Towards a smooth compactification of the space of curves in $\mathbb{P}^n$ (based on joint works w/ Yi Hu & Jun Li)

Friday, November 20, 2020   2:00 PM

Object: $\overline{\mathcal{M}}_g(\mathbb{P}^n, d) = \{ u : C \longrightarrow \mathbb{P}^n \mid C \text{ nodal},\ g(C) = g,\ \deg u = d,\ |\mathrm{Aut}(u)| < \infty \}$

Pros:
- a compact space containing
  $\mathcal{M}_g(\mathbb{P}^n, d) = \{ u : C \longrightarrow \mathbb{P}^n \mid C \text{ smooth}, \cdots \}$
- carries a virtual fundamental class

  $\rightsquigarrow$ Gromov-Witten invariants

Cons:
- boundary is to large: $\mathcal{M}_g(\mathbb{P}^n, d)$ is usually not dense in $\overline{\mathcal{M}}_g(\mathbb{P}^n, d)$.

  $\hookrightarrow$ how to describe the closure (i.e. the main component)?
- can be arbitrarily singular (when g and d vary).

  $\hookrightarrow$ how to resolve the singularities?

$\boxed{\text{Q1}}$: Can we construct $\widetilde{\mathcal{M}}_g(\mathbb{P}^n, d) \longrightarrow \overline{\mathcal{M}}_g(\mathbb{P}^n, d)$ s.t.

1). $\widetilde{\mathcal{M}}_g(\mathbb{P}^n, d)$ has smooth irreducible components and at worst normal crossing singularities, (i.e. $+$ ✓ $\prec$ ⊗)

2). proper,

3). birationally dominating the main component?

Remark. An affirmative answer could
- lead to direct approach to GW of CY 3-folds $\rightsquigarrow$ mirror symmetry
- shed some light on the resolution of other singular spaces.

Answer:

$g = 1$: Yes. (Vakil, Zinger '08; Hu, Li '10; Ranganathan, Santos-Parker, Wise '17; Hu, - '19)

$g = 2$: Yes. (Hu, Li, - '18; Hu, - '20; Battistella, Carocci '20)

$g \geq 3$: ?

- <u>Where are the singularities?</u>

- $u: C \longrightarrow \mathbb{P}^n$ can be described by $(C; s_0, \cdots, s_n)$ w/.

$$s_i = u^* x_i \in H^0(C, u^* \mathcal{O}_{\mathbb{P}^n}(1)).$$

(C, u) is singular whenever $h^0(C, u^* \mathcal{O}_{\mathbb{P}^n}(1))$ jumps.

- Let

$$\begin{array}{ccc} \mathcal{C} \xrightarrow{f} \mathbb{P}^n & \quad & (C, u, x) \xrightarrow{f} u(x) \\ \downarrow \pi & & \downarrow \pi \\ \overline{m}_g(\mathbb{P}^n, d) & & (C, u) \end{array}$$

be the universal family.

Then (C, u) is singular $\Leftrightarrow$ the sheaf $R^0 \pi_* f^* \mathcal{O}_{\mathbb{P}}(1)$ is not locally free.

- The local structure of $\overline{m}_g(\mathbb{P}^n, d)$ is encoded in the derived object

$$R^\bullet \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(1).$$

- Fix $(C, u) \in \overline{m}_g(\mathbb{P}^n, d)$ and a small open neighborhood $\mathcal{U}$ of (C, u).

  To study $\mathcal{U}$, we hope to
  - embed $\mathcal{U}$ in something smooth ($\mathcal{E}_V$ later on) and
  - write the local equation for $\mathcal{U}$ in $\mathcal{E}_V$

- How to construst such $\mathcal{E}_V$?

  Let $\mathcal{D}_g = \{ (C, D) \mid C \text{ nodal}, g(C) = g, D \text{ effective}, \text{stability} \}$

  $\mathcal{D}_g$ is a smooth algebraic stack.

