_\mathfrak{q} = B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q}$ be the respective residue fields of $\mathfrak{p}$ and $\mathfrak{q}$, and assume that $0 = (B \otimes_A M)_\mathfrak{q}$. Tensor with $\kappa_\mathfrak{q}$ over $B_\mathfrak{q}$ to get

$$0 = \kappa_\mathfrak{q} \otimes_{B_\mathfrak{q}} (B \otimes_A M)_\mathfrak{q} = \kappa_\mathfrak{q} \otimes_{B_\mathfrak{q}} B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p} = \kappa_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p}.$$

Since the tensor product is right exact, $\kappa_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p}$ surjects onto $\kappa_\mathfrak{q} \otimes_{A_\mathfrak{p}} (M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p})$. Therefore, the $\kappa_\mathfrak{p}$-vector space $\kappa_\mathfrak{q} \otimes_{A_\mathfrak{p}} (M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p})$ is zero. Since $\kappa_\mathfrak{q}$ is a field, $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} = 0$. Furthermore, $M_\mathfrak{p}$ is a finitely generated $A_\mathfrak{p}$-module and $\mathfrak{p}$ is the Jacobson radical of $A_\mathfrak{p}$. By Nakayama's Lemma, $M_\mathfrak{p} = 0$. Thus $\operatorname{Supp}(B \otimes_A M) \supseteq f^{*-1}(\operatorname{Supp}(M))$.

3.20

Let $f : A \to B$ be a ring homomorphism.

(i) If $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective, then every prime ideal of A is a contracted ideal. Conversely, let $\mathfrak{p} = \mathfrak{b}^c$ for some ideal $\mathfrak{b}$ in B then

$$\mathfrak{p}^{ec} = \mathfrak{b}^{cec} = \mathfrak{b}^c = \mathfrak{p}.$$

By Proposition 3.16, $\mathfrak{p}$ is the contraction of a prime ideal of B, and f^* is surjective.

(ii) Suppose that every prime ideal of B is an extended ideal, and let $\mathfrak{q}_1 = \mathfrak{a}_1^e, \mathfrak{q}_2 = \mathfrak{a}_2^e$ be prime ideals of B. If $f^*(\mathfrak{q}_1) = f^*(\mathfrak{q}_2)$, then

$$\mathfrak{q}_1 = \mathfrak{a}_1^e = \mathfrak{a}_1^{ece} = f^*(\mathfrak{q}_1)^e = f^*(\mathfrak{q}_2)^e = \mathfrak{a}_2^{ece} = \mathfrak{a}_2^e = \mathfrak{q}_2,$$

and f^* is injective.

For a counterexample to the converse, let

$$f : A = \mathbb{C} \to \mathbb{C}[x]/(x^2) = B$$

be the natural inclusion map. Since $x^2 = 0$ in B, every prime ideal of B contains x. On the other hand, (x) is a maximal ideal, so B has exactly one prime ideal. Similarly, A has only the prime ideal (0) because it is a field. Thus

$$f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

is the injective map between one point sets. Finally,

$$(x)^{ce} = (0_A)^e = (0_B)$$

shows that (x) cannot be an extended ideal.

3.21

(i) Let S be a multiplicatively closed subset of the ring A, and let $\phi : A \to S^{-1}A$ be the canonical homomorphism.

By Proposition 3.11 (iv), $\phi^* : \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A) = X$ is bijective onto its image, $S^{-1}X$. Furthermore, $S^{-1}X$ consists of prime ideals of A not meeting S. Thus $S^{-1}X = \bigcap_{s\in S} X_s$.

Now we will show that ϕ^* is an open map. Let $Y_{f/g}$ be an open set in $\mathrm{Spec}(S^{-1}A)$. Exercise 1.21 (i) allows us to calculate that

$$\begin{aligned} Y_{f/g} &= V(f/g)^c = V(fs)^c = Y_{fs} = Y_{\phi(fs)} = \phi^{*-1}(X_{fs}) \\ &= \phi^{*-1}(X_f \cap X_s) = \phi^{*-1}(X_f) \cap \phi^{*-1}(X_s), \end{aligned}$$

because gs is a unit in $S^{-1}A$, for every s in S. Thus

$$\begin{aligned} Y_{f/g} &= \bigcap_{s\in S} (\phi^{*-1}(X_f) \cap \phi^{*-1}(X_s)) \\ &= \phi^{*-1}(X_f) \cap \phi^{*-1}\left(\bigcap_{s\in S} X_s\right) \\ &= \phi^{*-1}(X_f) \cap \phi^{*-1}(S^{-1}X). \end{aligned}$$

Since $\phi^* : \mathrm{Spec}(S^{-1}A) \to S^{-1}X$ is bijective, $\phi^*(Y_{f/g}) = X_f \cap S^{-1}X$, and ϕ^* is an open map. Therefore, ϕ^* is a homeomorphism onto its image.

If S is the monoid generated by f, then $S^{-1}X = \bigcap_{n\geq 0} X_{f^n} = X_f$.

(ii) Let $f : A \to B$ be a ring homomorphism, $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B)$, and $f^* : Y \to X$ the associated mapping. The diagram,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ S^{-1}A & \xrightarrow{S^{-1}f} & S^{-1}B, \end{array}$$

obviously commutes. By Proposition 3.11 (iv), we may identify the prime ideals of A and B not meeting S with their extensions to $S^{-1}A$ and $S^{-1}B$, and

$$(S^{-1}f)^{-1}(S^{-1}\mathfrak{q}) = S^{-1}\mathfrak{p}$$

if and only if

$$f^{-1}(\mathfrak{q}) = \mathfrak{p},$$

by the commutativity of the diagram. Thus the restriction of f^* to $S^{-1}Y$ has image in $S^{-1}X$, and can be identified with the map

$$(S^{-1}f)^* : \mathrm{Spec}(S^{-1}B) \to \mathrm{Spec}(S^{-1}A).$$

If $f^*(\mathfrak{q}) = \mathfrak{p}$, then $S^{-1}(A/\mathfrak{p}) \to S^{-1}(B/\mathfrak{q})$ is injective, because localization is exact. Thus, $\mathfrak{p}$ must meet S, if $\mathfrak{q}$ does. Consequently, the only points of Y mapping into $S^{-1}X$ by f^* are those of $S^{-1}Y$. In other words, $S^{-1}Y = f^{*-1}(S^{-1}X)$.

(iii) Let $\mathfrak{a}$ be an ideal of A with extension $\mathfrak{b}$ to B, and let $\overline{f} : A/\mathfrak{a} \to B/\mathfrak{b}$ be the map obtained from $f : A \to B$ by passing to the quotient. If $\mathfrak{q}$ is a prime ideal of B containing $\mathfrak{b}$, then $\mathfrak{q}^c \supseteq \mathfrak{b}^c = \mathfrak{a}^{ec} \supseteq \mathfrak{a}$. Thus f^* restricted to $V(\mathfrak{b})$ has image in $V(\mathfrak{a})$.

By Exercise 1.21 (iv), the natural quotient maps $A \to A/\mathfrak{a}, B \to B/\mathfrak{b}$ induce homeomorphisms $\mathrm{Spec}(A/\mathfrak{a}) \to V(\mathfrak{a}), \mathrm{Spec}(B/\mathfrak{b}) \to V(\mathfrak{b})$. We must show that the diagram,

$$\begin{array}{ccc} \mathrm{Spec}(B/\mathfrak{b}) & \xrightarrow{\overline{f}^*} & \mathrm{Spec}(A/\mathfrak{a}) \\ \downarrow & & \downarrow \\ V(\mathfrak{b}) & \xrightarrow{f^*} & V(\mathfrak{a}), \end{array}$$

commutes. But the diagram,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A/\mathfrak{a} & \xrightarrow{\overline{f}} & B/\mathfrak{b}, \end{array}$$

already commutes. Pulling back a prime ideal from the lower right corner to the upper left along the two routes shows that the previous diagram of spectra commutes.

(iv) Let $\mathfrak{p}$ be a prime ideal of A, let $S = A \setminus \mathfrak{p}$, let $f : A \to B$ be a ring homomorphism, and $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B)$. Since $V(\mathfrak{p})$ is the set of prime ideals containing $\mathfrak{p}$ and $S^{-1}X$ is the set of prime ideals contained in $\mathfrak{p}$, $\{\mathfrak{p}\} = V(\mathfrak{p}) \cap S^{-1}X$. Therefore,

$$f^{*-1}(\mathfrak{p}) = f^{*-1}(V(\mathfrak{p}) \cap S^{-1}X) = V(\mathfrak{p}B) \cap S^{-1}Y,$$

by Exercise 1.21 (ii), and (iii) above. By (iii), we may identify $S^{-1}Y$ with $\mathrm{Spec}(B_\mathfrak{p})$. Under this identification, $V(\mathfrak{p}B) \cap S^{-1}Y = V(\mathfrak{p}B_\mathfrak{p})$. The latter space is canonically homeomorphic to $\mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ by Exercise 1.21 (iv). On the other hand,

$$\begin{aligned} B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} &= A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{p} \\ &= \kappa_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p} \otimes_A B \\ &= \kappa_\mathfrak{p} \otimes_A B. \end{aligned}$$

Therefore, $f^{*-1}(\mathfrak{p}) = \mathrm{Spec}(\kappa_\mathfrak{p} \otimes_A B)$.

3.22

Let $\mathfrak{p}$ be a prime ideal of the ring A. Every open neighborhood U of $\mathfrak{p}$ contains a basic open set X_s with s in $S = A \setminus \mathfrak{p}$. Thus,

$$\operatorname{Spec}(A_\mathfrak{p}) = S^{-1}X = \bigcap_{s\in S} X_s = \bigcap_{\mathfrak{p}\in U} U,$$

by Exercise 3.21 (i).

3.23

Let $X = \operatorname{Spec}(A)$ for some ring A and let $U = X_f$ for some f in A.

(i) Suppose that $U = X_g$. By Exercise 1.17 (iv), $\sqrt{(f)} = \sqrt{(g)}$. Thus there are positive integers m, n and elements a, b of A such that $f^m = ag$ and $g^n = bf$. Let $\phi : A \to A_f$ be the canonical map.

We will use Corollary 3.2 to get our desired canonical isomorphism. First,

$$f^{mn} = (ag)^n = a^n bf$$

shows that a, b are units in A_f, and moreover $ag = f^m$ shows that g is a unit in A_f. Second, $\phi(x) = 0$ implies that $f^k x = 0$ for some $k \geq 0$. Multiply through by b^k to get $0 = b^k f^k x = g^{nk} x$. Third,

$$\frac{x}{f^k} = \phi(x)\phi(f^k)^{-1} = \phi(x)\phi(b^k)\phi(b^k f^k)^{-1} = \phi(xb^k)\phi(g^{nk})^{-1}.$$

Therefore, there is a unique isomorphism $A_g \to A_f$ lifting the canonical homomorphism $A \to A_f$. Flipping the roles of f and g yields the inverse map. Thus $A(U)$ is a well defined ring.

(ii) Now let $X_g = U' \subseteq X_f = U$. By Exercise 1.17 (iv), $g \in \sqrt{(f)}$. Therefore, there is an integer $n \geq 0$ and a in A such that $g^n = af$. As a consequence, f is a unit in $A(U')$. By Proposition 3.1, there is a unique ring homomorphism $\rho : A(U) \to A(U')$ lifting the canonical map $A \to A(U')$.

(iii) If $U = U'$, then the uniqueness of ρ implies that it is the identity map.

(iv) The three arrows are all unique maps lifting the identity by Proposition 3.1. The uniqueness forces the composition of the lower to maps to equal the top map.

(v) Let $x = \mathfrak{p}$ be a point of X. If $x \in U$ and $U = X_f$, then $f \in A \setminus \mathfrak{p}$. Thus there is a unique map $\mu_U : A(U) \to A_\mathfrak{p}$ lifting the canonical homomorphism $A \to A_\mathfrak{p}$. Furthermore, $\mu_U = \mu_{U'} \circ \rho_{UU'}$ by uniqueness. By Exercise 2.16 there is a unique map $\rho : \varinjlim_{x\in U} A(U) \to A_\mathfrak{p}$ lifting the maps μ_U.

Given $\frac{x}{f} \in A_\mathfrak{p}$. We have that $\frac{x}{f}$ is in the image of μ_U for $U = X_f$. Thus ρ is surjective.

If $\rho(a) = 0$, then there is $f \in A \setminus \mathfrak{p}$ such that $fa = 0$ in A. Let $U = X_f$, and note that this implies that $a = 0$ in $A(U)$, and thus in $\varinjlim_{x\in U} A(U)$ as well. Therefore, ρ is injective.

3.24

Let $\{U_i\}_{i\in I}$ be a covering of X by basic open sets and let $s_i \in A(U_i)$ for each i such that $s_i = s_j$ in $A(U_i \cap U_j)$ for each i, j. For each i, let $U_i = X_{f_i}$ and $s_i = \frac{x_i}{f_i^{m_i}}$. Since $s_i = s_j$ in $A(U_i \cap U_j)$, there are integers $n_{ij} \geq 0$ such that $(f_i f_j)^{n_{ij}}(x_i f_j^{m_j} - x_j f_i^{m_i}) = 0$ in A for all i, j. On the other hand, there are finitely many f's such that $(f_1, \ldots, f_k) = (1)$ because X is quasi-compact. Raising the ideal $(f_1, \ldots, f_k)$ to a sufficiently large power yields $\sum_{i=1}^k a_i f_i^{m_i} = 1$ for some $a_1, \ldots, a_k$ in A such that $a_i \in (f_i)^{\max_{i\neq j}(n_{ij})}$. Define $s = \sum_{i=1}^k a_i x_i$. Then, in $A(U_j)$, we have

$$\begin{aligned}
s &= \sum_{i=1}^{k} a_i x_i \\
&= a_j s_j f_j^{m_j} + \sum_{i \neq j} a_i x_i \\
&= s_j \left(1 - \sum_{i \neq j} a_i f_i^{m_i} \right) + \sum_{i \neq j} a_i x_i \\
&= s_j + \sum_{i \neq j} a_i (x_i - s_j f_i^{m_i}) \\
&= s_j + \frac{1}{f_j^{m_f}} \sum_{i \neq j} a_i (x_i f_j^{m_j} - x_j f_i^{m_i}) \\
&= s_j,
\end{aligned}$$

for $j = 1, \ldots, k$.

Now, suppose that $i \in I$ is arbitrary. For each $j = 1, \ldots, k$, $s = s_j = s_i$ in $A(U_i \cap U_j)$. Therefore, there is $n \geq 0$ such that $(f_i^{m_i} s - x_i)(f_i f_j)^n = 0$ in A for each j. Taking powers of $(f_1, \ldots, f_k)$ as before, we may obtain an A-linear combination $\sum_{j=1}^{k} b_j f_j^n = 1$. Summing the equations

$$b_j (f_i^{m_i} s - x_i)(f_i f_j)^n = 0$$

in A yields $(f_i^{m_i} s - x_i)(f_i)^n = 0$. Thus $s = s_i$ in $A(U_i)$.

Now, suppose that $s' \in A$ also has image s_i in $A(U_i)$ for every $i \in I$. Given the $f_1, \ldots, f_k$ as before, we get the systems of equations $(s - s') f_j^n = 0$ for $j = 1, \ldots, k$. Raising $(f_1, \ldots, f_k)$ to a sufficiently high power and then summing our equations yields $s = s'$.

3.25

Let $f : A \to B, g : A \to C, h : A \to B \otimes_A C$ be ring homomorphisms such that $h(x) = f(x) \otimes g(x)$, and let X, Y, Z, T be the respective prime ideal spectra of $A, B, C, B \otimes_A C$. We will make multiple uses of Exercise 3.21 (iv). For any prime ideal $\mathfrak{p}$ of A, $\mathfrak{p} \in h^*(T)$ if and only if

$$h^{*-1}(\mathfrak{p}) = \operatorname{Spec}(\kappa_{\mathfrak{p}} \otimes (B \otimes_A C)) = \operatorname{Spec}((\kappa_{\mathfrak{p}} \otimes_A B) \otimes_{\kappa_{\mathfrak{p}}} (\kappa_{\mathfrak{p}} \otimes_A C))$$

is non-empty. This is equivalent to the condition that the $\kappa_{\mathfrak{p}}$-vector space $(\kappa_{\mathfrak{p}} \otimes_A B) \otimes_{\kappa_{\mathfrak{p}}} (\kappa_{\mathfrak{p}} \otimes_A C)$ is non-zero. In turn, this is equivalent to the condition that both $\kappa_{\mathfrak{p}} \otimes_A B$ and $\kappa_{\mathfrak{p}} \otimes_A C$ are non-zero $\kappa_{\mathfrak{p}}$-vector spaces. We may flip this again, to the condition that $f^{*-1}(\mathfrak{p}) = \operatorname{Spec}(\kappa_{\mathfrak{p}} \otimes_A B)$ and $g^{*-1}(\mathfrak{p}) = \operatorname{Spec}(\kappa_{\mathfrak{p}} \otimes_A C)$ are non-empty. Finally, this is equivalent to $\mathfrak{p} \in f^*(Y) \cap g^*(Z)$.

