Let Q_8 be the quaternion group. For our purposes, let it be the subgroup of $\mathrm{GL}_2(\mathbb{C})$ generated by the matrices

$$\widehat{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \widehat{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } \widehat{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where i is the imaginary unit. Furthermore, let $S = \{x, y, z\}$ be a set on three letters. The set map $S \to Q_8$ defined by $x \mapsto \widehat{i}, y \mapsto \widehat{j}, z \mapsto \widehat{k}$ yields a group homomorphism $f : F(S) \to Q_8$ by the Universal Property of Free Groups, where $F(S)$ is the free group on the set S. Furthermore, f is the unique group homomorphism from $F(S)$ to Q_8 which maps x, y, z to $\widehat{i}, \widehat{j}, \widehat{k}$ respectively.
Let $K = \ker(f)$. Since $\widehat{i}^2 = \widehat{j}^2 = \widehat{k}^2 = \widehat{i}\widehat{j}\widehat{k}$ and $\widehat{i}^4 = 1$, we know that $x^4, x^2y^{-2}, y^2z^{-2}, z^2x^{-2}$, and $x^2(xyz)^{-1}$ are in K. Let N be the smallest normal subgroup of $F(S)$ containing these five elements. Since K is normal in $F(S)$, we know that $N \leq K$. Let w be a non-empty word in $F(S)$. We will reduce in to a standard form modulo N. As follows:

- By hypothesis, $w = \prod_{a=1}^{b} c_a^{n_a}$, where $c_a \in \{x, y, z\}$ and $n_a \in \mathbb{Z}$ for every a.
- Since $x^2y^{-2}, y^2z^{-2}, z^2x^{-2}, x^2(xyz)^{-1} \in N$, we know that $x^2 \equiv y^2 \equiv z^2 \equiv xyz \mod N$. Let s be a coset representative for x^2N, so that $s \equiv x^2 \mod N$. Note that s commutes with x, y, z modulo N. Furthermore, $f(x^2) = f(x)^2 = \widehat{i}^2 \neq 1$. So $s \notin N$. However, $s^2 \in N$ because $s^2 \equiv x^4$ modulo N.
- (1) By Euclidean division, $n_a = 2q_a + r_a$ where $r_a \in \{0, 1\}$ for each a. Therefore,
$$w = \prod_{a=1}^{b} (c_a^2)^{q_a} c_a^{r_a} \equiv \prod_{a=1}^{b} s^{q_a} c_a^{r_a} \equiv s^{\sum_{a=1}^{b} q_a} \prod_{a=1}^{b} c_a^{r_a} \text{ modulo } N.$$
- Since $x^2 \equiv y^2 \equiv z^2 \equiv xyz \mod N$, the existence of inverses implies that $x \equiv yz \mod N$. Multiply on the right by x to get $s \equiv yzx \mod N$, then multiply on the left by y^{-1} to get $y \equiv zx \mod N$. Similarly we obtain $z \equiv xy \mod N$. On the other hand, take inverses to obtain $x^2 \equiv z^3y^3x^3 \mod N$. Thus $s \equiv s^3zyx$ mod N. Multiply by sz^3 to obtain $z^3 \equiv yx \mod N$. Left multiply by y^3 and right multiply by z to obtain $y^3 \equiv xz \mod N$. Similarly we may obtain $x^3 \equiv zy \mod N$.
- (2) The six relations we have derived allow us to collapse any adjacent pair of letters $c_a c_{a'}$ to a power of a single letter.
- Applying (1) lets us further reduce w to some power of s times a word where each letter has exponent 1. Applying (2) to such a word reduces the index b required. By alternating between process (1) and (2) we obtain a strictly decreasing sequence of positive integers b. By the well ordering principle, this process must terminate. Since the process only requires that $b \geq 2$, it terminates in a word of the form $s^p c^n$ where $p, n \in \{0, 1\}$ and $c \in \{x, y, z\}$.
- If $n = 1$, then there are six such words (two options for p, three options for c). If $n = 0$, then there are two such words. (We assumed w was non-empty, not necessarily non-trivial.) Therefore, every word of $F(S)$ is equivalent modulo N to one of these eight words (where the empty word is obviously the same as the word obtained from $p = n = 0$). We have not yet shown that they are inequivalent, but we do know that $F(S)/N$ is a finite group of order at most 8.
- Since $N \leq K$, the universal property of the quotient yields a unique map $\overline{f} : F(S)/N \to Q_8$ such that $\overline{f} \circ \pi = f$, where $\pi : F(S) \to F(S)/N$ is the natural quotient map. Since f is surjective, $\overline{f}$ is as well.
- Since Q_8 is the image of $\overline{f}$ (by our choice of model of Q_8 in the first paragraph and our construction of $\overline{f}$), and $F(S)/N$ is a finite group, we know that the order of $F(S)/N$ is at least 8 (just think about a surjective function between finite sets to see this). Therefore, $F(S)/N$ is actually of order 8. This forces our representatives from before to be distinct elements of $F(S)/N$. Furthermore, $\overline{f}$ is injective. So $N = K$.
- In summary, we have proved that $\langle x, y, z : x^4 = 1, x^2 = y^2 = z^2 = xyz \rangle$ is a presentation of Q_8.