  $\mathcal{U} \longrightarrow \mathcal{D}_g$, $\quad (C, u = [u_0 : \cdots : u_n]) \longrightarrow (C, \underbrace{u_0^{-1}(0)}_{\text{may assume simple}})$

  $(\subset \overline{m}_g(\mathbb{P}^n, d))$

  Let $V \to \mathcal{D}_g$ be a smooth chart containing the image of $\mathcal{U}$.

$$\begin{array}{ccc} \mathcal{C} = \{ (C, D, x) \mid (C, D) \in V, x \in C \} & \quad & \mathcal{D} = \{ (C, D, D) \mid (C, D) \in V \} \\ \rho \downarrow \quad \text{universal curve} & & \text{the universal divisor on } \mathcal{C}. \\ V & & \end{array}$$

We can choose g sections $A_1, \cdots, A_g$ of $\rho: \mathcal{C} \to \mathcal{V}$ s.t.

$$R^1 \rho_* \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \underbrace{\mathcal{A}}_{:= A_1 + \cdots + A_g}) = 0, \quad R^0 \rho_* \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{A}) \text{ locally free}$$

Let $\mathcal{E}_{\mathcal{V}}$: the total space of $\underbrace{\rho_* \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{A})^{\oplus \overbrace{n}^{\text{dim of } \mathbb{P}^n}}}_{\text{trivial vector bundle over } \mathcal{V}}$

$$\begin{array}{c} \mathcal{E}_{\mathcal{V}} \\ \downarrow \mathrm{pr} \\ \mathcal{V} \end{array}$$

$$F: \underbrace{\rho_* \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{A})^{\oplus n} \longrightarrow \rho_* \mathcal{O}_{\mathcal{C}}\left((\mathcal{D} + \mathcal{A})^{\oplus n} |_{\mathcal{A}}\right)}_{\text{trivial vector bundles over } \mathcal{V}} : \text{ tautological restriction}$$

Theorem : (Hu, Li '10):

There is a canonical open immersion $\mathcal{U} \to \ker F$.

- From now on, we will put $\mathcal{U} \subset \overline{\mathcal{M}}_g(\mathbb{P}^n, d)$ aside and focus on $\ker F$:

$$\boxed{M_\varphi \cdot \vec{w} = 0} \leftarrow \text{local equation for } \overline{\mathcal{M}}_g(\mathbb{P}^n, d)$$

where M_φ is a $g \times d$ matrix and $\vec{w}$ is a $d \times n$ matrix of free variables.

- To use the local equations and obtain a resolution

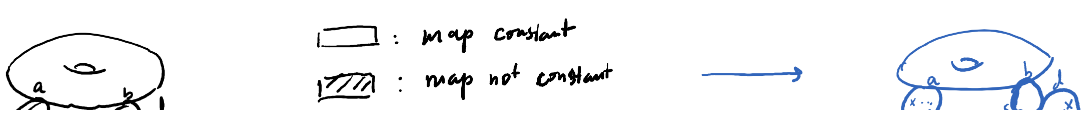
$$\widetilde{\mathcal{M}}_g(\mathbb{P}^n, d) \longrightarrow \overline{\mathcal{M}}_g(\mathbb{P}^n, d),$$

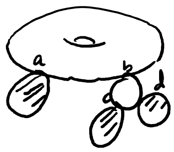
one needs

(I) to write M_φ as explicit as possible (this should suffice to get local resolution)

(II) to globalize (I) (nice for $g=1$; complicated for $g=2$).

Ex 1 $g = 1$



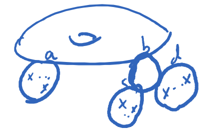


$(C,u) \in \overline{\mathcal{M}}_1(\mathbb{P}^n,d)$

▭ : map constant

▨ : map not constant

$\longrightarrow$



$(C,D) \in \mathcal{D}_1$

On the chart $\mathcal{V} \to \mathcal{D}_1$, $\exists$ local parameters $\zeta_a, \dots, \zeta_d$

$$\zeta_a = 0 \longleftrightarrow \text{node } a \text{ not smoothed.}$$

$$M_\varphi = [\, \zeta_a, \quad \zeta_b \zeta_c, \quad \zeta_b \zeta_d, \quad 0, \quad \cdots \,]$$

$$M_\varphi \cdot \vec{w} = 0 \iff \underbrace{\zeta_a w_a^j + \zeta_b \zeta_c w_c^j + \zeta_b \zeta_d w_d^j = 0}_{\text{singular.}}, \quad 1 \le j \le n. \quad \cdots ①$$

- Locally how to desingularize ① ?

blowing up along $\zeta_a = \zeta_b = 0$.