3.26

Let B be the direct limit of the direct system of rings $B_\alpha, g_{\alpha\beta}$, let $f_\alpha : A \to B_\alpha$ be A-algebras such that $g_{\alpha\beta} \circ f_\alpha = f_\beta$ for all $\alpha \leq \beta$, and let $f : A \to B$ be the induced map. By Exercise 3.21 (iv), $f^{*-1}(\mathfrak{p}) = \operatorname{Spec}(\kappa_{\mathfrak{p}} \otimes_A B)$ is empty if and only if $\kappa_{\mathfrak{p}} \otimes_A B = 0$. On the other hand,

$$\kappa_{\mathfrak{p}} \otimes_A B = \varinjlim_{\alpha} (\kappa_{\mathfrak{p}} \otimes_A B_\alpha)$$

by Exercise 2.20. By Exercise 2.15, a direct limit is 0 if and only if every element is eventually zero. In the case of a ring, this is equivalent to $1 = 0$ for some α. Now, $\kappa_{\mathfrak{p}} \otimes_A B_\alpha = 0$ if and only if $f_\alpha^{*-1}(\mathfrak{p}) = \operatorname{Spec}(\kappa_{\mathfrak{p}} \otimes_A B_\alpha)$ is empty. In summary, $\mathfrak{p}$ is in the image of f^* if and only if it is in the image of f_α^* for every α.

4 Primary Decomposition

4.1

Suppose $\mathfrak{a}$ is a decomposable ideal of A. The irreducible components of $\mathrm{Spec}(A/\mathfrak{a})$ are the sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal over $\mathfrak{a}$, by Exercise 1.20 (iv). By Proposition 4.6, these are the isolated prime ideals of $\mathfrak{a}$. Furthermore, Theorem 4.5 tells us that there are only finitely many such prime ideals.

4.2

Suppose $\mathfrak{a}$ is a decomposable ideal of A such that $\mathfrak{a} = \sqrt{\mathfrak{a}}$. Let $\bigcap_{i=1}^n \mathfrak{q}_i$ be an irredundant primary decomposition of $\mathfrak{a}$ with associated prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Then $\mathfrak{a} = \sqrt{\mathfrak{a}} = \bigcap_{i=1}^n \sqrt{\mathfrak{q}_i} = \bigcap_{i=1}^n \mathfrak{p}_i$ yields another primary decomposition of $\mathfrak{a}$. Since the initial decomposition was assumed to be irredundant, this decomposition is as well. Therefore $\mathfrak{a}$ has no embedded prime ideals.

4.3

Suppose A is absolutely flat and $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal. By Exercise 2.28, $A/\mathfrak{q}$ is absolutely flat, and non-units are zero-divisors. Since $\mathfrak{q}$ is primary, non-units are nilpotent. Therefore, $A/\mathfrak{q}$ is an absolutely flat local ring. By Exercise 2.28, $A/\mathfrak{q}$ is a field, and $\mathfrak{q}$ is a maximal ideal as a consequence.

4.4

Define $\mathfrak{m} = (2, t), \mathfrak{q} = (4, t)$ as ideals of $\mathbb{Z}[t]$. Since $\mathbb{Z}[t]/\mathfrak{m} \cong \mathbb{F}_2$, $\mathfrak{m}$ is a maximal ideal. Note that $2^2, t \in \mathfrak{q}$. Thus $\mathfrak{m} \subseteq \sqrt{\mathfrak{q}}$, and we have equality because $\mathfrak{m}$ is maximal. By Proposition 4.2, $\mathfrak{q}$ is $\mathfrak{m}$-primary.

On the other hand, $t \in \mathfrak{q} \setminus \mathfrak{m}^2$. So $\mathfrak{m}^k \subseteq \mathfrak{m}^2 \subset \mathfrak{q} \subset \mathfrak{m}$ for all integers $k \geq 2$. Therefore, $\mathfrak{q}$ is not a power of $\mathfrak{m}$.

4.5

Define $\mathfrak{p}_1 = (x, y), \mathfrak{p}_2 = (x, z), \mathfrak{m} = (x, y, z)$ in the polynomial ring $K[x, y, z]$, where K is a field and x, y, z are indeterminates. Since $K[x, y, z]/\mathfrak{p}_1 \cong K[z]$, $K[x, y, z]/\mathfrak{p}_2 \cong K[y]$, and $K[x, y, z]/\mathfrak{m} \cong K$, we know that $\mathfrak{p}_1, \mathfrak{p}_2$ are prime ideal and $\mathfrak{m}$ is maximal.

Define $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2 = (x^2, xy, xz, yz)$. Clearly $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are respectively $\mathfrak{p}_1$-primary and $\mathfrak{p}_2$-primary. By Proposition 4.2, $\mathfrak{m}^2$ is $\mathfrak{m}$-primary. Since $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (x, yz)$ and $\mathfrak{m}^2 = (x^2, xy, xz, yz, y^2, z^2)$, we know that

$$\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2 = \mathfrak{a}$$

is a primary decomposition. Furthermore, the primary components $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{m}^2$ have distinct associated prime ideals, and

$$x \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \setminus \mathfrak{m}^2, y^2 \in \mathfrak{p}_1 \cap \mathfrak{m}^2 \setminus \mathfrak{p}_2, z^2 \in \mathfrak{p}_2 \cap \mathfrak{m}^2 \setminus \mathfrak{p}_1$$

shows that the decomposition is irredundant.

Let $\mathfrak{p}$ be a prime ideal containing $\mathfrak{a}$, then $x \in \mathfrak{p}$ because $x^2 \in \mathfrak{a} \subseteq p$. Similarly, $yz \in \mathfrak{p}$ implies that $y \in \mathfrak{p}$ or $z \in \mathfrak{p}$. Thus $\mathfrak{p}_1, \mathfrak{p}_2$ are the isolated prime ideals of $\mathfrak{a}$ and $\mathfrak{m}$ is an embedded prime ideal.

4.6

Let X be a compact Hausdorff space where (0) is a decomposable ideal of $C(X)$, and let $(0) = \bigcap_{i=1}^n \mathfrak{q}_i$ be an irredundant primary decomposition of (0).

Suppose $\mathfrak{a}$ is an ideal contained in distinct maximal ideals $\mathfrak{m}_x, \mathfrak{m}_y$. Since X is Hausdorff, x and y have disjoint open neighborhoods U, V. By Urysohn's Lemma, there are $f, g \in C(X)$ such that $f(x) = g(y) = 1$, $f|_{X \setminus U} = 0$, and $g|_{X \setminus V} = 0$. Thus, $fg = 0 \in \mathfrak{a}$, but $f \notin \mathfrak{m}_y$ and $g \notin \mathfrak{m}_x$. Therefore, each primary ideal of $C(X)$ is contained in a unique maximal ideal.

For each i, let $\mathfrak{m}_i$ be the maximal ideal over $\mathfrak{q}_i$. By Exercise 1.26, X is homeomorphic to

$$\operatorname{Max}(C(X)) = \operatorname{Max}(C(X)) \cap V(0) = \operatorname{Max}(C(X)) \cap \left(\bigcup_{i=1}^{n} V(\mathfrak{q}_i) \right) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}.$$

Therefore, X must be finite.

4.7

Let $A[x]$ be the ring of polynomials in one indeterminate over the ring A, and let $\mathfrak{a}[x]$ be the ideal of polynomials with coefficients in the ideal $\mathfrak{a}$ of A.

(i) Clearly, $\mathfrak{a}^e \subseteq \mathfrak{a}[x]$. On the other hand, $\mathfrak{a}x^n \subseteq \mathfrak{a}^e$ for every non-negative integer n. Thus, $\mathfrak{a}^e = \mathfrak{a}[x]$.

(ii) Lift the natural map $A \to A/\mathfrak{p}$ to obtain a map $A[x] \to (A/\mathfrak{p})[x]$ with kernel $\mathfrak{p}[x]$. Therefore, $A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$, and $\mathfrak{p}[x]$ is a prime ideal of $A[x]$ whenever $\mathfrak{p}$ is a prime ideal of A.

(iii) Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal of A. As in (ii), $(A/\mathfrak{q})[x] \cong A[x]/\mathfrak{q}[x]$. Let $f \in (A/\mathfrak{q})[x]$ be a zero-divisor. By Exercise 1.2, there is $b \in A \setminus \mathfrak{q}$ such that $bf \in \mathfrak{q}$. Therefore, the coefficients of f are zero-divisors in $A/\mathfrak{q}$. Since $\mathfrak{q}$ is primary, the coefficients of f are nilpotent in $A/\mathfrak{q}$. Again we cite Exercise 1.2 to see that f is nilpotent. Thus $\mathfrak{q}[x]$ is a primary ideal of $A[x]$. Furthermore, $f^k \in \mathfrak{q}[x]$ for some positive integer k if and only if the coefficients of f are in $\sqrt{\mathfrak{q}} = \mathfrak{p}$. So $\sqrt{\mathfrak{q}[x]} = \mathfrak{p}[x]$, and $\mathfrak{q}[x]$ is in fact $\mathfrak{p}[x]$-primary.

(iv) Let $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ be an irredundant primary decomposition of an ideal $\mathfrak{a}$ of A. The prime ideals $(\mathfrak{p}_i[x])_{i=1}^n$ of $A[x]$ are distinct because they have distinct contractions to A. Furthermore, for each j we have

$$\bigcap_{i \neq j} (\mathfrak{q}_i[x]) = \left(\bigcap_{i \neq j} \mathfrak{q}_i \right) [x] \not\subseteq \mathfrak{q}_j[x]$$

because $\bigcap_{i \neq j} \mathfrak{q}_i \not\subseteq \mathfrak{q}_j$. Therefore,

$$\mathfrak{a} = \left(\bigcap_{i=1}^{n} \mathfrak{q}_i \right) [x] = \bigcap_{i=1}^{n} \mathfrak{q}_i[x]$$

is an irredundant primary decomposition of $\mathfrak{a}$.

(v) Suppose that $\mathfrak{p}$ is an isolated prime ideal of $\mathfrak{a}$ and let $\mathfrak{p}'$ be a prime ideal of $A[x]$ such that $\mathfrak{a}[x] \subseteq \mathfrak{p}' \subseteq \mathfrak{p}[x]$. Taking contractions yields $\mathfrak{a} \subseteq (\mathfrak{p}')^c \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is minimal, $(\mathfrak{p}')^c = \mathfrak{p}$. On the other hand, $\mathfrak{p}' \supseteq (\mathfrak{p}')^{ce} = \mathfrak{p}[x]$. Thus $\mathfrak{p}' = \mathfrak{p}[x]$. So $\mathfrak{p}[x]$ is an isolated prime ideal of $\mathfrak{a}[x]$.

4.8

Let k be a field and define ideals $\mathfrak{p}_i = (x_1, \dots, x_i)$ of $k[x_1, \dots, x_n]$ for each $i = 1, \dots, n$. The map $k[x_1, \dots, x_n] \to k[x_{i+1}, \dots, x_n]$ obtained by evaluating each x_j at 0 for $1 \leq j \leq i$ is surjective with kernel $\mathfrak{p}_i$. Since $k[x_{i+1}, \dots, x_n]$ is a domain, $\mathfrak{p}_i$ is prime ideal. If $i = n$, then $\mathfrak{p}_i$ is a maximal ideal. By Proposition 4.2, the powers of $\mathfrak{p}_i$ are $\mathfrak{p}_i$-primary in $k[x_1, \dots, x_i]$. By Exercise 4.7 (iii), the powers of $\mathfrak{p}_i$ are $\mathfrak{p}_i$-primary in $k[x_1, \dots, x_n]$ as well.

4.9

Let A be a ring and define

$$D(A) = \{\mathfrak{p} \in \operatorname{Spec}(A) : \mathfrak{p} \supseteq (0 : a) \text{ is a minimal for some } a \in A\}.$$

If $x \in A$ is a non-zero, zero-divisor, then $x \in (0 : a)$ for some non-zero $a \in A$. If $\mathfrak{p}$ is a minimal prime ideal above $(0 : a)$, then $x \in \mathfrak{p}$ and $\mathfrak{p} \in D(A)$.

Let x be a non-zero element of A and suppose that there is a prime ideal $\mathfrak{p}$ such that $x \in \mathfrak{p} \in D(A)$. By the definition of $D(A)$, there is $a \in A$ such that $\mathfrak{p}$ is a minimal prime ideal above $(0 : a)$. Thus x is nilpotent in $(A/(0 : a))_{\mathfrak{p}}$. In other words, there is $s \in A \setminus \mathfrak{p}$ and a positive integer n such that $x^n s \in (0 : a)$. Since $s \notin \mathfrak{p}$, we know that sa is non-zero. If n is the smallest such integer, then $x^{n-1}sa$ is non-zero, but $x \cdot x^{n-1}sa = 0$. Therefore, x is a zero-divisor.

Let S be a multiplicatively closed subset of A. By definition, $S^{-1}\mathfrak{p} \in D(S^{-1}A)$ if and only if $S^{-1}\mathfrak{p}$ is a minimal prime ideal above $(0 : a/s)$ for some $a \in A$ and $s \in S$. By Proposition 3.11 (ii), $(0 : a/s) = S^{-1}(0 : a)$. Therefore, by Proposition 3.11 (iv), this is equivalent to the condition that $\mathfrak{p}$ is a minimal prime ideal above $(0 : a)$ for some $a \in A$ and $S \cap \mathfrak{p} = \varnothing$. In turn, this is holds precisely when $\mathfrak{p} \in D(A) \cap \operatorname{Spec}(S^{-1}A)$.

Suppose that (0) is a decomposable ideal and $\mathfrak{p}$ is one of its associated prime ideals. By Theorem 4.5, $\mathfrak{p} = \sqrt{(0 : a)}$ for some $a \in A$. Thus $\mathfrak{p} \in D(A)$.

On the other hand, suppose that $\mathfrak{p} \in D(A)$ is a minimal prime ideal above $(0 : a)$ for some $a \in A$. Then $\mathfrak{p}_{\mathfrak{p}} \in D(A_{\mathfrak{p}})$. So $\mathfrak{p}_{\mathfrak{p}}$ is a minimal prime ideal above $(0 : a)_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ is local, $\mathfrak{p}_{\mathfrak{p}} = \sqrt{(0 : a)_{\mathfrak{p}}}$. By Proposition 4.2, $(0 : a)_{\mathfrak{p}}$ is $\mathfrak{p}_{\mathfrak{p}}$-primary. Taking contractions yields

$$(0 : a) = \left(0 : (a)_{\mathfrak{p}}^{c}\right) \supseteq (0 : a)_{\mathfrak{p}}^{c} \supseteq (0 : a).$$

Thus $(0 : a)$ is a primary ideal. Furthermore, by Proposition 4.8, $(0 : a)$ is $\mathfrak{p}$-primary. So $\mathfrak{p}$ is an associated prime ideal of (0).

4.10

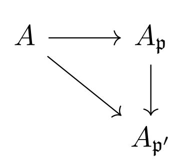
For every prime ideal $\mathfrak{p}$ of A, let $S_{\mathfrak{p}}(0)$ be the kernel of the natural map $A \to A_{\mathfrak{p}}$.

(i) Since $(0)_{\mathfrak{p}} \subseteq \mathfrak{p}_{\mathfrak{p}}$, $S_{\mathfrak{p}}(0) \subseteq \mathfrak{p}$ by Proposition 3.11.

(ii) If $\sqrt{S_{\mathfrak{p}}(0)} = \mathfrak{p}$, then $\mathfrak{p}_{\mathfrak{p}} = \left(\sqrt{S_{\mathfrak{p}}(0)}\right)_{\mathfrak{p}} = \sqrt{(S_{\mathfrak{p}}(0))_{\mathfrak{p}}} = \sqrt{0}$ in $A_{\mathfrak{p}}$. Therefore, $A_{\mathfrak{p}}$ has only one prime ideal, and $\mathfrak{p}$ is a minimal prime ideal of A.

If $\mathfrak{p}$ is a minimal prime ideal of A, then $\mathfrak{p}_{\mathfrak{p}}$ is the nilradical of $A_{\mathfrak{p}}$. By Exercise 1.18, $\sqrt{S_{\mathfrak{p}}(0)} = \mathfrak{p}$.

