

GENUS TWO STABLE MAPS, LOCAL EQUATIONS AND MODULAR RESOLUTIONS

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ABSTRACT. We geometrically describe a canonical sequence of modular blowups of the relative Picard stack over the Artin stack of genus two weighted curves. The resulted stack locally diagonalizes certain tautological derived objects. This implies a resolution of the primary component of the moduli space of genus two stable maps to projective space and meanwhile makes the entire space admit only normal crossing singularities. Our approach should extend to higher genera.

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1. INTRODUCTION

Moduli problems are of central importance in algebraic geometry. Among them, the moduli spaces $\overline{M}_{g,k}(\mathbb{P}^n, d)$ of genus g degree d stable maps into projective space with k marked points are particularly important. For example, the Gromov-Witten theory is an intersection theory on these fundamental spaces (introduced and constructed in [11, 1, 2]).

This article focuses on the deep geometry of the moduli space $\overline{M}_{g,k}(\mathbb{P}^n, d)$. By Murphy's law of Vakil ([13]), the moduli space $\overline{M}_{g,k}(\mathbb{P}^n, d)$ is arbitrarily singular: every singularity of finite type over \mathbb{Z} appears for some g and d . The resolution of singularity is arguably one of the hardest problems in algebraic geometry ([3, 4, 10]). To date, the problem remains open for positive characteristics. However, several moduli problems offer platforms for geometric solutions, and $\overline{M}_{g,k}(\mathbb{P}^n, d)$ is one of them (this is *the pivotal motivation* for us to study the moduli $\overline{M}_{g,k}(\mathbb{P}^n, d)$). When the genus g is 0, the moduli space $\overline{M}_{0,k}(\mathbb{P}^n, d)$ is smooth. For $g=1$, the moduli space $\overline{M}_{1,k}(\mathbb{P}^n, d)$ is singular and the resolution of the primary component of $\overline{M}_{1,k}(\mathbb{P}^n, d)$ was

constructed by Vakil and Zinger in [14], followed by an algebraic approach to the entire moduli $\overline{M}_1(\mathbb{P}^n, d)$ (without marked points) by Hu and Li [5].

Our current article begins to treat the more difficult cases of general genera. We start with the moduli space $\overline{M}_2(\mathbb{P}^n, d)$; for conciseness and without loss of generality in terms of singularity, we still work with curves without marked points. The goal of this paper is to *geometrically* describe a canonical sequence of blowups of $\overline{M}_2(\mathbb{P}^n, d)$ such that the resulted stack admits only normal crossing singularities and every of its irreducible components is smooth. This is the best that one can hope for as far as smoothness is concerned.

In the subsequent works [7, 8], the first and third named authors plan to prove that resulted blowup stacks in both [5] and the current article are moduli stacks. This approach has the benefit that it is almost ready for a generalization to arbitrary genus.

Although not a primary motivation of this series of works, the smooth blowup of $\overline{M}_2(\mathbb{P}^n, d)$ obtained in this paper can be used, for instance, to prove the “hyperplane” property of the Gromov-Witten invariants of quintic Calabi-Yau threefolds conjectured in [12] (along with other possible calculations). This is also a reason that we work with unmarked curves. Further, the method provided in this paper may also be helpful for the resolution of other singular moduli used in the recent progresses in mirror symmetry.

To state our main results, we denote by \mathfrak{M}_2 the Artin stack of genus 2 nodal curves C and by \mathfrak{P}_2 the relative Picard stack over \mathfrak{M}_2 : its objects are pairs (C, L) , where C are genus 2 weighted curves and L are line bundles over C . Let

$$(1.1) \quad \mathfrak{r} : \overline{M}_2(\mathbb{P}^n, d) \longrightarrow \mathfrak{P}_2; \quad [C, u] \rightarrow (C, u^* \mathcal{O}_{\mathbb{P}^n}(1)).$$

be the representable morphism.

We denote by

$$\pi : \mathcal{C} \longrightarrow \overline{M}_2(\mathbb{P}^n, d), \quad \mathfrak{f} : \mathcal{C} \longrightarrow \mathbb{P}^n$$

the universal family of the moduli $\overline{M}_2(\mathbb{P}^n, d)$. The main technical goal of this article is to locally diagonalize the derived objects $\mathbf{R}\pi_* \mathfrak{f}^* \mathcal{O}_{\mathbb{P}^n}(k)$, $k \geq 1$.

Theorem 1. *There is a canonical sequence of blowups $\tilde{\mathfrak{P}}_2 \rightarrow \mathfrak{P}_2$ such that for all integer $k \geq 1$, the pullback of the derived object $\mathbf{R}\pi_* \mathfrak{f}^* \mathcal{O}_{\mathbb{P}^n}(k)$ to*

$$\tilde{M}_2(\mathbb{P}^n, d) = \overline{M}_2(\mathbb{P}^n, d) \times_{\mathfrak{r}, \mathfrak{P}_2} \tilde{\mathfrak{P}}_2$$

becomes locally diagonalizable.