The local blowup is embedded in $\mathcal{V} \times \underbrace{\mathbb{P}^1}_{[u_a : u_b]}$, covered by two charts:

On the chart $\mathcal{V} \times \{u_a = 1\}$,

$$M_\varphi \text{ pulls back to } \zeta_a [\, 1, \; u_b \zeta_c, \; u_b \zeta_d, \; 0, \; \cdots \,]$$

$\Rightarrow$ ① pulls back to

$$\underbrace{\underbrace{\zeta_a}_{\text{smooth, bdry}} \cdot \big(\underbrace{w_a^j + u_b \zeta_c w_c^j + u_b \zeta_d w_d^j}_{\text{smooth, main}}}_{\text{normal crossing}} \big) = 0 \qquad 1 \le j \le n$$

On the chart $\mathcal{V} \times \{u_b = 1\}$,

$$M_\varphi \text{ pulls back to } \zeta_b [\, u_a, \; \zeta_c, \; \zeta_d, \; 0, \; \cdots \,]$$

$\Rightarrow$ ① pulls back to

$$\zeta_b \cdot \big(\underbrace{u_a w_a^j + \zeta_c w_c^j + \zeta_d w_d^j}_{\text{not smooth yet along } u_a = \zeta_c = \zeta_d = 0} \big) = 0 \qquad 1 \le j \le n \quad \cdots ②$$

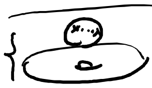
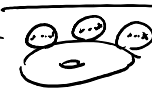
So we continue to blow up along $u_a = \zeta_c = \zeta_d = 0$.

The local blowup is embedded in $(V \times \{u_b = 1\}) \times \underbrace{\mathbb{P}^2}_{[v_a : v_c : v_d]}$, covered by

charts
$$\begin{cases} \underline{V \times \{u_b = 1\} \times \{v_a = 1\}}, & \text{on which } ② \text{ pulls back to} \\ M_\varphi = \varsigma_a u_a [1, v_c, v_d, \cdots] & \underbrace{\underbrace{\varsigma_b \cdot u_a}_{\text{smooth bdries}} \cdot (\underbrace{w_a^j + v_c w_c^j + v_d w_d^j}_{\text{smooth main}})}_{\text{normal crossing}} = 0 \\ V \times \{u_b = 1\} \times \{v_c = 1\}, & \text{similar} \\ V \times \{u_b = 1\} \times \{v_d = 1\}, & \text{similar} \end{cases}$$

✓

- These local blowups are global and have clear geometric descriptions:

blowing up $\mathcal{D}_1$ along $\overline{\{}$  $\overline{\}}$, then $\overline{\{}$  $\overline{\}}$, then $\overline{\{}$  $\overline{\}}$, ...

Theorem (Vakil, Zinger '08, Hu, Li '10, Ranganathan, Santos-Parker, Wise '17 (log geom.))
The above blowups give an affirmative answer to Q 1 for genus $= 1$.

Remark. The local equations $M_\varphi \cdot \vec{w} = 0$ also lead to a description of
$(C, u) \in \text{Closure}(M_1(\mathbb{P}^n, d)) \setminus M_1(\mathbb{P}^n, d)$
(Battistella, Carocci, Manolache '18; originally by Zinger via symplectic topology.)

- Next, consider $g = 2$.

recall: the general idea is still to
(I). describe M_φ explicitly $\rightsquigarrow$ know how to desingularize locally
(II). globalize (I).

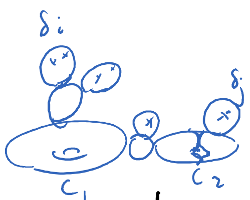
- (I) is fine.