(iii) If $\mathfrak{p} \supseteq \mathfrak{p}'$, then the universal property of localization gives a unique lift of $A \to A_{\mathfrak{p}'}$ to $A_{\mathfrak{p}} \to A_{\mathfrak{p}'}$ such that the following diagram commutes:



Therefore, $S_{\mathfrak{p}}(0) \subseteq S_{\mathfrak{p}'}(0)$.

(iv) Suppose $x \in \bigcap_{\mathfrak{p} \in D(A)} S_{\mathfrak{p}}(0)$, and $\mathfrak{p}$ is a minimal prime ideal above $(0 : x)$. By definition, $\mathfrak{p} \in D(A)$. Thus $x \in S_{\mathfrak{p}}(0)$ by hypothesis. If $x \neq 0$, then there is $s \notin \mathfrak{p}$ such that $sx = 0$. In turn, this implies that $s \in (0 : x) \subseteq \mathfrak{p}$, which is a contradiction. Therefore, $x = 0$.

4.11

Let $\mathfrak{p}$ be a minimal prime ideal of A and let $\mathfrak{q}$ be $\mathfrak{p}$-primary. Since $\mathfrak{p}$ is minimal, $\mathfrak{p}_{\mathfrak{p}}$ is the unique prime ideal ideal of $A_{\mathfrak{p}}$. Therefore, $(0)_{\mathfrak{p}}$ is $\mathfrak{p}_{\mathfrak{p}}$-primary by Proposition 4.2, and its contraction, $S_{\mathfrak{p}}(0)$, is $\mathfrak{p}$-primary by Proposition 4.8. Furthermore, by Exercise 1.18 and Proposition 4.8, $(0) \subseteq \mathfrak{q}$ implies that $S_{\mathfrak{p}}(0) \subseteq \mathfrak{q}$.

Let $\mathfrak{a}$ be the intersection of all $S_{\mathfrak{p}}(0)$ for which $\mathfrak{p}$ is a minimal prime ideal of A. Then $\mathfrak{a} \subseteq S_{\mathfrak{p}}(0) \subseteq \mathfrak{p}$ for each minimal prime ideal. As a consequence, $\mathfrak{a} \subseteq \bigcap_{\mathfrak{p} \text{ minimal}} \mathfrak{p} = \sqrt{(0)}$.

Suppose (0) is a decomposable ideal of A. Then A has only finitely many minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ by Corollary 4.11.

If $\mathfrak{a} = (0)$, then $\bigcap_{i=1}^n S_{\mathfrak{p}_i}(0)$ is a primary decomposition of (0). Note that the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are distinct. If $S_{\mathfrak{p}_j}(0) \supseteq \bigcap_{i \neq j} S_{\mathfrak{p}_i}(0)$ for some j, then take radicals to obtain $\mathfrak{p}_j \supseteq \bigcap_{i \neq j} \mathfrak{p}_i$. By Proposition 1.11 (ii), $\mathfrak{p}_j \supseteq \mathfrak{p}_i$ for some i. Since $\mathfrak{p}_j$ is a minimal prime ideal, we have equality. This contradicts the distinctness of the minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Therefore, $\bigcap_{i=1}^n S_{\mathfrak{p}_i}(0)$ is an irredundant primary decomposition of (0). By Theorem 4.5, $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are the associated prime ideals of (0). So every associated prime ideal of (0) is isolated.

On the other hand, if every associated prime ideal of (0) is isolated, then $(0) = \bigcap_{\mathfrak{p} \in D(A)} S_{\mathfrak{p}}(0) = \bigcap_{i=1}^n S_{\mathfrak{p}_i}(0) = \mathfrak{a}$ by Problems 4.9 and 4.10 (iv).

4.12

Let S be a multiplicatively closed subset of the ring A and for each ideal $\mathfrak{a}$ of A define the saturation of $\mathfrak{a}$, $S(\mathfrak{a})$, as the contraction of $S^{-1}\mathfrak{a}$ to A.

(i) By Proposition 3.11 (v) and Exercise 1.18,

$$S(\mathfrak{a}) \cap S(\mathfrak{b}) = \left(S^{-1}\mathfrak{a}\right)^c \cap \left(S^{-1}\mathfrak{b}\right)^c = \left(S^{-1}\mathfrak{a} \cap S^{-1}\mathfrak{b}\right)^c = \left(S^{-1}\left(\mathfrak{a} \cap \mathfrak{b}\right)\right)^c = S(\mathfrak{a} \cap \mathfrak{b}).$$

(ii) By Proposition 3.11 (iv) and Exercise 1.18,

$$S\left(\sqrt{\mathfrak{a}}\right) = \left(S^{-1}\sqrt{\mathfrak{a}}\right)^c = \left(\sqrt{S^{-1}\mathfrak{a}}\right)^c = \sqrt{\left(S^{-1}\mathfrak{a}\right)^c} = \sqrt{S(\mathfrak{a})}.$$

(iii) By Proposition 1.17, $S(\mathfrak{a}) = (1)$ if and only if $S^{-1}\mathfrak{a} = S^{-1}A$. By Proposition 3.11 (ii), this is equivalent to $S \cap \mathfrak{a} \neq \varnothing$.

(iv) Let $f : A \to S^{-1}A$ be the natural map and note that $x \in S(\mathfrak{a})$ if and only if $f(x) \in S^{-1}\mathfrak{a}$. This is equivalent to the existence of $s \in S$ such that $xs \in \mathfrak{a}$. Therefore,

$$\begin{aligned} x \in S_1(S_2(\mathfrak{a})) &\text{ if and only if } xs_1 \in S_2(\mathfrak{a}) \text{ for some } s_1 \in S_1 \\ &\text{ if and only if } xs_1s_2 \in \mathfrak{a} \text{ for some } s_1 \in S_1, s_2 \in S_2 \\ &\text{ if and only if } x \in (S_1S_2)(\mathfrak{a}). \end{aligned}$$

Suppose $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ is an irredundant primary decomposition. By Proposition 4.9, $S(\mathfrak{a}) = \bigcap_{j=1}^m \mathfrak{q}_{i_j}$ is a irredundant primary decomposition for some $1 \leq i_1, \ldots, i_m \leq n$. Since there are only finitely many subsets of $\{1, \ldots, n\}$, there are only finitely many ideals $S(\mathfrak{a})$.

4.13

Let $\mathfrak{p}$ be a prime ideal in a ring A, and define the nth symbolic power of $\mathfrak{p}$ as $\mathfrak{p}^{(n)} = S_{\mathfrak{p}}(\mathfrak{p}^n)$, where $S_{\mathfrak{p}} = A \setminus \mathfrak{p}$.

(i) By Proposition 3.11 (v), $\sqrt{S_{\mathfrak{p}}^{-1}\mathfrak{p}^n} = S_{\mathfrak{p}}^{-1}\sqrt{\mathfrak{p}^n} = S_{\mathfrak{p}}^{-1}\mathfrak{p}$. Thus $S_{\mathfrak{p}}^{-1}\mathfrak{p}^n$ is an $S_{\mathfrak{p}}^{-1}\mathfrak{p}$-primary ideal by Proposition 4.2. Since $\mathfrak{p}^{(n)}$ is the contraction of $S_{\mathfrak{p}}^{-1}\mathfrak{p}^n$, it is a $\mathfrak{p}$-primary ideal by Proposition 4.8 (ii).

(ii) Suppose that $\mathfrak{p}^n = \bigcap_{i=1}^m \mathfrak{q}_i$ is an irredundant primary decomposition and J is the set of indices j for which $\mathfrak{q}_j \subseteq \mathfrak{p}$. By Proposition 4.9, $\mathfrak{p}^{(n)} = S_{\mathfrak{p}}(\mathfrak{p}^n) = \bigcap_{j \in J} \mathfrak{q}_j$. By (i), $\mathfrak{p} = \sqrt{\mathfrak{p}^{(n)}} \subseteq \sqrt{\mathfrak{q}_j}$ for each $j \in J$. Therefore, $\mathfrak{q}_j$ is $\mathfrak{p}$-primary for each $j \in J$. Since the decomposition is irredundant by hypothesis, there is only one index j in J, and $\mathfrak{q}_j = \mathfrak{p}^{(n)}$. Thus $\mathfrak{p}^{(n)}$ is the $\mathfrak{p}$-primary component of $\mathfrak{p}^n$.

(iii) Suppose that $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)} = \bigcap_{i=1}^k \mathfrak{q}_i$ is an irredundant primary decomposition and J is the set of indices j for which $\mathfrak{q}_j \subseteq \mathfrak{p}$. Since $S_{\mathfrak{p}}^{-1}\mathfrak{p}^{(m)}\mathfrak{p}^{(n)} = S_{\mathfrak{p}}^{-1}\mathfrak{p}^{m+n}$, Proposition 4.8 (ii) implies that $S_{\mathfrak{p}}\left(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}\right) = \mathfrak{p}^{(m+n)}$. On the other hand, Proposition 4.9 tells us that $S_{\mathfrak{p}}\left(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}\right) = \bigcap_{j \in J} \mathfrak{q}_j$. By (i), $\mathfrak{p} = \sqrt{\mathfrak{p}^{(m+n)}} \subseteq \sqrt{\mathfrak{q}_j}$ for each $j \in J$.

Therefore, $\mathfrak{q}_j$ is $\mathfrak{p}$-primary for each $j \in J$. Since the decomposition is irredundant by hypothesis, there is only one index j in J, and $\mathfrak{q}_j = \mathfrak{p}^{(m+n)}$. Therefore, $\mathfrak{p}^{(m+n)}$ is the $\mathfrak{p}$-primary component of $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$.

(iv) If $\mathfrak{p}^{(n)} = \mathfrak{p}^n$, then $\mathfrak{p}^n$ is $\mathfrak{p}$-primary because $\mathfrak{p}^{(n)}$ is by (i). If $\mathfrak{p}^n$ is $\mathfrak{p}$-primary, then $\mathfrak{p}^n = \mathfrak{p}^{(n)}$ by (ii).

4.14

Let $\mathfrak{a}$ be a decomposable ideal in a ring A and let $\mathfrak{p}$ be a maximal element of the set of ideals of the form $(\mathfrak{a} : x)$ where $x \in A \setminus \mathfrak{a}$. If $yz \in \mathfrak{p}$ and $y \notin \mathfrak{p}$, then $z \in (\mathfrak{a} : xy)$ and $\mathfrak{p} \subseteq (\mathfrak{a} : xy)$. Since $\mathfrak{p}$ is maximal, $z \in (\mathfrak{a} : xy) = \mathfrak{p}$. So $\mathfrak{p}$ is a prime ideal and $\mathfrak{p} = \sqrt{(\mathfrak{a} : x)}$ is an associated prime ideal of $\mathfrak{a}$ by Theorem 4.5.

4.15

Let $\mathfrak{a}$ be a decomposable ideal in a ring A, let Σ be an isolated set of prime ideals associated to $\mathfrak{a}$, let $\mathfrak{q}_\Sigma$ be the intersection of the corresponding primary components. Suppose that there is $f \in A$ such that $f \in \mathfrak{p}$ if and only $\mathfrak{p} \notin \Sigma$, for each associated prime ideal $\mathfrak{p}$ of A, and let S_f be the multiplicative monoid generated by f.

By Proposition 4.9, $S_f(\mathfrak{a}) = \bigcap_{i=1}^m \mathfrak{q}_i = \mathfrak{q}_\Sigma$, where $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\} = \Sigma$. By Proposition 3.11 (ii), $S_f(\mathfrak{a}) = \bigcup_{j\geq 0}(\mathfrak{a} : f^j)$. Since $f \in \mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ for $i = m+1, \dots, n$, there is some k such that $f^k \in \bigcap_{i=m+1}^n \mathfrak{q}_i$. Therefore, $S_f(\mathfrak{a}) \subseteq (\mathfrak{a} : f^k) \subseteq \bigcup_{j\geq 0}(\mathfrak{a} : f^j) = S_f(\mathfrak{a})$.

4.16

Let A be a ring in which every ideal is decomposable, let S be a multiplicatively closed subset, and let $\mathfrak{b}$ be an ideal of $S^{-1}A$. By Proposition 3.11 (i), $\mathfrak{b} = S^{-1}\mathfrak{a}$ for some ideal $\mathfrak{a}$ of A. Let $\bigcap_{i=1}^n \mathfrak{q}_i$ be an irredundant primary decomposition of $\mathfrak{a}$. By Proposition 4.9, $S^{-1}\mathfrak{a} = \bigcap_{i=1}^m S^{-1}\mathfrak{q}_i$ is an irredundant primary decomposition of $\mathfrak{b}$.

4.17

(L1) For every proper ideal $\mathfrak{a}$ and every prime ideal $\mathfrak{p}$, there exists $x \notin \mathfrak{p}$ such that $S_\mathfrak{p}(\mathfrak{a}) = (\mathfrak{a} : x)$.

Let A be a ring with the property (L1), and suppose $\mathfrak{a}$ is a proper ideal of A. Let $\mathfrak{p}_1$ be a minimal prime ideal above $\mathfrak{a}$. The ideal $\mathfrak{q}_1 = S_{\mathfrak{p}_1}(\mathfrak{a})$ is the contraction of the $\overline{\mathfrak{p}_1}$-primary ideal $S_{\overline{\mathfrak{p}_1}}(\overline{0}) \subseteq A/\mathfrak{a}$ to A, so $\mathfrak{q}_1$ is $\mathfrak{p}_1$-primary by Exercise 4.11. By (L1), $\mathfrak{q}_1 = (\mathfrak{a} : x)$ for some $x \notin \mathfrak{p}_1$. Clearly, $\mathfrak{a} \subseteq \mathfrak{q}_1 \cap (\mathfrak{a} + (x))$. If $f = a + bx \in \mathfrak{q}_1 \cap (\mathfrak{a} + (x))$, then $fx - ax = bx^2 \in \mathfrak{a}$, because $f \in \mathfrak{q}_1 = (\mathfrak{a} : x)$ and $a \in \mathfrak{a}$. Thus $b \in S_{\mathfrak{p}_1}(\mathfrak{a}) = \mathfrak{q}_1 = (\mathfrak{a} : x)$. So $f \in \mathfrak{a}$ and we have $\mathfrak{a} = \mathfrak{q}_1 \cap (\mathfrak{a} + (x))$. Let $\mathfrak{a}_1$ be an ideal maximal among those for which $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{a}_1$ and $x \in \mathfrak{a}_1$. Now we have $\mathfrak{a} \subsetneq \mathfrak{a}_1 \nsubseteq \mathfrak{p}_1$.

Given $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_n \cap \mathfrak{a}_n$ where $\mathfrak{a}_{j-1} \subsetneq \mathfrak{a}_j \nsubseteq \mathfrak{p}_j$ for each $j = 1, \dots, n$, if $\mathfrak{a}_n$ is a proper ideal, then we may repeat the process of the previous paragraph to extend the strictly increasing sequence of ideals $\mathfrak{a} \subsetneq \mathfrak{a}_1 \subsetneq \dots \subsetneq \mathfrak{a}_n$.

4.18

(L2) For every ideal $\mathfrak{a}$ and every descending chain $S_1 \supseteq S_2 \supseteq S_3 \supseteq \dots$ of multiplicatively closed subsets, there exists a positive integer N such that $S_n(\mathfrak{a}) = S_N(\mathfrak{a})$ for every $n \geq N$.

(i)$\Rightarrow$(ii) Let $\mathfrak{a}$ be an ideal of A, let $\mathfrak{p}$ be a prime ideal, and let $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ be an irredundant primary decomposition of $\mathfrak{a}$ with associated prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Let $\Sigma_\mathfrak{p}$ be the set of associated prime ideals of $\mathfrak{a}$ contained in $\mathfrak{p}$. Clearly, $\Sigma_\mathfrak{p}$ is an isolated set of prime ideals. Without loss of generality, $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\} = \Sigma_\mathfrak{p}$ and $\mathfrak{p} \nsupseteq \mathfrak{p}_{m+1} \cdots \mathfrak{p}_n$. Thus there is $f \in \mathfrak{p}_{m+1} \cdots \mathfrak{p}_n \setminus \mathfrak{p}$. By Exercise 4.15 and Proposition 4.9,

$$S_\mathfrak{p}(\mathfrak{a}) = \mathfrak{q}_{\Sigma_\mathfrak{p}} = (\mathfrak{a} : f^k)$$

for some positive integer k. Therefore, A satisfies (L1).

Let $S_1 \supseteq S_2 \supseteq \dots$ be a descending chain of multiplicatively closed subsets of A. Without loss of generality, the first m_j associated prime ideals of $\mathfrak{a}$ meet S_j, for every $j \geq 1$. Since $m_1, m_2, m_3, \dots$ is a decreasing sequence of positive integers, it has a least element m_N. By Proposition 4.9, if $j \geq N$, then $S_j(\mathfrak{a}) = \bigcap_{i=m_j}^n \mathfrak{q}_i = \bigcap_{i=m_N}^n \mathfrak{q}_i = S_N(\mathfrak{a})$. Therefore, A satisfies (L2).