Theorem 1 is proved in §5; see Theorem 5.1.1. We remark here that in order to make, for example, $\mathbf{R}\pi_* \mathfrak{f}^* \mathcal{O}_{\mathbb{P}^n}(5)$ locally diagonalizable (hence to make $\pi_{N*} \mathfrak{f}_N^* \mathcal{O}_{\mathbb{P}^n}(5)$ locally free), it suffices to perform a part of the blowups mentioned in Theorem 1.

The primary components of $\overline{M}_2(\mathbb{P}^n, d)$ and $\tilde{M}_2(\mathbb{P}^n, d)$ are respectively the irreducible components whose general points are stable maps with smooth

domain curves. The $k = 1$ case of Theorem 1 leads to the following conclusion.

Corollary 2. *For $d > 2$, $\widetilde{M}_2(\mathbb{P}^n, d)$ has normal crossing singularities, and its primary component is smooth and of the expected dimension $d(n+1) - n + 3$.*

Corollary 2 is proved in §6; see Theorem 6.2.2. Theorem 1 also gives rise to the following statement on the desingularization of the sheaves $\pi_* \mathfrak{f}^* \mathcal{O}_{\mathbb{P}^n}(k)$. It follows directly from Theorem 1 and Proposition 2.1.3.

Corollary 3. *Let N be a fixed irreducible component of $\widetilde{M}_2(\mathbb{P}^n, d)$, $d > 2$, and $(\widetilde{\pi}_N, \widetilde{\mathfrak{f}}_N)$ be the pullback to N of the universal family (π, \mathfrak{f}) . Then the direct image sheaf $\widetilde{\pi}_{N*} \widetilde{\mathfrak{f}}_N^* \mathcal{O}_{\mathbb{P}^n}(k)$ is locally free for all $k \geq 1$. It is of rank $kd - 1$ if N is the primary component of $\widetilde{M}_2(\mathbb{P}^n, d)$.*

Our approach improves the technique of [5] and combines with the idea of the derived modular blowups in [6]. To explain this, we introduce the Artin stack $\mathfrak{M}_2^{\text{div}}$ of pairs (C, D) , where C are genus 2 nodal curves and D are effective divisors supported on the smooth loci of C . The stack $\mathfrak{M}_2^{\text{div}}$ is equipped with a universal curve $\rho : \mathcal{C} \rightarrow \mathfrak{M}_2^{\text{div}}$ and a universal divisor \mathcal{D} . There is a canonical morphism

$$\mathfrak{M}_2^{\text{div}} \longrightarrow \mathfrak{P}_2, \quad (C, D) \rightarrow (C, \mathcal{O}_C(D));$$

via this morphism, we study the parallel diagonalization problem of the derived object $\rho_* \mathcal{O}_{\mathcal{C}}(D)$. Locally over a smooth chart \mathcal{V} , the object $\rho_* \mathcal{O}_{\mathcal{C}}(D)|_{\mathcal{V}}$ can be represented by a two term structural homomorphism

$$(1.2) \quad \varphi : \mathcal{O}_{\mathcal{V}}^{\oplus m+1} \longrightarrow \mathcal{O}_{\mathcal{V}}^{\oplus 2},$$

where m is the degree of D .

We provide a detailed analysis of this homomorphism in terms of local modular parameters in §2 and §5. The analysis of φ suggests our construction of the modular blowups in §4. The procedure involves three rounds, the first and the last of which further contain several phases. When the degree d is fixed, each round or phase consists of finitely many steps. For conciseness, we hereafter write

$$(1.3) \quad \mathbf{r}_i, \quad \mathbf{p}_j, \quad \mathbf{s}_\ell$$

respectively for the i -th round, the j -th phase, and the ℓ -th step whenever applicable. We point out here that in order to locally diagonalize $\mathbf{R}\pi_* \mathfrak{f}^* \mathcal{O}_{\mathbb{P}^n}(k)$ with $k \geq 2$, $\mathbf{r}_1 \mathbf{p}_5$ and $\mathbf{r}_3 \mathbf{p}_4$ can be omitted; to locally diagonalize $\mathbf{R}\pi_* \mathfrak{f}^* \mathcal{O}_{\mathbb{P}^n}(k)$ with $k \geq 3$, $\mathbf{r}_3 \mathbf{p}_3$ can be omitted as well.