Theorem (Hu, Li, - '18)

Theorem (Hu, Li, - 18)

$$\underbrace{M_\varphi}_{2\times d} = \begin{pmatrix} \cdots & c_{1i} \prod\limits_{\substack{\text{nodes } q_e \text{ between} \\ \delta_i \text{ and } C_1}} \zeta_e & \cdots & c_{1j} \prod\limits_{\substack{\text{nodes } q_e \text{ between} \\ \delta_j \text{ and } C_1}} \zeta_e & \cdots \\ \cdots & c_{2i} \prod\limits_{\substack{\text{nodes } q_e \text{ between} \\ \delta_i \text{ and } C_2}} \zeta_e & \cdots & c_{2j} \prod\limits_{\substack{\text{nodes } q_e \text{ between} \\ \delta_j \text{ and } C_1}} \zeta_e & \cdots \end{pmatrix}$$

Column $i \leftrightarrow \delta_i$ Column $j \leftrightarrow \delta_j$ $(D = \delta_1 + \cdots + \delta_d)$

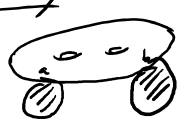


where c_{ai} are nowhere vanishing functions on $\mathcal{V}$, satisfying

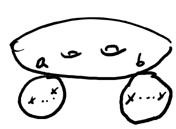
$$\det\begin{pmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j} \end{pmatrix} = \underbrace{k_{ij}}_{\substack{\text{a local function measuring} \\ \text{how far } \delta_j \text{ is from being} \\ \text{conjugate to } \delta_i}} \cdot \underbrace{\prod_{\substack{\text{Common nodes } q_e \text{ between } \delta_i, \delta_j \text{ and} \\ \text{the smallest conn. genus 2 subcurve}}} \zeta_e}_{\substack{\text{a local function measuring how far apart} \\ \delta_i \text{ and } \delta_j \text{ are}}}$$

- (II) is complicated.

Ex 2. $\underline{g = 2}$



$\longrightarrow$



$(C, u) \in \overline{\mathcal{M}}_2(\mathbb{P}^n, d)$ $(C, D = \delta_1 + \cdots + \delta_d) \in \mathcal{D}_2$

(assume a, b in general position, i.e. not Weierstrass / Conjugate)

Under suitable trivialization,

$$M_\varphi = \begin{bmatrix} \zeta_a & \zeta_a & \zeta_b & \cdots \\ 0 & \zeta_a^2 & \zeta_b & \cdots \end{bmatrix} \quad \cdots ③$$

$$= \begin{bmatrix} \zeta_b & \zeta_b & \zeta_a & \cdots \\ 0 & \zeta_b^2 & \zeta_a & \cdots \end{bmatrix} \quad \cdots ④$$

[illegible] $\zeta_a = \zeta_b = 0$ $\leftrightarrow$ globally along { [illegible] }

Locally, blow up $s_a = s_b = 0$ $\longleftrightarrow$ globally, along $\{$  $\}$.

The local blowup is embedded in $V \times \underbrace{\mathbb{P}^1}_{[u_a : u_b]}$, covered by two charts:

On the chart $V \times \{u_a = 1\}$, ③ pulls back to

$$\begin{array}{cc} s_a & \begin{bmatrix} 1 & * & * & \cdots \\ 0 & \underbrace{s_a}_{s_a = 0 \,\updownarrow\, \text{exceptional divisor}} & u_b & \cdots \end{bmatrix} \end{array}$$

The 2nd Row still does not give smooth equation,

$\Rightarrow$ Locally, need to blow up $\underline{s_a = u_b = 0}$. $\leftarrow$ the locus $[1:0]$ in the exceptional divisor

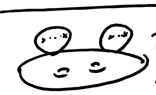
$\longleftrightarrow$ globally, along $\mathbb{P}(L_a \oplus 0) \subset \mathbb{P}(\underbrace{L_a \oplus L_b}_{\text{exceptional divisor}})$

$\downarrow$

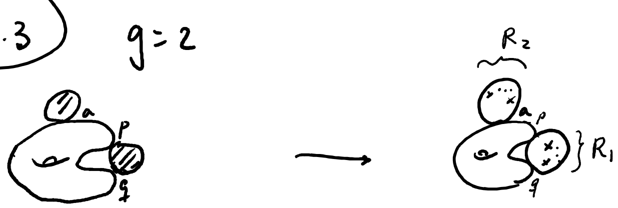
$\{$  $\}$

On the chart $V \times \{u_b = 1\}$, use ④ instead of ③.