(ii)⇒(i) Let $\mathfrak{a}$ be an ideal of A. Using the property (L1), obtain the prime ideals $\mathfrak{p}_i$ and ideals $\mathfrak{a}_i$, where we are using the notation of Exercise 4.17. Now, define $S_n = S_{\mathfrak{p}_1} \cap \dots \cap S_{\mathfrak{p}_n}$ for each positive integer n. By (L2), there exists a positive integer N such that $n \geq N$ implies that $S_n(\mathfrak{a}) = S_N(\mathfrak{a})$.

4.19

Let $\mathfrak{p}$ be a prime ideal in a ring A, and let $\mathfrak{q}$ be $\mathfrak{p}$-primary. Since $(0) \subseteq \mathfrak{q}$, $S_{\mathfrak{p}}(0) \subseteq S_{\mathfrak{p}}(\mathfrak{q}) = \mathfrak{q}$ by Proposition 4.8.

Suppose that for every prime ideal $\mathfrak{p}$ of A, $S_{\mathfrak{p}}(0)$ is the intersection of all $\mathfrak{p}$-primary ideals of A, and let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be distinct non-minimal prime ideals of A. If $n = 1$, then $\mathfrak{a} = \mathfrak{p}_1$ has $\mathfrak{p}_1$ as its only associated prime ideal. Suppose that $\mathfrak{p}_n$ is maximal among the $\mathfrak{p}_i$, and that there is a decomposable ideal $\mathfrak{b}$ with irredundant primary decomposition $\bigcap_{i=1}^{n-1} \mathfrak{q}_i$, where each $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. Let $\mathfrak{p}$ be a minimal prime ideal of A contained in $\mathfrak{p}_n$. If $\mathfrak{b} \subseteq S_{\mathfrak{p}_n}(0)$, then $\mathfrak{b} \subseteq S_{\mathfrak{p}_n}(0) \subseteq S_{\mathfrak{p}}(0) \subseteq \mathfrak{p}$ by Exercise 4.10 (iii). By taking radicals, this implies that $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some i. This contradicts our hypothesis that the $\mathfrak{p}_i$'s are non-minimal prime ideals. By our assumption on A, this implies that there is a $\mathfrak{p}_n$-primary ideal, $\mathfrak{q}_n$, such that $\mathfrak{b} \not\subseteq \mathfrak{q}_n$. Since $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ were assumed to be distinct, we have that $\mathfrak{a} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_n$ is an irredundant primary decomposition.

4.20

Let N be a submodule of an A-module, M. Define the M-radical of N as

$$\sqrt[M]{N} = \{x \in A : x^q M \subseteq N \text{ for some } q > 0\}.$$

It follows directly from the definition that, $\sqrt[M]{N} = \sqrt{(N : M)} = \sqrt{\operatorname{Ann}(M/N)}$.

Analogues of Exercise 1.13: Let N, N' be submodules of M and let $\mathfrak{a}$ be an ideal of A. It is unclear what analogues the author is requesting. The following are what come to my mind.

(i) $\sqrt[M]{N} \supseteq (N : M)$ (The analogy is that $\mathfrak{a}$ is an annihilator of $A/\mathfrak{a}$)

(ii) $\sqrt{\sqrt[M]{N}} \supseteq \sqrt[M]{N}$

(iii) I will break this into two formulas. The first deals with intersections of submodules. The second deals with the product of an ideal and a submodule.

$$\begin{aligned}\sqrt[M]{N \cap N'} &= \sqrt{(N \cap N' : M)} \\ &= \sqrt{(N : M) \cap (N' : M)} \\ &= \sqrt{(N : M)} \cap \sqrt{(N' : M)} \\ &= \sqrt[M]{N} \cap \sqrt[M]{N'}\end{aligned}$$

Let $x \in \sqrt[M]{\mathfrak{a}M} \cap \sqrt[M]{N}$, then there are positive integers k_1, k_2 such that $x^{k_1} \in (\mathfrak{a}M : M)$ and $x^{k_2} \in (N : M)$. If $m \in M$, then $x^{k_1} m = \sum_{i=1}^k a_i m_i$ for some $a_1, \dots, a_k \in \mathfrak{a}$ and $m_1, \dots, m_k \in M$. Thus,

$$x^{k_1 + k_2} m = x^{k_2} \sum_{i=1}^k a_i m_i = \sum_{i=1}^k a_i (x^{k_2} m_i) \in \mathfrak{a}N.$$

Therefore, $\sqrt[M]{\mathfrak{a}M} \cap \sqrt[M]{N} \subseteq \sqrt[M]{\mathfrak{a}N}$.

On the other hand,
$$\sqrt[M]{\mathfrak{a}N} = \sqrt{(\mathfrak{a}N : M)} \subseteq \sqrt{(\mathfrak{a}M \cap N : M)} = \sqrt[M]{\mathfrak{a}M} \cap \sqrt[M]{N}.$$
Given the previous paragraph, we have established
$$\sqrt[M]{\mathfrak{a}N} = \sqrt[M]{\mathfrak{a}M} \cap \sqrt[M]{N}.$$

(iv) $\sqrt[M]{N} = (1)$ if and only if $1 \in \operatorname{Ann}(M/N)$.

(v) Suppose $x^k = y + z \in \sqrt[M]{N} + \sqrt[M]{N'}$, and get k_1, k_2 so that $y^{k_1} \in (N : M)$ and $z^{k_2} \in (N' : M)$. Then $x^{k(k_1+k_2)} \in (N + N' : M)$. Therefore, $\sqrt{\sqrt[M]{N} + \sqrt[M]{N'}} \subseteq \sqrt[M]{N + N'}$.

(vi) If $\mathfrak{p}$ is a prime ideal, then
$$\sqrt[M]{\mathfrak{p}^n M} = \sqrt[M]{\mathfrak{p}\left(\mathfrak{p}^{n-1}M\right)} = \sqrt[M]{\mathfrak{p}M} \cap \sqrt[M]{\mathfrak{p}^{n-1}M} = \sqrt[M]{\mathfrak{p}^{n-1}M}$$
for every positive integer n. By induction, $\mathfrak{p} \subseteq \sqrt[M]{\mathfrak{p}M} = \sqrt[M]{\mathfrak{p}^n M}$. Now suppose that $M/\mathfrak{p}M$ and $\sqrt[M]{\mathfrak{p}M}$ are finitely generated A-modules. By Propositions 3.11 (v) and 3.14, we have $\left(\sqrt[M]{\mathfrak{p}M}\right)_{\mathfrak{p}} = \sqrt{\operatorname{Ann}\left((M/\mathfrak{p}M)_{\mathfrak{p}}\right)} = (0)$ because $(M/\mathfrak{p}M)_{\mathfrak{p}}$ is a $\kappa_{\mathfrak{p}}$-vector space. By Exercise 3.1, there is $s \notin \mathfrak{p}$ such that $s \cdot \sqrt[M]{\mathfrak{p}M} = (0) \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime ideal, we have $\sqrt[M]{\mathfrak{p}^n M} = \sqrt[M]{\mathfrak{p}M} \subseteq p$.

In particular, if A is Noetherian, and $M/\mathfrak{p}M$ is a finitely generated A-module, then $\sqrt[M]{\mathfrak{p}^n M} = \mathfrak{p}$.

4.21

A proper submodule Q of M is primary if and only if every zero-divisor of M/Q is nilpotent. Suppose that Q is a primary submodule of M and that $xy \in (Q : M)$, but $y \notin (Q : M)$. Let $m \in M$ such that $ym \notin Q$. Then x is a zero-divisor of M/Q. Since Q is primary, $x^n \in (Q : M)$ for some positive integer n. Therefore, $(Q : M)$ is a primary ideal of A.

Analogue of Lemma 4.3: Let $Q_1, \ldots, Q_n$ be $\mathfrak{p}$-primary submodules of M and let $Q = \bigcap_{i=1}^n Q_i$. By Lemma 4.3, $(Q : M) = \bigcap_{i=1}^n (Q_i : M)$ is a $\mathfrak{p}$-primary ideal. Therefore, Q is a $\mathfrak{p}$-primary submodule of M.

Analogue of Lemma 4.4: Let Q be a $\mathfrak{p}$-primary submodule of M.

(i) If $m \in Q$, then $(Q : m) = (1)$ because Q is an A-module.

(ii) If $m \notin Q$ and $xy \in (Q : m)$ with $y \notin (Q : m)$, then $xym \in Q$ and $ym \notin Q$. Since Q is primary, there is a positive integer n such that $x^n \in (Q : M) \subseteq (Q : m)$. Letting $y = 1$, we see that $(Q : M) \subseteq (Q : m) \subseteq \sqrt{(Q : M)} = \mathfrak{p}$. Therefore, $(Q : m)$ is $\mathfrak{p}$-primary.

(iii) If $x \notin \mathfrak{p}$ and $m \in (Q : x)$, then $xm \in Q$, but x is not nilpotent in M/Q. Therefore, $m \in Q$, and we see that $(Q : x) = Q$.

4.22

Analogue of Theorem 4.5: Let $N = \bigcap_{i=1}^n Q_i$ be an irredundant primary decomposition. For every $m \in M$,
$$\sqrt{(N : m)} = \bigcap_{i=1}^{n} \sqrt{(Q_i : m)} = \bigcap_{m \notin Q_j} \mathfrak{p}_j.$$
If $\sqrt{(N : m)}$ is prime ideal, then it equals $\mathfrak{p}_j$ for some Q_j not containing m, by Proposition 1.11 (ii). On the other hand, since the decomposition is irredundant, we may choose $m_j \in \bigcap_{i \neq j} Q_i \setminus Q_j$ for each index j. Thus $\sqrt{(N : m_j)} = \mathfrak{p}_j$ for each j.

By the Lattice Isomorphism theorem, the natural map $M \to M/N$ yields an order preserving bijection between the submodules of M containing N and the submodules of M/N. In particular, sums, intersections, inclusions, and quotients are preserved. Thus the associated prime ideals of N in M are the associated prime ideals of $\overline{0}$ in M/N.

4.23

By the Lattice Isomorphism Theorem, we may assume that N is the zero submodule.

Analogue of Proposition 4.6: Suppose 0 is a decomposable submodule of M, let $0 = \bigcap_{i=1}^n Q_i$ be an irredundant primary decomposition, and let $\mathfrak{p}$ be a prime ideal of A. If $\mathfrak{p} \supseteq \operatorname{Ann}(M) = \bigcap_{i=1}^n (Q_i : M)$, then taking radicals shows that $\mathfrak{p} \supseteq \bigcap_{i=1}^n \mathfrak{p}_i$. By Proposition 1.11 (ii), $\mathfrak{p}$ contains an associated prime ideal of 0. Therefore, every minimal prime ideal of $A/\operatorname{Ann}(M)$ is an isolated prime ideal of $0 \leq M$.

Analogue of Proposition 4.7: Suppose 0 is a decomposable submodule of M, and let D be the set of zero-divisors of M. Then $x \in D$ if and only if there exists a non-zero $m \in M$ such that $xm = 0$. In particular, $D = \bigcup_{0 \neq m \in M} \sqrt{(0 : m)}$. If $\mathfrak{p}$ is an associated prime ideal of 0, then Exercise 4.22 implies that $\mathfrak{p} = \sqrt{(0 : m)}$ for some $m \in M$. Therefore, $\mathfrak{p} \subseteq D$. On the other hand, for every non-zero $m \in M$, there is some associated prime ideal, $\mathfrak{p}$, of 0 such that

$$\sqrt{(0 : m)} = \bigcap_{m \notin Q_j} \sqrt{(Q_j : m)} = \bigcap_{m \notin Q_j} \mathfrak{p}_j \subset \mathfrak{p}.$$

Thus $D \subseteq \bigcup_{i=1}^n \mathfrak{p}_i$, and we see that the set of zero-divisors of M is the union of the associated prime ideals of 0 in M.

Analogue of Proposition 4.8: Let S be a multiplicatively closed subset of A and let Q be a $\mathfrak{p}$-primary submodule of M.

(i) If $s \in S \cap \mathfrak{p}$, then $s^n \in S \cap (Q : M)$ for some positive integer n. Therefore, $S^{-1}M/S^{-1}Q = S^{-1}(M/Q) = 0$.

(ii)

Analogue of Proposition 4.9:

Analogue of Theorem 4.10:

Analogue of Corollary 4.11:

5 Integral Dependence and Valuations

5.1

Let $f : A \to B$ be an integral map of rings and let $\mathfrak{b}$ be an ideal of B. By Exercise 1.21 (iii), $f^*(V(\mathfrak{b})) \subseteq V(f^{-1}(\mathfrak{b}))$. Let $\mathfrak{p} \in V(f^{-1}(\mathfrak{b}))$. By Proposition 5.6 (i), $B/\mathfrak{b}$ is integral over $A/f^{-1}(\mathfrak{b})$. By Proposition 5.10, there is $\mathfrak{q} \in \operatorname{Spec}(B/\mathfrak{b}) = V(\mathfrak{b})$ such that $f^*(\mathfrak{q}) = \mathfrak{p}$. Therefore, $f^*(V(\mathfrak{b})) = V(f^{-1}(\mathfrak{b}))$, and f^* is a closed map.

5.2

Let B/A be an integral extension of rings, let Ω be an algebraically closed field, and let $f : A \to \Omega$ be a ring map. Let $\mathfrak{m} = \ker(f)$. By Proposition 5.10, there is a prime ideal $\mathfrak{n}$ of B over $\mathfrak{m}$. Since Ω is a field, $\mathfrak{m}$ is maximal. By Corollary 5.8, $\mathfrak{n}$ is maximal. By Proposition 5.6 (i), $\kappa_{\mathfrak{n}}/\kappa_{\mathfrak{m}}$ is an integral, hence algebraic, extension of fields. Therefore, $\overline{f} : \kappa_{\mathfrak{m}} \to \Omega$ lifts to some $\overline{g} : \kappa_{\mathfrak{n}} \to \Omega$. Precompose $\overline{g}$ with the natural map $B \to \kappa_{\mathfrak{n}}$ to get $g : B \to \Omega$.

Our construction yields the following diagram:

$$\begin{array}{ccccc} A & \longrightarrow & \kappa_{\mathfrak{m}} & \xrightarrow{\overline{f}} & \Omega \\ \downarrow & & \downarrow & & \downarrow \scriptstyle{\text{id}} \\ B & \longrightarrow & \kappa_{\mathfrak{n}} & \xrightarrow{\overline{g}} & \Omega, \end{array}$$

where the vertical maps are the extension maps $A \to B, \kappa_{\mathfrak{m}} \to \kappa_{\mathfrak{n}}$, and the identity map. The horizontal composites are f and g, respectively, and the right square commutes by construction. On the other hand, the left square commutes since the central map is the quotient of the map $A \to B$. Therefore the entire diagram commutes and we see that g is an extension of f.

5.3

Let $f : B \to B'$ be a map of A-algebras and let C be an A-algebra. Suppose f is integral. If $x \in B'$, then $B[x] = \oplus_{i=0}^{n-1} Bx^i$ as a B-module, for some non-negative integer n. Therefore,

$$\begin{aligned}(B \otimes_A C)[x \otimes 1] &= B[x] \otimes_A C \\ &= \oplus_{i=0}^{n-1} (Bx^i \otimes_A C) \\ &= \oplus_{i=0}^{n-1} (B \otimes_A C)(x \otimes 1)^i\end{aligned}$$

is a finitely generated $B \otimes_A C$-module. Thus $x \otimes 1$ is integral over $B \otimes_A C$ by Proposition 5.1. Since $B' \otimes_A C$ is the $B \otimes_A C$-span of $\{x \otimes 1\}_{x \in B'}$, $f \otimes \text{id}_C$ is an integral map of A-algebras.

5.4

Let $A = k[y], B = k[x]$, where k is a field and x is a root of $t^2 - (y+1)$. Note that $t^2 - (y+1) \in A[t]$ is irreducible by Eisenstein's criterion. Let $\mathfrak{n} = (x-1)$ and $\mathfrak{m} = (y)$ be maximals ideal so B and A respectively, and note that $\mathfrak{n} \cap A = \mathfrak{m}$. Additionally, let $z = \frac{1}{x+1} \in B_{\mathfrak{n}}$ and let $K = A_{(0)}$.