Remark. Ex.2. suggests

- only blowing up along $\{$  $\}$, $\{$  $\}$, $\cdots$ is not enough. needs more rounds of blowups.
- The blowup centers of some rounds lie in the exceptional divisors of previous rounds.

Ex.3 $g = 2$



assume $\deg D|_{R_1} \geq 3$

$(C,u) \in \overline{M}_2(\mathbb{P}^n, d)$ $\qquad\qquad$ $(C,D) \in \mathfrak{D}_2$

under suitable trivialization

$$M_\varphi = \Bigg[\underbrace{\begin{matrix} 1 & * & * & \cdots \\ 0 & k_{12} & k_{13} & \cdots \end{matrix}}_{R_1} \; \underbrace{\begin{matrix} \varsigma_a & \cdots \\ \varsigma_a & \cdots \end{matrix}}_{R_2} \Bigg]$$

$$= \begin{bmatrix} 1 & * & * & \cdots & \varsigma_a & \cdots \\ 0 & f_2\varsigma_p + g_2\varsigma_q & f_3\varsigma_p + g_3\varsigma_q & \cdots & \varsigma_a & \cdots \end{bmatrix},$$

$\boxed{f_i, g_i, \begin{vmatrix} f_2 & g_2 \\ f_3 & g_3 \end{vmatrix} \text{ nowhere vanishing on } \mathcal{V}}$

Locally, blow up along $\varsigma_p = \varsigma_q = \varsigma_a = 0$ $\longleftrightarrow$ globally, along {  }

The blowup is embedded in $\mathcal{V} \times \underbrace{\mathbb{P}^3}_{[u_p : u_q : u_a]}$

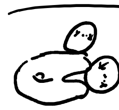
On the chart $\mathcal{V} \times \{u_a = 1\}$, Similar to Ex 1 & Ex 2.

On the chart $\mathcal{V} \times \{u_p = 1\}$, M_φ pulls back to

$$\begin{bmatrix} 1 & 0 \\ 0 & \varsigma_p \end{bmatrix} \begin{bmatrix} 1 & * & * & \cdots & * & \cdots \\ 0 & \underline{f_2 + g_2 u_q} & \underline{f_3 + g_3 u_q} & \cdots & u_a & \cdots \end{bmatrix}$$

at least one of them is nowhere vanishing @ (C,D) (hence on $\mathcal{V}$) b/c of

On the chart $\mathcal{V} \times \{u_q = 1\}$, similar.

Remark. Ex 3 suggests a round of blow ups: {  }, then {  }, ...