Since $(x+1)z = 1$, we have $yz^2 + 2z - 1 = 0$. Therefore, $m_K^z(t) = t^2 + \frac{2}{y}t - \frac{1}{y}$. Let $g(t) = yt^2 + 2t - 1$, and suppose z is integral over $A_{\mathfrak{m}}$. Then $f(z) = 0$ for some $f(t) = c_0 t^n + \ldots + c_n \in A[t]$ with $c_0 \notin \mathfrak{m}$. Without loss of generality, f is primitive of minimum positive degree. Since $f \in K[t]$ and $f(z) = 0$, $f(t) = m_K^z(t)q(t)$ for some $q(t) \in K[t]$. Thus $yf(t) = g(t)q(t)$. By Gauss' Lemma, $q(t) \in A[t]$ and y divides $q(t)$. Therefore, $f(t) = g(t)h(t)$ for some $h(t) \in A[t]$. However, this implies that y divides c_0, contrary to hypothesis.

5.5

Let B/A be an integral extension of rings.

(i) Suppose $x \in A$ and $x^{-1} \in B$. Then $x^{-n}+a_1x^{n-1}+\ldots+a_n = 0$ implies $x^{-1} = -(a_1+a_2x+\ldots+a_nx^{n-1}) \in A$.

(ii) By Corollary 5.8, every maximal ideal of B contracts to a maximal ideal of A. By Theorem 5.10, every maximal ideal of A is the contraction of a prime ideal of B, which is maximal by Corollary 5.8. Thus the contraction of the Jacobson radical of B is the Jacobson radical of A.

5.6

Let $B_1, \ldots, B_n$ be integral A-algebras and let $B = \prod_{i=1}^n B_i$. Then B is clearly an A-algebra by the diagonal map $A \to B$. Let $x = (x_1, \ldots, x_n) \in B$ and let $m_i(t)$ be the minimal polynomial of x_i over A for each i. Define $p(t) = \prod_{i=1}^n m_i(t) \in A[t]$. By construction, $p(t)$ is monic and

$$p(x) = (p(x_1), \ldots, p(x_n)) = 0.$$

Thus x is integral over A.

5.7

Let A be a subring of B such that $B \setminus A$ is closed under multiplication, and let $x \in B$ be integral over A with minimal polynomial $t^n + a_1t^{n-1} + \ldots + a_n$. If $x \notin A$, then

$$x(x^{n-1} + a_1x^{n-2} + \ldots + a_{n-1}) = -a_n \in A$$

implies that $x^{n-1} + a_1x^{n-2} + \ldots + a_{n-1} = a' \in A$. Thus

$$x^{n-1} + a_1x^{n-2} + \ldots + a_{n-1} - a' = 0.$$

Since n was assumed to be minimal, we must have $n = 1$ and the minimal polynomial of x is $t + a_1$. So $x \in A$, contrary to hypothesis. Therefore, $x \in A$.

5.8

Let A be a subring of B, and let C be the integral closure of A in B.

(i) Suppose B is a domain, let $f, g \in B[x]$ be monic such that $fg \in C[x]$, and let K be a splitting field of f, g over $B_{(0)}$. Then we may factor as $f(x) = \prod_{i=1}^n (x - \xi_i)$ and $g(x) = \prod_{j=1}^m (x - \eta_j)$. Since $fg \in C[x]$, the ξ's and η's are integral over C. Therefore coefficients of $f, g \in B[x]$ are polynomials in elements of K, integral over C. Since C is integrally closed in B, we have $f, g \in C[x]$.

(ii) Let $h \in B[x]$ be monic, let $B' = B[x]/hB[x]$, let $i : B \to B[x]$ be the inclusion, and let $q : B[x] \to B'$ be the quotient map. We claim that $q \circ i$ is injective. Note that $a \in hB[x]$ if and only if there exists some $b \in B[x]$ such that $a = bh$. Since h is monic, a has positive degree. Since the domain of $q \circ i$ contains no elements of $B[x]$ of positive degree, $q \circ i$ is injective. Therefore, B' is a finite B-algebra with an injection $B \to B'$. Furthermore, $h(t) = (t - x)h_1(t)$ over B', for some $h_1(t) \in B'[t]$ (it is here that we are using the fact that a non-unique form of Euclidean division lets us divide by a monic polynomial over an arbitrary ring).

Let $f, g \in B[x]$ be monic such that $fg \in C[x]$. Repeat the preceding construction with f and g to obtain a finite B-algebra B' such that f and g split over B'. The same argument as in (i) concerning polynomials in the roots of f and g establishes that $f, g \in C[x]$.

5.9

Let A be a subring of B, and let C be the integral closure of A in B. Let $f \in B[x]$ be integral over $A[x]$ with minimal polynomial $p(t) = t^n + a_1 t^{n-1} + \ldots + a_n$, where $a_1, \ldots, a_n \in A[x]$. Let r be an integer greater than $\deg a_1, \ldots, \deg a_n, \deg f$, and n. If $f_0 = f - x^r$, then

$$0 = p(f) = p(f_0 + x^r) = f_0^n + h_1 f_0^{n-1} + \ldots + h_{n-1} f_0 + p(x^r).$$

Therefore,

$$f_0, f_0^{n-1} + h_1 f_0^{n-2} + \ldots + h_{n-1} \in B[x]$$

and

$$f_0(f_0^{n-1} + h_1 f_0^{n-2} + \ldots + h_{n-1}) = -p(x^r) \in C[x].$$

By Exercise 5.8, $f_0 \in C[x]$.

5.10

Let $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ be the map associated to a ring homomorphism $f : A \to B$.

(i) (a)⇒(b) Suppose that f^* is closed, $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ are prime ideals of A, and $\mathfrak{q}_1$ is a prime ideal of B over $\mathfrak{p}_1$. Then

$$f^*(V(\mathfrak{q}_1)) = V(f^*(\mathfrak{q}_1)) = V(\mathfrak{p}_1)$$

and $\mathfrak{p}_2 \in V(\mathfrak{p}_1)$. Thus there is $\mathfrak{q}_2 \in V(\mathfrak{q}_1)$ such that $f^*(\mathfrak{q}_2) = \mathfrak{p}_2$.

(b)⇔(c) The map $\overline{f}^* : \operatorname{Spec}(B/\mathfrak{q}) \to \operatorname{Spec}(A/\mathfrak{p})$ is surjective if and only if every prime ideal of A containing $\mathfrak{p}$ has a lift to a prime ideal of B over $\mathfrak{q}$.

(ii) (a)⇒(b) Suppose that f^* is open, $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$ are prime ideals of A, and $\mathfrak{q}_1$ is a prime ideal of B over $\mathfrak{p}_1$. Let U be an open neighborhood of $\mathfrak{q}_1$ in $\operatorname{Spec} B$. Since $f^*(\mathfrak{q}_1) = \mathfrak{p}_1$ and $f^*(U)$ is open, there is $s \in A \setminus \mathfrak{p}_1$ such that X_s is an open neighborhood of $\mathfrak{p}_1$ in $f^*(U)$. Since $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$, we also have $\mathfrak{p}_2 \in X_s \subseteq f^*(U)$.

By Exercise 3.26,

$$f^*(\operatorname{Spec} B_{\mathfrak{q}_1}) = \bigcap_{t \notin \mathfrak{q}_1} f^*(\operatorname{Spec} B_t) = \bigcap_{t \notin \mathfrak{q}_1} f^*(Y_t).$$

Since $\mathfrak{p}_2 \in f^*(Y_t)$ for all $t \notin \mathfrak{q}_1$, there is some $\mathfrak{q}_2 \subseteq \mathfrak{q}_1$ over $\mathfrak{p}_2$.

(b)⇔(c) The map $\widehat{f}^* : \operatorname{Spec} B_{\mathfrak{q}} \to \operatorname{Spec} A_{\mathfrak{p}}$ is surjective if and only if every prime ideal of A contained in $\mathfrak{p}$ has a lift to a prime ideal of B contained in $\mathfrak{q}$.

5.11

Let $f : A \to B$ be a flat ring homomorphism. By Exercise 3.18, $f^* : \operatorname{Spec} B_{\mathfrak{q}} \to \operatorname{Spec} A_{\mathfrak{p}}$ is surjective whenever $\mathfrak{q}$ is a prime ideal of B and $\mathfrak{p} = \mathfrak{q}^c$. By Exercise 5.10 (ii), f has the going-down property.

5.12

Let G be a finite group of automorphisms of a ring A, let A^G be the ring of G-invariants, and let $x \in A$. Define $p(t) = \prod_{\sigma \in G}(t - \sigma(x)) \in A[t]$. Since the coefficients of $p(t)$ are symmetric polynomials in the G-orbit of x, $p(t) \in A^G[t]$. Furthermore, $p(x) = 0$. So x is integral over A^G.

Let S be a G-stable multiplicatively closed subset of A and let $S^G = S \cap A^G$. For any $\sigma \in G$, the universal property of localization lifts the map $A \xrightarrow{\sigma} A \to S^{-1}A$ uniquely to a map $\sigma : S^{-1}A \to S^{-1}A$ such that $\sigma\left(\frac{a}{s}\right) = \frac{\sigma(a)}{\sigma(s)}$. This is the desired group action.

Since $A^G \subseteq A$ and $S^G \subseteq S$, the universal property of localization yields a unique map $f : (S^G)^{-1}A^G \to S^{-1}A$ extending the inclusion $A^G \to A$ such that $f\left(\frac{a}{s}\right) = \frac{a}{s}$. Given $\frac{a}{s} \in (S^G)^{-1}A^G$ and $\sigma \in G$,

$$\sigma\left(f\left(\frac{a}{s}\right)\right) = \sigma\left(\frac{a}{s}\right) = \frac{\sigma(a)}{\sigma(s)} = \frac{a}{s} = f\left(\frac{a}{s}\right).$$

So the image of f is in $(S^{-1}A)^G$.

Let $\frac{a}{s} \in (S^{-1}A)^G$, and define $s' = \prod_{\sigma\in G} \sigma(s) \in S^G$.

5.13

Let $\mathfrak{p}$ be a prime ideal of A^G and let P be the set of prime ideals of A over $\mathfrak{p}$. Suppose that $\mathfrak{p}_1, \mathfrak{p}_2 \in P$ and $x \in \mathfrak{p}_1$. Then we have

$$\prod_{\sigma\in G} \sigma(x) \in \mathfrak{p}_1 \cap A^G = \mathfrak{p} \subseteq \mathfrak{p}_2.$$

Since $\mathfrak{p}_2$ is prime ideal, there is $\sigma \in G$ such that $x \in \sigma(\mathfrak{p}_2)$. Therefore, $\mathfrak{p}_1 \subseteq \bigcup_{\sigma\in G} \sigma(\mathfrak{p}_2)$. By the prime ideal Avoidance Lemma, $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ for some σ. Since A is integral over A^G and $\mathfrak{p}_1 \cap A^G = \mathfrak{p}_2 \cap A^G$, we know that $\mathfrak{p}_1 = \sigma(\mathfrak{p}_2)$.

5.14

Let A be an integrally closed domain with $K = A_{(0)}$, L/K a finite, separable, normal field extension, and let B be the integral closure of A in L, and let $G = \mathrm{Gal}(L/K)$. Suppose $b \in B$ has minimal polynomial $p(x) = x^n + a_1x^{n-1} + \ldots + a_n$. Given $\sigma \in G$, $0 = \sigma(0) = \sigma(p(b)) = p(\sigma(b))$, because G fixes A. Since B is the integral closure of A in L, B is G-stable. Furthermore, we have shown that σ permutes the roots $p(x)$. Therefore, there is $b' \in B$ such that $\sigma(b') = b$.

If $b \in B^G$, then $b \in K$ because it is fixed, and b is integral over A because it is in B. Since A is integrally closed, $b \in A$. Therefore, $B^G \subseteq A$.

5.15

Let A be an integrally closed domain with $K = A_{(0)}$, L/K a finite extension of fields, and let B be the integral closure of A in L.

5.16

Let k be an infinite field and let A be a finitely generated k-algebra. Then $A = k[x_1, \ldots, x_n]/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of $k[x_1, \ldots, x_n]$. If $\mathfrak{a} = (0)$ or $n = 0$, then the lemma holds trivially. Suppose that $\mathfrak{a} \neq (0)$, $n > 0$, and that the lemma holds for any number of generators less than n. Let $f \in \mathfrak{a}$ be non-zero. Without loss of generality, x_n is one of the variables appearing in f. Let $F \in k[x_1, \ldots, x_n]$ be the homogeneous component of f of maximum degree, d. Since k is infinite, there are scalars $\lambda_1, \ldots, \lambda_{n-1}$ such that $F(\lambda_1, \ldots, \lambda_{n-1}, 1) \neq 0$. Define the variables, $x_i' = x_i - \lambda_i x_n$ for $1 \leq i < n$. For an arbitrary monomial, we have

$$\prod_{i=1}^{n} x_n^{\nu_i} = x_n^{\nu_n} \prod_{i=1}^{n-1} (x_i' + \lambda_i x_n)^{\nu_i} = \left(\prod_{i=1}^{n-1} \lambda_i^{\nu_i}\right) x_n^d + a_1 x_n^{d-1} + \ldots + a_d,$$

where $a_1, \ldots, a_d \in k[x_1', \ldots, x_{n-1}']$. Therefore, by change of coordinates, we may write

$$f = F(\lambda_1, \ldots, \lambda_{n-1}, 1)x_n^d + b_1 x_n^{d-1} + \ldots + b_d,$$

with $b_1, \dots, b_d \in k[x'_1, \dots, x'_{n-1}]$. Let $\mathfrak{a}'$ be the kernel of the natural map $k[x'_1, \dots, x'_{n-1}] \to A$, and let $A' = k[x'_1, \dots, x'_{n-1}]/\mathfrak{a}'$. Note that the map $A' \to A$ is injective. By induction, there $y_1, \dots, y_r \in A'$ that are algebraically independent over k, and $k[y_1, \dots, y_r] \to A'$ is finite, injective, and $y_1, \dots, y_r$ are k-linear combinations of $x'_1, \dots, x'_{n-1}$. By Proposition 2.16, $k[y_1, \dots, y_r] \to A$ is finite, injective, and $y_1, \dots, y_r$ are k-linear combinations of $x_1, \dots, x_n$. (Practically, the process of proof can be repeated until $\mathfrak{a} = 0$. At that point we have obtained the desired algebraically independent set of generators.)

5.17

Let k be an algebraically closed field and let $X \subseteq k^n$ be an affine algebraic set with ideal $I(X) \neq (1)$. Let $P(X) = k[t_1, \dots, t_n]/I(X)$ be the coordinate ring of X. Since $I(X) \neq (1)$, we know that $P(X)$ is not the zero ring. By Exercise 1.27, we know that any maximal ideal $\mathfrak{m}$ of $P(X)$ has the form $(t_1 - a_1, \dots, t_n - a_n)$ for some $a_1, \dots, a_n \in k$. Let $f \in I(X)$. Since $\mathfrak{m}$ is a maximal ideal of $P(X)$, we know that $f \in \mathfrak{m}$. Therefore, $f(a_1, \dots, a_n) = 0$. Since this is true for every $f \in I(X)$, the point $(a_1, \dots, a_n)$ is in X.

5.18

Let k be a field and let B be a finitely generated k-algebra that is also a field. Let B be generated by $x_1, \dots, x_n$ over k. If $n = 1$, then

$$\frac{1}{x_1} = x_1^n + a_1 x_1^{n-1} + \ldots + a_n$$

for some $a_1, \dots, a_n \in k$ shows that x_1 is integral over k. Therefore, B/k is a finite field extension. Therefore, we may assume that $n > 1$.

Let $A = k[x_1]$ and let $K = A_{(0)}$. By the inductive hypothesis, B/K is a finite field extension. Therefore, $x_2, \dots, x_n$ are integral over K, hence algebraic over A. Let f be the product of the lead coefficients of the minimal polynomials of $x_2, \dots, x_n$ over A. Then $x_2, \dots, x_n$ are integral over A_f. By the generation assumption on B, this implies that B, hence K is integral over A_f. If x_1 is transcendental over k, then A is a unique factorization domain, hence integrally closed. By Proposition 5.12, A_f is integrally closed. Since K/A_f is integral, $K = A_f$.

5.19 Undone

5.20

Let B be a finitely generated faithful A-algebra that is a domain, and let $S = A \setminus (0)$. Then $S^{-1}B$ is a finitely generated K-algebra, where $K = S^{-1}A$ is the field of fractions of A. By Noether's Normalization Lemma, there are $x_1, \dots, x_n \in S^{-1}B$, algebraically independent over K, such that $S^{-1}B$ is integral over $K[x_1, \dots, x_n]$. Let $z_1, \dots, z_m$ generate B as an A-algebra, and let $m_i(t) = t^{d_i} + \frac{a_{i1}}{s_{i1}} t^{d_i - 1} + \ldots + \frac{a_{id_i}}{s_{id_i}}$ be the minimal polynomial of z_i over $K[x_1, \dots, x_n]$ for each $1 \leq i \leq m$. For each $1 \leq k \leq n$, let $s_k \in S$ be such that $s_k x_k \in B$. Define $s = \left(\prod_{i,j} s_{ij}\right)\left(\prod_k s_k\right)$, define $y_k = s x_k \in B$, and let $\widehat{m}_i(t) \in A[x_1, \dots, x_n][t]$ be monic polynomials such that $\widehat{m}_i(st) = s m_i(t)$ for each i. Then $y_1, \dots, y_n$ are algebraically independent over K, hence over A.

Let $B' = A[y_1, \dots, y_n]$. For each i, polynomial $\widehat{m}_i(t)$ gives an equation of integrality for $s z_i$ over $A[x_1, \dots, x_n]$. Since $y_i = \frac{x_i}{s}$, $z_i \in B_s$ is integral over B'_s. Since B_s is generated by $z_1, \dots, z_n$ over B'_s, the ring extension is integral.

5.21

Let B be a finitely generated faithful A-algebra that is a domain. By Exercise 5.20, there exist $y_1, \dots, y_n \in B$ that are algebraically independent over A and a non-zero $s \in A$ such that B_s is integral over $A[y_1, \dots, y_n]_s$. Let Ω be an algebraically closed field and let $f : A \to \Omega$ be a ring homomorphism for which $f(s) \neq 0$. Since the $y_1, \dots, y_n$ are algebraically independent over A, f has an extension to $\overline{g} : A[y_1, \dots, y_n] \to \Omega$ given by mapping each y_i to 0. Furthermore, $f(s) \neq 0$ implies that $\overline{g}(s)$ is a unit. By the universal property of localization, $\overline{g}$ has a lift to a map $\widehat{g} : A[y_1, \ldots, y_n]_s \to \Omega$. Since B_s is finitely generated and integral over $A[y_1, \ldots, y_n]_s$, it is a finite $A[y_1, \ldots, y_n]_s$-module. Let $z_1, \ldots, z_m$ be a list of generators. We can successively extend $\widehat{g}$ by mapping each z_i to a root of its minimal polynomial in Ω. Since there are only finitely many z_i's, this process terminates in a map $g : B \to \Omega$.

5.22 Undone

Let B be a finitely generated faithful A-algebra that is a domain.

5.23

Let A be a ring.

(i)⇒(ii) By the First Isomorphism Theorem, homomorphic images of A may be identified with quotients. Let $\mathfrak{a}$ be an ideal of A. The nilradical of $A/\mathfrak{a}$ is the intersection of all prime ideals of $A/\mathfrak{a}$. By the Lattice Isomorphism Theorem we may identify ideals of A containing $\mathfrak{a}$ with ideals of $A/\mathfrak{a}$ in such a way that containment, primality, maximality, and intersections are preserved. If $\mathfrak{p}$ is a prime ideal of A containing $\mathfrak{a}$, then there is a family of maximal ideals $\{\mathfrak{m}_i\}_{i \in I_\mathfrak{p}}$ such that $\mathfrak{p} = \bigcap_{i \in I_\mathfrak{p}} \mathfrak{m}_i$. By the lattice isomorphism theorem, this intersection is preserved on passing to the quotient. Therefore the nilradical of $A/\mathfrak{a}$ is an intersection of maximal ideals. Since the Jacobson radical is the intersection of all maximal ideals, the Jacobson radical is contained in the nilradical of $A/\mathfrak{a}$. The reverse containment holds in general, so the Jacobson radical and nilradical of $A/\mathfrak{a}$ are the same.

(ii)⇒(iii) Let $\mathfrak{p}$ be a non-maximal prime ideal of A. We apply the Lattice Isomorphism Theorem. The nilradical of $A/\mathfrak{p}$ corresponds to the ideal $\mathfrak{p}$. The Jacobson radical of $A/\mathfrak{p}$ corresponds to the intersection of all maximal ideals containing $\mathfrak{p}$. By hypothesis, these are equal. If $\mathfrak{q}$ is a prime ideal of A strictly containing $\mathfrak{p}$, then $\mathfrak{q}$ is contained in a maximal ideal. Therefore, the intersection of all prime ideals strictly containing $\mathfrak{p}$ is an ideal containing the nilradical of $A/\mathfrak{p}$ and contained in the Jacobson radical of $A/\mathfrak{p}$. Since these are equal, the intersection of all prime ideals strictly containing $\mathfrak{p}$ is $\mathfrak{p}$.

(iii)⇒(i) Suppose A has a prime ideal ideal which is not an intersection of maximal ideals. By passing to the quotient we may assume that A is a domain whose Jacobson radical is non-zero. Let f be a non-zero element of the Jacobson radical of A. Since A is a domain, $A_f \neq 0$. Let $\mathfrak{p}$ be the contraction of a maximal ideal of A_f to A. Then $\mathfrak{p}$ is a prime ideal of A not containing f and is maximal with respect to this property. Therefore, the intersection of all prime ideals strictly containing $\mathfrak{p}$ contains f, and is not equal to $\mathfrak{p}$.

5.24

Let A be a Jacobson ring and let B be an A-algebra.

(i) Suppose that B is integral over A. Let $\mathfrak{b}$ be an ideal of B. We want to show that $\sqrt{\mathfrak{b}}$ is the intersection of maximal ideals of B containing $\mathfrak{b}$. Without loss of generality, let $\mathfrak{b} = \sqrt{\mathfrak{b}}$, and let $\mathfrak{a}$ be the contraction of $\mathfrak{b}$ to A. Since taking the radical commutes with taking the contraction, $\mathfrak{a} = \sqrt{\mathfrak{a}}$ as well. By Proposition 5.6 (i), $B/\mathfrak{b}$ is integral over $A/\mathfrak{a}$. Let J be the Jacobson radical of $B/\mathfrak{b}$. By Exercise 5.5 (ii), J^c is the Jacobson radical of $A/\mathfrak{a}$. Since A is a Jacobson ring, $J^c = (0)$.

Suppose there is $f \in J$ that is not nilpotent and f has minimal polynomial $t^n + a_1 t^{n-1} + \ldots + a_n$ over $A/\mathfrak{a}$. Since $f^n + a_1 f^{n-1} + \ldots + a_n = 0$, $a_n \in J$, hence $a_n \in J^c = (0)$. Since f is not nilpotent, $f \notin \mathfrak{q}$ for some prime ideal $\mathfrak{q}$ of $B/\mathfrak{b}$, then

$$f \cdot (f^{n-1} + a_1 f^{n-2} + \ldots + a_{n-1}) = 0 \in \mathfrak{q}$$

implies that $f^{n-1} + a_1 f^{n-2} + \ldots + a_{n-1} \in \mathfrak{q}$. Let $\mathfrak{p}$ be the contraction of $\mathfrak{q}$ to $A/\mathfrak{a}$. Passing to the quotient gives $B/\mathfrak{q}$ integral over $A/\mathfrak{p}$, f is in the Jacobson radical of $B/\mathfrak{q}$ with minimal polynomial of degree $n-1$, the contraction of the J to $A/\mathfrak{p}$ is the Jacobson radical of $A/\mathfrak{p}$ and is (0), and f is again not nilpotent. As long as $n > 1$ we may repeat this process until we arrive at the case that f has minimal polynomial $t + a_1$. Since f is in the Jacobson radical of $B/\mathfrak{q}$ and $f = -a_1 \in A/\mathfrak{p}$, f is in the Jacobson radical of $A/\mathfrak{p}$. Therefore, $f \in \mathfrak{p} \subseteq \mathfrak{q}$. This contradicts the hypothesis on $\mathfrak{q}$. Therefore, the Jacobson radical of $B/\mathfrak{b}$ is equal to the nilradical, and B is a Jacobson ring.

(ii) Suppose that B is finitely generated over A. Let $\mathfrak{q}$ be a non-maximal prime ideal of B and let $\mathfrak{p}$ be the contraction of $\mathfrak{q}$ to A. By passing to the quotient we may assume that B is a finitely generated faithful A-algebra which is also a domain and that the Jacobson radical of A is (0). By Exercise 5.22, we know that the Jacobson radical of B is (0). Since every non-trivial prime ideal of B is contained in a maximal ideal, the intersection of all non-trivial prime ideals is contained in the Jacobson radical of B, and is therefore (0). This establishes criterion (iii) of Exercise 5.23.

Since $\mathbb{Z}$ is a Jacobson ring and every field is a Jacobson ring, finitely generated rings and finitely generated k-algebras are Jacobson rings.

5.25

Let A be a ring.

(i)⇒(ii) Let A be a Jacobson ring and let B be a field which is a finitely generated A-algebra. Since B is a field, the kernel of $A \to B$ is a prime ideal ideal. Passing to the quotient, we may assume that A is a domain whose Jacobson radical is (0), and B is a faithful A-algebra. Let $s \in A$ be as in Exercise 5.21. Since $s \neq 0$, there is a maximal ideal $\mathfrak{m}$ of A not containing s. Let $k = A/\mathfrak{m}$ and let Ω be the algebraic closure of k. Compose the natural maps $A \to k$ and $k \to \Omega$ to obtain a map $f : A \to \Omega$ such that $f(s) \neq 0$. By Exercise 5.21, f has an extension to some $g : B \to \Omega$. Since B is a field, g is injective. Therefore, B is an algebraic extension of k. Since B is finitely generated over k, B/k is a finite field extension. Given $x \in B$, let $\overline{m}(t) = t^n + \overline{a_1}t^{n-1} + \ldots + \overline{a_n} \in (A/\mathfrak{m})[t]$ be the minimal polynomial of x over k, where $a_1, \ldots, a_n \in A$. Then $\overline{m}(t)$ lifts to some monic $m(t) \in A[x]$ such that $m(x) = a' \in \mathfrak{m}$. Thus x satisfies the polynomial $m(t) - a'$ and is integral over A. Therefore, B is a finite A-algebra.

(ii)⇒(i) Let $\mathfrak{p}$ be a non-maximal prime ideal of A and let $B = A/\mathfrak{p}$. Let f be a non-zero element of B. Then B_f is a finitely generated A-algebra. Suppose that B_f is a field. By hypothesis, it is a finite A-algebra. Thus it is a finite B-algebra, hence integral over B. By Proposition 5.7, B is a field, in contradiction to our hypothesis on $\mathfrak{p}$. Therefore, B_f is not a field. Contract a maximal ideal of B_f to B to obtain a prime ideal $\mathfrak{q}$ not containing f and maximal with this property. Therefore, for every $f \in \backslash\mathfrak{p}$, there is a prime ideal strictly containing $\mathfrak{p}$ but not f. So $\mathfrak{p}$ is the intersection of all prime ideals strictly containing it.

5.26

Let X be a topological space. Suppose $U^{\text{open}}, C^{\text{closed}} \subseteq X$. We claim that $U \cap C$ is open in its closure. Since $\overline{U}$ and C are closed, and $U \cap C \subseteq \overline{U} \cap C$, we have $\overline{U \cap C} \subseteq \overline{U} \cap C$. Thus

$$\overline{U \cap C} \cap C^c \subseteq \overline{U} \cap C \cap C^c = \varnothing.$$

Therefore,

$$\begin{aligned}\overline{U \cap C} \setminus (U \cap C) &= \overline{U \cap C} \cap (U^c \cup C^c) \\ &= (\overline{U \cap C} \cap U^c) \cup (\overline{U \cap C} \cap C^c) \\ &= (\overline{U \cap C} \cap U^c),\end{aligned}$$

which is closed. Thus $U \cap C$ is open in $\overline{U \cap C}$.

On the other hand, suppose that $S \subseteq X$ is open in $\overline{S}$. Then there is $U^{\text{open}} \subseteq X$ such that $S = U \cap \overline{S}$, and we see that S is the intersection of an open and a closed set.

Let X_0 be a subset of X.

(1)⇒(2) Suppose that every non-empty locally closed subset of X meets X_0, and let $E^{\text{closed}} \subseteq X$. Let $x \in E$ and let U be a neighborhood of x. Since $U \cap E$ is locally closed, $U \cap E \cap X_o \neq \varnothing$. Therefore, x is in the closure of $E \cap X_0$. The reverse containment holds in general.

(2)⇒(3) Suppose that $\overline{E \cap X_0} = E$ for every $E^{\text{closed}} \subseteq X$, and let U, V be open subsets of X such that $U \cap X_0 = V \cap X_0$. Taking X_0-complements yields $U^c \cap X_0 = V^c \cap X_0$. Now take closures in X to obtain

$$U^c = \overline{U^c \cap X_0} = \overline{V^c \cap X_0} = V^c.$$

Thus $U = V$ and the correspondence is injective. On the other hand, the correspondence is surjective by the definition of the subspace topology.

(3)⇒(1) Suppose that the correspondence $U \mapsto U \cap X_0$ from the topology of X to that of X_0 is bijective, and let S be a non-empty locally closed subset of X. By definition, there are $U^{\text{open}}, C^{\text{closed}} \subseteq X$ such that $S = U \cap C$. Since $\varnothing \mapsto \varnothing$ and the correspondence is bijective, $U \cap X_0 \neq \varnothing$. Since S is non-empty, $X \mapsto X_0$, and the correspondence is bijective, $C^c \cap X_0 \neq X_0$. Therefore, $S \cap X_0 \neq \varnothing$.

Let A be a ring, let $X = \operatorname{Spec} A$, and let X_0 be the set of maximal ideals of A.

(i)⇒(ii) Suppose that A is a Jacobson ring. Let $\mathfrak{a}$ be an ideal of A and let $V(\mathfrak{b}) = \overline{V(\mathfrak{a}) \cap X_0}$. By construction, $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$. On the other hand, $V(\mathfrak{a}) \cap X_0 \subseteq V(\mathfrak{b})$ is the set of maximal ideals containing $\mathfrak{a}$. Therefore, $\mathfrak{a}$ and $\mathfrak{b}$ are contained in exactly the same maximal ideals. Since A is a Jacobson ring, $\sqrt{\mathfrak{a}}, \sqrt{\mathfrak{b}}$ are each the intersection of the maximal ideals containing them, and are thus equal. In summary, $\overline{V(\mathfrak{a}) \cap X_0} = V(\mathfrak{a})$, and $V(\mathfrak{a})$ was an arbitrary closed set. So X_0 is very dense in X.

(ii)⇒(iii) Suppose that X_0 is very dense in X and suppose that $\{x\}$ is locally closed. By criterion (1), $\{x\} \cap X_0$ is non-empty. Therefore, x is a maximal ideal of A, and $\{x\}$ is closed.

(iii)⇒(i) Suppose that every locally closed subset of X consisting of single point is closed. Let $\mathfrak{p}$ be a non-maximal prime ideal of A and let $f \in A \setminus \mathfrak{p}$. Since $X_f \cap V(\mathfrak{p})$ is locally closed and $\{\mathfrak{p}\}$ is not closed, $X_f \cap V(\mathfrak{p})$ contains a point $\mathfrak{q}$ other than $\mathfrak{p}$, and $f \notin \mathfrak{q}$. Since f was arbitrary, $\mathfrak{p}$ is equal to the intersection of prime ideal ideals strictly containing it, and A is a Jacobson ring.

5.27

Let K be a field and let Σ be the set of all local subrings of K. Since $K \in \Sigma$, Σ is non-empty. Order Σ by domination and let $A_1 \leq A_2 \leq \ldots$ be a chain. Let $A = \bigcup_{i \geq 1} A_i$ and let $\mathfrak{m} = \bigcup_{i \geq 1} \mathfrak{m}_i$, where $\mathfrak{m}_i$ is the maximal ideal of A_i for each i. Then A is a ring and $\mathfrak{m}$ is an ideal. Suppose $f \in A \setminus \mathfrak{m}$. Then $f \in A_i \setminus \mathfrak{m}_i$ for some i. Since A_i is a local ring, $f^{-1} \in A_i \subseteq A$. Therefore, A is a local ring and $\mathfrak{m}$ is its maximal ideal. By Zorn's Lemma, Σ has maximal elements.

Let $A, \mathfrak{m}$ be a maximal element of Σ and let $f \in K \setminus A$. Then A is a subring of $A[f]$. If $\mathfrak{m}A[f]$ is a proper ideal, then it is contained in some maximal ideal $\mathfrak{n}$ of $A[f]$. Since we are working inside of an ambient field, A is a subring of the local ring $(A[f])_{\mathfrak{n}}$ and $\mathfrak{m} = \mathfrak{n} \cap A$. Since A is maximal, $A = (A[f])_{\mathfrak{n}}$, contrary to our assumption on f. Therefore, $1 \in \mathfrak{m}A[f]$, $f^{-1} \in \mathfrak{m} \subseteq A$, and we see that A is a valuation ring of K.