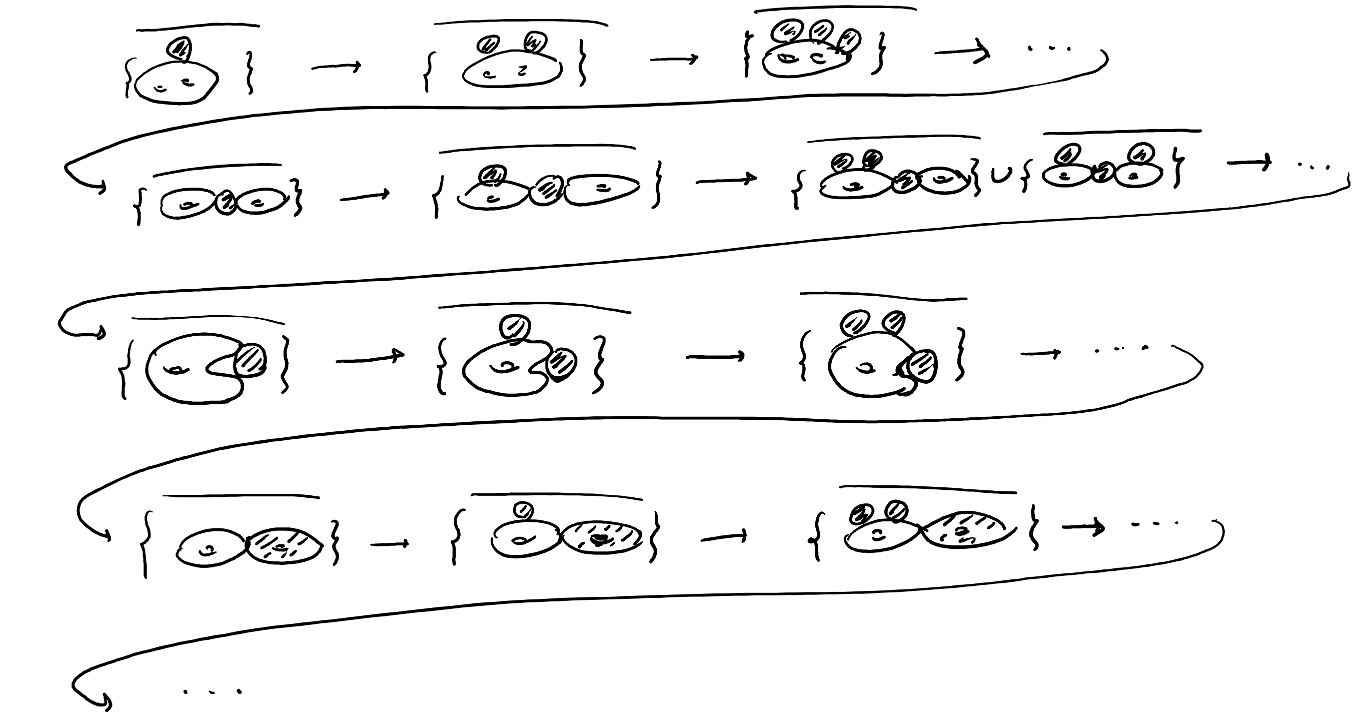
which is different from Ex. 2.

So the order of these rounds of blowups needs attention.

Theorem (Hu, Li, - '18):

For genus $= 2$, a 9-round blowup gives an affirmative answer of Q1.

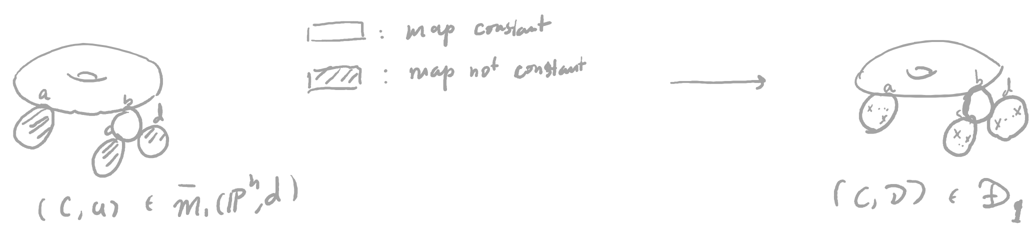
Remark. The first few rounds are along



- As the exceptional divisors are accumulating, we need a better way to go from local equations to global blowups. To this end, we introduce the theory of "stacks with twisted fields" to reorganize the $g=1,2$ cases and hopefully to treat higher genera.

- To get some idea, let's revisit Ex. 1. from another prospective:

Ex. 4. (Ex. 1 revisited):



$M_\varphi = [\ \xi_a,\quad \xi_b\xi_c,\quad \xi_b\xi_d,\quad 0,\quad \cdots\ ]$

- Notice that each $(C,D) \in \mathcal{D}_1$ corresponds to a unique rooted tree $\tau_{(C,D)}$ as follows:

$(C, \mathcal{D})$ $\longleftrightarrow$ $\tau_{(C,\mathcal{D})} = \tau_{(C',\mathcal{D}')}$ $\longleftrightarrow$ $(C', \mathcal{D}')$ $\in \mathcal{D}_{\lambda}$