Suppose that A is a valuation ring of K. By Proposition 5.18 (i), $A, \mathfrak{m}$ is a local ring. Suppose that $B, \mathfrak{n}$ is a local subring of K dominating $A, \mathfrak{m}$. If $f \in B \setminus A$, then $f^{-1} \in A$ is a non-unit. So $f^{-1} \in \mathfrak{m} \subseteq \mathfrak{n}$. Thus, $1 \in \mathfrak{n}$, contrary to hypothesis. Therefore, $A, \mathfrak{m}$ is a maximal element of Σ.

5.28

Let A be a domain with fraction field K.

(i)⇒(ii) Suppose that A is a valuation ring of K with ideals $\mathfrak{a}, \mathfrak{b}$, let $f \in \mathfrak{b} \setminus \mathfrak{a}$, and let $0 \neq g \in \mathfrak{a}$. Since A is a valuation ring, $\frac{f}{g} \in A$ or $\frac{g}{f} \in A$. If $\frac{f}{g} \in A$, then $f = \frac{f}{g} \cdot g \in \mathfrak{a}$, contrary to hypothesis. Therefore, $\frac{g}{f} \in A$ and $g = \frac{g}{f} \cdot f \in \mathfrak{b}$. Thus $\mathfrak{a} \subseteq \mathfrak{b}$.

(ii)⇒(i) Suppose that the ideals of A are totally ordered by containment and let $\frac{f}{g} \in K$ with $f, g \in A$. Then either $(f) \subseteq (g)$ or $(f) \supseteq (g)$. So $f = ag$ for some $a \in A$ or $bf = g$ for some $b \in A$. Therefore, $\frac{f}{g} = a \in A$ or $\frac{g}{f} = b \in A$, and A is a valuation ring.

If A is a valuation ring with prime ideal $\mathfrak{p}$, then the ideals of A are totally ordered by containment. Since the respective lattices of ideals of $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ are sublattices of the lattice of ideals of A, they are also totally ordered by containment. Therefore, $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ are valuation rings of their fields of fractions.

5.29

Let A be a valuation ring of a field K and let B be a subring of K containing A. Since A is a valuation ring of K, B is also a valuation ring of K by Proposition 5.18 (ii). By Proposition 5.188 (i), B is a local ring with maximal ideal $\mathfrak{n}$. Let $\mathfrak{p} = \mathfrak{n} \cap A$. If $s \in A \setminus \mathfrak{p}$, then $s \in B \setminus \mathfrak{n}$. So s is a unit in B. By the universal property of localization, the inclusion $A \to B$ lifts uniquely to a map $A_{\mathfrak{p}} \to B$. Furthermore, this map is injective because A is a domain. If $f \in B \setminus A$, then $f^{-1} \in A$ because A is a valuation ring. Since $f \in B$, f^{-1} is not in $\mathfrak{n}$, hence not in $\mathfrak{p}$. Therefore, $f = (f^{-1})^{-1} \in A_{\mathfrak{p}}$. Therefore, the localization map is surjective as well.

5.30

Let A be a valuation ring of a field K, let $U \leq K^*$ be the group of units of A, and let $\Gamma = K^*/U$. For $\xi, \eta \in \Gamma$ define $\xi \geq \eta$ by $xy^{-1} \in A$ for some coset representatives $x, y \in K^*$ of $\xi, \eta \in \Gamma$. We claim that this is a well-defined total ordering on Γ that is compatible with the group structure.

(Well-Defined) Let $a, b \in U$ and let $x, y \in K^*$ such that $xy^{-1} \in A$. Then

$$(ax)(by)^{-1} = ab^{-1}xy^{-1} \in A.$$

So the relation is well-defined on elements of Γ.

(Total) Since A is a valuation ring, $xy^{-1} \in A$ or $yx^{-1} \in A$ for every $x, y \in K^*$.

(Reflexive) Since $xx^{-1} = 1 \in A$, the relation is reflexive.

(Anti-Symmetric) If $xy^{-1}, yx^{-1} \in A$, then $xy^{-1} \in U$. Therefore, $\xi\eta^{-1} = 1$, and $\xi = \eta$.

(Transitive) Let $\zeta \in \Gamma$ with coset representative $z \in K^*$, and suppose that $xy^{-1}, yz^{-1} \in A$. Then $xz^{-1} = xy^{-1} \cdot yz^{-1} \in A$.

(Compatible) If $xy^{-1} \in A$, then $(xz)(yz)^{-1} = xy^{-1} \in A$.

Let $v : K^* \to \Gamma$ be the quotient map. Since A is a valuation ring, $xy^{-1} \in A$ or $yx^{-1} \in A$. Without loss of generality, $xy^{-1} \in A$. So $\min(v(x), v(y)) = v(y)$. Therefore,

$$(x + y)y^{-1} = xy^{-1} + 1 \in A,$$

and we see that $v(x + y) \geq \min(v(x), v(y))$.

5.31

Let Γ be a totally ordered abelian group (written additively) and let K be a field. A valuation of K with values in Γ is a map $v : K^* \to \Gamma$ such that (i) $v(xy) = v(x) + v(y)$ and (ii) $v(x + y) \geq \min(v(x), v(y))$.

Let $A \subseteq K$ be the set of $x \in K$ such that $x = 0$ or $v(x) \geq 0$. By (i) and (ii), A is closed under addition and multiplication. By (i), $v(1) = v(1) + v(1)$, which implies that $v(1) = 0$. Therefore, $1 \in A$. On the other hand, $0 = v(1) = v(-1) + v(-1)$ implies that $v(-1) = 0$ (even if K has characteristic 2). Thus $v(-x) = v(x)$ for all $x \in A$ by (i). So A is closed under the taking of additive inverses, and A is a ring.

5.32 Undone

5.33 Undone

5.34 Undone

5.35 Undone

6 Chain Conditions

6.1

Let M be an A-module and let $u : M \to M$ be A-linear.

(i) Suppose that u is surjective and M is Noetherian. Then the ascending chain, $0 \leq \ker u \leq \ker u^2 \leq \ldots$, stabilizes at some index $n \geq 1$. Since u is surjective, u^n is as well. If $x \in \ker u$ and $x = u^n(y)$ for some $y \in M$, then

$$0 = u(x) = u^{n+1}(y).$$

Thus $y \in \ker u^{n+1} = \ker u^n$. So $x = u^n(y) = 0$. Therefore, u is injective.

(ii) Suppose that u is injective and M is Artinian. Then the descending chain, $0 \geq \operatorname{im} u \geq \operatorname{im} u^2 \geq \ldots$, stabilizes at some index $n \geq 1$. Since u is injective, it lifts to an injective map $\operatorname{coker} u^i \to \operatorname{coker} u^{i+1}$ for each i. If $\overline{x} \in \operatorname{coker} u$, then $u^n(\overline{x}) \in \operatorname{coker} u^{n+1} = \operatorname{coker} u^n$. Thus $u^n(\overline{x}) = \overline{0}$. Therefore, $\overline{x} = \overline{0}$, and u is surjective.

6.2

Let M be an A-module such that every non-empty set of finitely generated submodules has a maximal element, let N be a submodule of M, and let Σ be the set of finitely generated submodules of N. Since $0 \in \Sigma$, it is non-empty. Let N' be a maximal element of Σ. Since $N' \in \Sigma$, N' is finitely generated. If $x \in N$, then $Ax + N'$ is a finitely generated submodule of N containing N'. Since N' is maximal, $N' = Ax + N'$. Since x was arbitrary, $N' = N$. We have proven that every submodule of M is finitely generated. Therefore, M is Noetherian.

6.3

Let N_1, N_2 be submodules of an A-module M.

(Noetherian) Suppose $M/N_1, M/N_2$ are Noetherian, and let

$$M_1 \leq M_2 \leq M_3 \leq \ldots$$

be an ascending chain of submodules of M containing $N_1 \cap N_2$. For $j = 1, 2$, adjoin N_j and pass to the quotient to obtain an ascending chain,

$$(M_1 + N_j)/N_j \leq (M_2 + N_j)/N_j \leq \ldots,$$

of submodules of M/N_j. Since M/N_j is Noetherian, the ascending chain terminates at some index n_j. Let $n = \max(n_1, n_2)$. Then we have the two identities,

$$(M_n + N_1)/N_1 = (M_{n+1} + N_1)/N_1$$

and

$$(M_n + N_2)/N_2 = (M_{n+1} + N_2)/N_2.$$

Let $f \in M_{n+1}$. By the first identity, there is $f_1 \in N_1$ such that $f + f_1 \in M_n$. By the second identity, there is $f_2 \in N_2$ such that $f + f_1 = f + f_2$. Therefore, $f_1 = f_2 \in N_1 \cap N_2$, and $f \in M_n + N_1 \cap N_2$. Since f was arbitrary, $M_{n+1} + N_1 \cap N_2 = M_n + N_1 \cap N_2$. Therefore, the chain,

$$M_1/(N_1 \cap N_2) \leq M_2/(N_1 \cap N_2) \leq M_3/(N_1 \cap N_2) \leq \ldots,$$

of submodules of $M/(N_1 \cap N_2)$ terminates. So $M/(N_1 \cap N_2)$ is Noetherian.

(Artinian) If $M/N_1, M/N_2$ are Artinian, the same proof works with the reverse ordering.

6.4

Let M be a Noetherian A-module, let $\mathfrak{a} = \operatorname{Ann}(M)$, and suppose that $\mathfrak{a}_1 \leq \mathfrak{a}_2 \leq \dots$ is an ascending chain of ideals of A containing $\mathfrak{a}$. Since M is Noetherian, it is finitely generated by some $x_1, \dots, x_k$. Again, since M is Noetherian, for each i the ascending chain of submodules, $\mathfrak{a}_1 x_i \leq \mathfrak{a}_2 x_i \leq \dots$, stabilizes for some index n_i. Let $n = \max(n_1, \dots, n_k)$. Then $\mathfrak{a}_n x_i = \mathfrak{a}_{n+1} x_i$ for each i. Hence, $\mathfrak{a}_n M = \mathfrak{a}_{n+1} M$. Let $a \in \mathfrak{a}_{n+1}$. Since $\mathfrak{a}_n M = \mathfrak{a}_{n+1} M$, the multiplication by a map $M/\mathfrak{a}_n M \to M/\mathfrak{a}_n M$ is the zero map. Since $\mathfrak{a} \subseteq \mathfrak{a}_n$, the annihilator of $M/\mathfrak{a}_n M$ is $\mathfrak{a}_n$. Therefore, $a \in \mathfrak{a}_n$. Since a was arbitrary, $\mathfrak{a}_{n+1} \subseteq \mathfrak{a}_n$, and the chain stabilizes. Therefore, $A/\mathfrak{a}$ is a Noetherian ring.

Let $M = \mathbb{Z}_2/\mathbb{Z}$ as a $\mathbb{Z}$-module (that is M consists of fractions whose denominators are powers of 2, modulo 1). Let N be a submodule of M. If $\frac{n}{2^k} \in N$ is in lowest terms, then there are $a, b \in \mathbb{Z}$ such that

$$a \cdot \frac{n}{2^k} = \frac{an + b2^k}{2^k} = \frac{1}{2^k} \in N.$$

Therefore, $(2^{-k}\mathbb{Z})/\mathbb{Z} \subseteq N$. If N contains elements of arbitrarily large denominator, then $N = M$. Otherwise, $N = (2^{-k}\mathbb{Z})/\mathbb{Z}$ for some $k \geq 0$. Now, having classified all of the $\mathbb{Z}$-submodules of M, we can see that M is Artinian. If $n \in \mathbb{Z}$ annihilates M, then $2^k | n$ for all $k \geq 0$. Thus $\operatorname{Ann}(M) = (0)$. However, $\mathbb{Z}/(0) \cong \mathbb{Z}$ is not Artinian because $(2) \supsetneq (4) \supsetneq (8) \supsetneq \dots$ is a non-terminating descending chain of ideals.

6.5

Let X be a Noetherian topological space, let Y be a subspace, and let $U_1 \subseteq U_2 \subseteq \dots$ be an ascending chain of open subsets of Y. By the definition of the subspace topology, there is some open subset V_1 of X such that $V_1 \cap Y = U_1$. Given open subsets $V_1 \subseteq V_2 \subseteq \dots \subseteq V_{n-1}$ of X such that $V_i \cap Y = U_i$ for each i, let V_n' be an open subset of X such that $V_n' \cap Y = U_n$, and define $V_n = V_{n-1} \cup V_n'$. Then we have $V_{n-1} \subseteq V_n$ and

$$V_n \cap Y = (V_{n-1} \cap Y) \cup (V_n' \cap Y) = U_{n-1} \cup U_n = U_n,$$

so the inductive selection of the family of V's may continue. Since X is a Noetherian topological space, there is an index, N, at which the chain of V's stabilizes. Therefore,

$$U_N = V_N \cap Y = V_n \cap Y = U_n,$$

for all $n \geq N$. So Y is a Noetherian topological space.

Let $\{W_i\}_{i \in I}$ be an open cover of X. Without loss of generality, I is an ordinal. For each $i \in \omega \cap I$, define $Y_i = \bigcup_{j<i} W_j$. Then $Y_1 \subseteq Y_2 \subseteq \dots$ is an ascending chain of open subsets of X. Since X is Noetherian, this chain stabilizes at some index $n \in \mathbb{N}$. Since $\{W_i\}_{i \in I}$ is an open cover, $X \subseteq Y_n = \bigcup_{j<n} W_j$. Therefore, X is quasi-compact.

6.6

Let X be a topological space.

(i)$\Rightarrow$(iii) If X is Noetherian, and Y is a subspace of X, then Y is Noetherian by Exercise 6.5. Furthermore, Exercise 6.5 implies that Y is quasi-compact.

(iii)$\Rightarrow$(ii) This is obvious.

(ii)$\Rightarrow$(i) Suppose that every open subspace of X is quasi-compact, let $U_1 \subseteq U_2 \subseteq \dots$ be an ascending chain of open subsets of X, and define $U = \bigcup_{i \geq 1} U_i$. Since U is a union of open subsets it is open, hence quasi-compact, and $\{U_i\}_{i \geq 1}$ is an open cover of U. Therefore, there is some n such that $U_1, \dots, U_n$ cover U, and the chain stabilizes.

6.7

6.8

Let A be a Noetherian ring, let $X = \operatorname{Spec} A$, and let $V_1 \supseteq V_2 \supseteq \dots$ be a descending chain of closed subsets of X. By the definition of the Zariski topology, there are radical ideals $\mathfrak{a}_i$ of A such that $V_i = V(\mathfrak{a}_i)$ for each i and $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ is an ascending chain of ideals. Since A is Noetherian, this chain terminates at some index, n. Therefore, the descending chain of closed subspaces of X terminates at n as well. Therefore, X is Noetherian.

By our method of proof one can see that if X is Noetherian, then A satisfies the ACC on the set of radical ideals.

6.9 Undone

6.10 Undone

6.11 Undone

6.12 Undone

7 Noetherian Rings

7.1

8 Artin Rings

9 Discrete Valuation Rings and Dedekind Domains

10 Completions

11 Dimension Theory

12 Conventions

1. Don't use "$\in$".
2. Don't use "\$".
3. Direct proofs when possible.
4. Give constructive and non-constructive proof when possible.
5. Colon for definitions.
6. Reader should not need to look anything up.

13 Remarks

Remark 1. I would like to prove an analogue of Exercise 1.2 (iii) by the following induction: If f is a zero-divisor, then by Exercise 1.2 (iii), there is an element a of A_{r-1} such that $af = 0$. Then claim that the lead coefficient of a as a polynomial in x_{r-1} kills f as well. However, the argument in Exercise 1.2 (iii) made use of the hypothesis that g was of minimum degree, and we have no obvious such control on the degree of the lead coefficient of a with respect to a single variable. If you find an example showing that this method of proof will not work, or if you find a way of making it work, please send me an email.

Remark 2. The following proof is for the case that A is reduced. I would like to extend this approach to the non-reduced case, but I am skeptical that this is possible. In particular, a regular function on a reduced scheme is uniquely determined by its values. This is no longer true for non-reduced schemes. I suspect that this feature is at the heart of the matter.

Remark 3. The following proof uses machinery from Chapter 3. The merit of the proof is that it has a geometric interpretation: $\mathbb{Z}/m\mathbb{Z}$ corresponds to a module supported on the finite closed subset $V(m)$ of $\operatorname{Spec}\mathbb{Z}$. If m, n are coprime, then $V(m) \cap V(n)$ is empty. Therefore the tensor product module has empty support.

Remark 4. There are two other proofs of this fact that are somewhat straightforward, albeit tedious. You can prove that $M \otimes_A N$ satisfies the universal property of the direct sum or you can prove that $\bigoplus_{i\in I} (M_i \otimes_A N)$ satisfies the universal property of the tensor product. Since universal objects are unique up to unique isomorphism, in either case you have the desired result.

Remark 5. More generally, suppose M is an A-module with endomorphism ϕ such that $\phi^2 = \phi$. Then

$$M = (\ker \phi) \oplus (\operatorname{im} \phi)\,.$$

This can be seen by applying ϕ and $\mathrm{id}_M - \phi$ to the decomposition $m = (m - \phi(m)) + \phi(m)$. Such an endomorphism is called a *linear projection.*

Suppose that $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a short exact sequence of A-modules. If there is $\sigma : N \to M$ such that $g \circ \sigma = \mathrm{id}_N$, then $\sigma \circ g$ is a projection on M. If there is $\tau : M \to K$ such that $\tau \circ f = \mathrm{id}_K$, then $f \circ \tau$ is a projection on M. In either case, we obtain an identification of M with $K \oplus N$.

Remark 6. Here we give an alternative construction of the direct limit of the directed system $(M_i, \mu_{ij} : M_i \to M_j)$. Let $X = \coprod_{i\in I} M_i$ be the disjoint union of the sets $(M_i)_{i\in I}$. Define the relation $\sim$ on X by $(x, i) \sim (y, j)$ if there is $k \geq i, j$ such that $\mu_{ik}(x) = \mu_{jk}(y)$.

We claim that $\sim$ is an equivalence relation on X. Reflexivity and symmetry follow immediately from those properties of equality. Suppose that $(x, i) \sim (y, j)$ and $(y, j) \sim (z, k)$. Then there are $m, n \in I$ such that $m \geq i, j$; $n \geq j, k$; $\mu_{im}(x) = \mu_{jm}(y)$; and $\mu_{jn}(y) = \mu_{kn}(z)$. Let $p \in I$ such that $p \geq m, n$. Then we have

$$\mu_{ip}(x) = \mu_{mp}(\mu_{im}(x)) = \mu_{mp}(\mu_{jm}(y)) = \mu_{jp}(y) = \mu_{np}(\mu_{jn}(y)) = \mu_{np}(\mu_{kn}(z)) = \mu_{kp}(z).$$

That is, $(x, i) \sim (z, k)$.

Let $M = X/\sim$. We will put the structure of an A-module on M. Let $[(x, i)], [(y, j)] \in M$. Define

$$[(x, i)] + [(y, j)] := [(\mu_{ik}(x) + \mu_{jk}(y), k)],$$

where $k \in I$ is such that $k \geq i, j$. Let us show that addition is well-defined. Suppose that $l \geq i, j$. Then there is $m \geq k, l$ such that

$$\mu_{km}(\mu_{ik}(x) + \mu_{jk}(y)) = \mu_{im}(x) + \mu_{jm}(y) = \mu_{lm}(\mu_{il}(x) + \mu_{jl}(y)).$$

So the equivalence class $[(\mu_{ik}(x) + \mu_{jk}(y), k)]$ does not depend on the choice of k.

Suppose $(u, p) \sim (x, i)$ and $(v, q) \sim (y, j)$. Pick $k \geq i, j, p, q$ such that $\mu_{pk}(u) = \mu_{ik}(x)$ and $\mu_{qk}(v) = \mu_{jk}(y)$. Then

$$\mu_{ik}(x) + \mu_{jk}(y) = \mu_{pk}(u) + \mu_{qk}(v).$$

Therefore, addition does not depend on the choice of class representatives.

Let $a \in A$ and define

$$a \cdot [(x,i)] := [(ax,i)].$$

If $(x,i) \sim (y,j)$, let $k \geq i,j$. Then

$$\mu_{ik}(ax) = a\mu_{ik}(x) = a\mu_{jk}(y) = \mu_{jk}(ay).$$

So scalar multiplication is well-defined.

The axioms for M to be an A-module follow from the maps μ_{ij} being A-linear and from each M_i being an A-module. The can be tediously verified.

Each map $\mu_i : M_i \to M$ is defined as the inclusion $M_i \to X$ followed by the quotient $X \to M$. It is easy to check that these maps are A-linear. Let $i \leq j$ and let $x \in M_i$. Then $\mu_j(\mu_{ij}(x)) = [(\mu_{ij}(x), j)]$ and $\mu_i(x) = [(x,i)]$. Take $k = j$ in the definition of $\sim$ to see that $[(x,i)] = [(\mu_{ij}(x), j)]$.

Remark 7. I have a suspicion that there is a problem with the given proof.

Remark 8. A useful consequence of Exercise 2.16 is that the construction of the direct limit given in the text of Exercise 2.14 and that in Remark 6 are canonically isomorphic.

Remark 9. Let A be a ring and let $(M_i)_{i\in I}, (\mu_{ij})_{i,j\in I}$ be a directed system of A-modules. Let $i_0 \in I$ and let J_{i_0} be the set of indices $j \in I$ such that $j \geq i_0$. Then $\varinjlim_{j\in J_{i_0}} M_j$ is canonically isomorphic to $\varinjlim_{i\in I} M_i$ as an A-module. Below is a proof.

First, we need to define the structure maps $\mu'_i : M_i \to \varinjlim_{j\in J_{i_0}} M_i$ to see $\varinjlim_{j\in J_{i_0}} M_i$ as a direct limit over I. Let $(\mu_i)_{i\in J_{i_0}}$ be the structure maps of $\varinjlim_{j\in J_{i_0}} M_i$ from Exercise 2.14. If $i \in I$, then there is $j \in J_{i_0}$ such that $i \leq j$. Define $\mu'_i := \mu_j \circ \mu_{ij}$. The identity stated in the text at the end of Exercise 2.14 shows that μ'_i is well-defined.

We will use the characterization of $\varinjlim M_i$ given in Exercise 2.16. Let N be an A-module and let $(\alpha_i : M_i \to N)_{i\in I}$ be a family of A-linear maps such that $\alpha_i = \alpha_j \circ \mu_{ij}$ for all $i \leq j$ in I. Let $\mu'_i(x) \in \varinjlim_{j\in J_{i_0}} M_i$. Define $\alpha(\mu'_i(x)) := \alpha_i(x)$.

Let us show that α is well-defined. Let $\mu'_i(x) = \mu'_j(y)$ and let $k \geq i,j$. Then

$$\mu'_k(\mu_{ik}(x)) = \mu'_i(x) = \mu'_j(y) = \mu'_k(\mu_{jk}(y)).$$

So $\mu'_k(\mu_{ik}(x) - \mu_{jk}(y)) = 0$. By Exercise 2.15 there is $l \geq k$ such that $\mu_{il}(x) = \mu_{jl}(y)$. Therefore,

$$\alpha_i(x) = \alpha_l(\mu_{il}(x)) = \alpha_l(\mu_{jl}(y)) = \alpha_j(y),$$

and α is well-defined. On the other hand, the definition of α was forced by the condition $\alpha \circ \mu'_i = \alpha_i$. So α is unique.

Remark 10. Here is a characterization of the Tor functor due to J. P. May (we have adapted it to commutative rings):

For every $n \in \mathbb{Z}$ there is a functor

$$\mathrm{Tor}_n^A : \underline{A - \mathbf{Mod}} \times \underline{A - \mathbf{Mod}} \to \underline{A - \mathbf{Mod}}$$

such that for every A-module M and short exact sequence of A-modules

$$0 \to N' \to N \to N'' \to 0$$

There are connecting homomorphisms

$$\partial : \mathrm{Tor}_n^A(N'', M) \to \mathrm{Tor}_{n-1}^A(N', M) \text{ and } \partial : \mathrm{Tor}_n^A(M, N'') \to \mathrm{Tor}_{n-1}^A(M, N'),$$

natural in M and in the choice of short exact sequence that satisfy the following properties:

1. $\mathrm{Tor}_n^A(M,N) = 0$ for all $n < 0$.

2. $\mathrm{Tor}_0^A(M,N)$ is naturally isomorphic to $M \otimes_A N$.
3. $\mathrm{Tor}_n^A(M,N) = 0$ for $n > 0$ if either M or N is projective.
4. The following sequences are exact:

$$\cdots \to \mathrm{Tor}_{n+1}^A(N'',M) \to \mathrm{Tor}_n^A(N',M) \to \mathrm{Tor}_n^A(N,M) \to \mathrm{Tor}_n^A(N'',M) \to \mathrm{Tor}_{n-1}^A(N',M) \to \cdots$$
$$\cdots \to \mathrm{Tor}_{n+1}^A(M,N'') \to \mathrm{Tor}_n^A(M,N') \to \mathrm{Tor}_n^A(M,N) \to \mathrm{Tor}_n^A(M,N'') \to \mathrm{Tor}_{n-1}^A(M,N') \to \cdots$$

Furthermore, for each fixed M, the functors $\mathrm{Tor}_n^A(M,\cdot)$ and $\mathrm{Tor}_n^A(\cdot,M)$ with their associated connecting homomorphisms are uniquely determined up to isomorphism by properties 1-4.

Here are some further properties from Weibel's *An Introduction to Homological Algebra*:

- (Weibel 3.2.8) Let M,N be A-modules and let $F_* \to M$ be a flat resolution of M. Then

  $$\mathrm{Tor}_*^A(M,N) \cong H_*(F_* \otimes_A N) \text{ and } \mathrm{Tor}_*^A(N,M) \cong H_*(N \otimes_A F_*).$$

  Note that free $\Rightarrow$ projective $\Rightarrow$ flat.

-

Remark 11. Let $\mathfrak{a}$ be an ideal of A and let M be an A-module. Tensor the short exact sequence

$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$$

with M to get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
\mathrm{Tor}_1^A(A,M) & \longrightarrow & \mathrm{Tor}_1^A(A/\mathfrak{a},M) & \longrightarrow & \mathfrak{a}\otimes_A M & \longrightarrow & A\otimes_A M & \longrightarrow & (A/\mathfrak{a})\otimes_A M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathfrak{a}M & \longrightarrow & M & \longrightarrow & M/\mathfrak{a}M & \longrightarrow & 0
\end{array}$$

Taking the kernel complex yields the exact sequence

$$0 = \mathrm{Tor}_1^A(A,M) \to \mathrm{Tor}_1^A(A/\mathfrak{a},M) \to \ker(\mathfrak{a}\otimes_A M \to \mathfrak{a}M) \to 0.$$

As a consequence, we have the following chain of implications:

"M is flat"

$\Rightarrow$ "$\mathfrak{a}\otimes_A M = \mathfrak{a}M$ for all ideals $\mathfrak{a}$ of A"

$\Rightarrow$ "$\mathfrak{a}\otimes_A M = \mathfrak{a}M$ for all finitely generated ideals of A"

$\Rightarrow$ "M is flat."

Remark 12. A consequence of Exercise 2.27: Every finitely generated ideal of an absolutely flat ring is projective.

Remark 13. Here I will develop an alternative construction of the localization of a ring A with respect to a multiplicative subsemigroup S.

Construction of A_s: Let $s \in A$. Define $A_s := A[x]/(sx-1)$ and let $f : A \to A_s$ be the composition of the canoncial maps $A \to A[x]$ and $A[x] \to A_s$. In the ring A_s, we will denote the class of x as s^{-1}.

Representation of Elements: Every element of A_s has a representative of the form as^{-n} for some $n \geq 0$ and $a \in A/$

Given any polynomial $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ in $A[x]$ we have the following congruence mod $(sx-1)$:

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 \equiv a_n x^n + a_{n-1}sx^n + \ldots a_0 s^n x^n = (a_n + a_{n-1}s + \ldots a_0 s^n)x^n.$$

Universal Property of A_s: If B is an A-algebra, then $\#\hom_{A-alg}(A_s, B) \leq 1$, with equality if and only if s is invertible in B.

Let $g \in \hom_{A-alg}(A_s, B)$. Then

$$1 = g(1) = g(ss^{-1}) = sg(s^{-1})$$

implies that s is invertible in B. Furthermore, let $f : A \to A_s$, $h : A \to B$ be the structure maps. Then

$$g(f(a)f(s)^{-1}) = h(a)h(s)^{-1}$$

shows that g is uniquely determined by the structure map of B. Therefore, $\#\hom_{A-alg}(A_s, B) \leq 1$.

Conversely, suppose B is an A algebra such that s is invertible in B. By the Universal Property of Polynomial Rings, there is a unique A-algebra homomorphism $A[x] \to B$ sending x to s^{-1}. Clearly, $sx - 1$ is in the kernel. So this A-algebra homomorphism lifts uniquely to an A-algebra homomorphism $A_s \to B$ such that $s^{-1} \mapsto s^{-1}$.

Characterization of $\ker(A \to A_s)$: $a \in \ker(A \to A_s)$ if and only if $s \in \sqrt{\mathrm{Ann}_A(a)}$.

Suppose $a \in A$ maps to 0 in A_s. Then there is

$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \ldots b_0 \in A[x]$$

such that

$$a = (sx - 1)q(x) \text{ in } A[x]. \tag{4}$$

Looking at the degree $m + 1$ term on the right hand side of (4) shows that $sb_m = 0$. Therefore,

$$sa = (sx - 1)(sb_{m-1}x^{m-1} + \ldots + sb_0).$$

Repeating this same reasoning a total of m times yields

$$s^m a = (sx - 1)s^m b_0 = s^{m+1}b_0 x - s^m b_0.$$

By comparing degrees, we get

$$s^m a = -s^m b_0 \text{ and } s^{m+1}b_0 = 0.$$

Therefore,

$$s^{m+1}a = -s^{m+1}b_0 = 0.$$

Conversely, suppose that $a \in A$ and $s^m a = 0$ for some $m \geq 1$. Then

$$a \equiv s^m a x^m \mod (sx - 1) = 0.$$

Construction of $S^{-1}A$: We will construct $S^{-1}A$ as a direct limit of A-algebras.

Let $S \subseteq A$ be a multiplicatively closed set containing 1. Let $\mathcal{S}'$ be the category with objects $\{A_s\}_{s \in S}$ and A-algebra homomorphisms as morphisms. By the "Universal Property 1" each hom-set has cardinality at most 1. Let $\mathcal{S}$ be a subcategory of $\mathcal{S}'$ obtained by selecting a single representative of each isomorphism class of $\mathcal{S}'$. Then $\mathcal{S}$ is a partial order. If $s \in S$, then A_s is uniquely isomorphic to a unique object $A_{s'}$ of $\mathcal{S}$ and we may identify these A-algebras without further comment. By abuse of notation, we will refer to an object A_s of $\mathcal{S}$ as s. In particular, $s_1 \leq s_2$ if and only if there is an A-algebra homomorphism $\sigma_{s_1 s_2} : A_{s_1} \to A_{s_2}$.

Suppose that $s_1, s_2 \in \mathcal{S}$. In $A_{(s_1 s_2)}$ we have the identities $s_1 \cdot \big(s_2(s_1 s_2)^{-1}\big) = 1$ and $s_2 \cdot \big(s_1(s_2 s_1)^{-1}\big) = 1$. Therefore, $s_1, s_2 \leq s_1 s_2$. So $\mathcal{S}$ is a directed set. Properties (1) and (2) of Exercise 2.14 follow immediately from the cardinality of hom-sets. Therefore, $\{A_s\}_{s \in \mathcal{S}}, \{\sigma_{s_1 s_2} : A_{s_1} \to A_{s_2}\}_{s_1 \leq s_2}$ is a directed system of A-algebras. We define

$$S^{-1}A := \varinjlim_{s \in \mathcal{S}} A_s.$$

If we picked a different subcategory $\mathcal{S}$ of $\mathcal{S}'$, then our previous remarks would show that we obtain an isomorphic partially ordered set and all objects are uniquely isomorphic as A-algebras. Therefore, $S^{-1}A$ has been constructed up to unique isomorphism.

Universal Property of $S^{-1}A$: (Proposition 3.1) If B is an A-algebra, then $\# \hom_{A-alg}(S^{-1}A, B) \leq 1$, with equality if and only if every element of S is invertible in B.

By the Universal Property of A_s and the identification,

$$\hom_{A-alg}(S^{-1}A, B) = \varprojlim_{s \in \mathcal{S}} \hom_{A-alg}(A_s, B),$$

we can see that the set $\hom_{A-alg}(S^{-1}A, B)$ has at most one element and is non-empty precisely when every element of S is invertible in B.

A Third Construction: Let A be a ring with multiplicatively closed subset S. Let $B := A[\{x_s\}_{s \in S}]$ be the polynomial ring over A with an indeterminate for each element of S. Let I be the ideal of B generated by the family $\{sx_s - 1\}_{s \in S}$, and define $S^{-1}A := B/I$. It is straightforward to show that the ring B/I satisfies the universal property of $S^{-1}A$ by applying the Universal Property of Polynomial Rings and using the First Isomorphism Theorem.