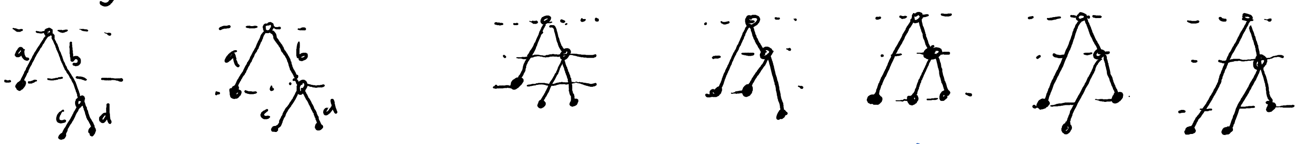
This gives a stratification : $\mathcal{D}_1 = \bigsqcup \mathcal{D}_\tau$

- $\mathcal{D}_\lambda \cap V = \{ \varsigma_a = \varsigma_b = \varsigma_c = \varsigma_d = 0 \}$ they give rise to line bundles $L_a, L_b, L_c, L_d \to \mathcal{D}_\lambda$

- In order to desingularize $\ker M_\varphi$, we need to compare how fast

$$\varsigma_a, \quad \varsigma_b \varsigma_c, \quad \varsigma_b \varsigma_d \quad \rightsquigarrow 0$$

by adding levels to λ:



For each leveled tree, we add

$$\begin{array}{ccccc} \mathbb{P}(L_a) & \mathring{\mathbb{P}}(L_a \oplus L_b) & \cdots & \cdots & \mathbb{P}(L_b) \times_{\mathcal{D}_\lambda} \mathring{\mathbb{P}}(L_a \oplus L_b \otimes L_c \oplus L_b \otimes L_d) & \cdots \\ \downarrow & \downarrow & & & \downarrow & \\ \mathcal{D}_\lambda & \mathcal{D}_\lambda & & & \mathcal{D}_\lambda & \end{array}$$

a bunch of strata.

For all rooted trees and all level structures on them, we obtain a number of strata.

Theorem. (Hu, – '19)

These strata can be glued together using smooth charts in a canonical way to form a smooth $\mathcal{D}_1^{tf}$ $(\longrightarrow \mathcal{D}_1)$, birationally and properly dominating $\mathcal{D}_1$.
This construction desingularizes $\overline{M}_1(\mathbb{P}^n, d)$.

Remark. This construction is actually isomorphic to the Vakil-Zinger / Hu-Li blowup construction.

The advantage (in the sense of resolution of singularities) is we do not need to describe the blowup centers.

- This theory does not only work for $\mathcal{D}_1$. For any smooth $\mathcal{M}$, if it has a stratification $\mathcal{M} = \coprod_{\alpha \in I} \mathcal{M}_\alpha$ by smooth subspaces (substacks) satisfying
  1). each $\mathcal{M}_\alpha$ corresponds to a unique tree τ_α,
  2). each edge corresponds to a line bundle $L_e \to \mathcal{M}_\alpha$ (locally gives a normal direction of $\mathcal{M}_\alpha$ in $\mathcal{M}$),
  3). some compatibility conditions,

  Then $\underbrace{\mathcal{M}^{tf}}_{\text{smooth}} \underbrace{\longrightarrow}_{\text{birat'l, proper}} \mathcal{M}$ can be constructed by mimicking $\mathcal{D}_1^{tf} \to \mathcal{D}_1$.

- Moreover, just as $\mathcal{D}_1^{tf}$, the new $\mathcal{M}^{tf}$ has a natural stratification. Often, there are choices of assigning rooted trees to the strata of $\mathcal{M}^{tf}$, satisfying 1) - 3) above. This suggests a recursive construction:

$$\cdots \to (\mathcal{M}^{tf})^{tf} \to \mathcal{M}^{tf} \to \mathcal{M}$$

Theorem. (Hu, - '20)

We can apply this construction 8 times, starting from $\mathcal{D}_2$, to obtain an affirmative answer for $\boxed{Q1}$ when genus $= 2$.

Thank you!