Many sequences of our blowups are performed in the Artin stack $\mathfrak{M}_2^{\text{wt}}$ of weighted curves of genus 2 (i.e. genus 2 nodal curves whose irreducible components are decorated by weights in $\mathbb{Z}_{\geq 0}$; cf. §2.3). There is a canonical morphism

$$\mathfrak{P}_2 \longrightarrow \mathfrak{M}_2^{\text{wt}}, \quad (C, L) \rightarrow (C, c_1(L))$$

and a canonical representable morphism

$$(1.4) \quad \varrho : \overline{M}_2(\mathbb{P}^n, d) \longrightarrow \mathfrak{M}_2^{\text{wt}}, \quad [u, C] \rightarrow (C, c_1(u^* \mathcal{O}_{\mathbb{P}^n}(1)))$$

that factors through (1.1). Each blowup $\widetilde{\mathfrak{M}}' \rightarrow \mathfrak{M}_2^{\text{wt}}$ determines the blowup $\widetilde{\mathfrak{P}}' = \mathfrak{P}_2 \times_{\mathfrak{M}_2^{\text{wt}}} \widetilde{\mathfrak{M}}'_2$ of \mathfrak{P}_2 .

The first round of blowups. We first blow up $\mathfrak{M}_2^{\text{wt}}$ successively along the proper transforms of the substacks that are illustrated in Figure 1 and described in §4.3. The description of these substacks are “topological”; they are determined by the dual graphs and the distribution of weights.

When r_1 terminates, we denote the final stack by $\widetilde{\mathfrak{M}}^{r_1}$. Every point of $\mathfrak{M}_2^{\text{wt}}$ has a neighborhood \mathcal{V} so that the pullback $\widetilde{\varphi}^{r_1}$ of the structural homomorphism φ of (1.2) to $\widetilde{\mathcal{V}}^{r_1} = \mathcal{V} \times_{\mathfrak{M}_2^{\text{wt}}} \widetilde{\mathfrak{M}}^{r_1}$ has one row that contains an element that divides all other entries of $\widetilde{\varphi}^{r_1}$ evenly. However, $\widetilde{\varphi}^{r_1}|_{\widetilde{\mathcal{V}}^{r_1}}$ may *not* be diagonalizable (c.f. Definition 2.1.1) due to the existence of (1) some distinguished directions in the exceptional divisors obtained in r_1 and/or (2) the Weierstrass and conjugate points of genus 2 curves.

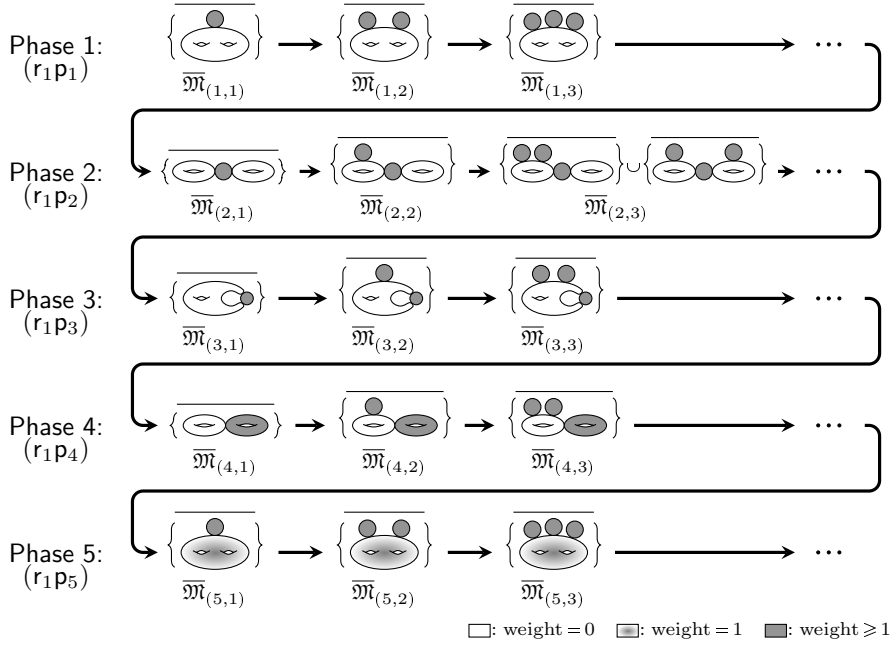


FIGURE 1. The first round (r_1) of the modular blowups

The second round of blowups. The blowup loci of r_2 lie in the proper transforms of the exceptional divisors obtained in r_1 , hence are more complicated. They are illustrated in Figure 2 and described in §4.4. We blow up $\widetilde{\mathfrak{M}}^{r_1}$ successively along these substacks and obtain $\widetilde{\mathfrak{M}}^{r_2}$. After this round, the only

obstacle to the pullback of φ being locally diagonalizable is the existence of the Weierstrass and conjugate points. They will be dealt with in the next (i.e. the last) round of blowups.

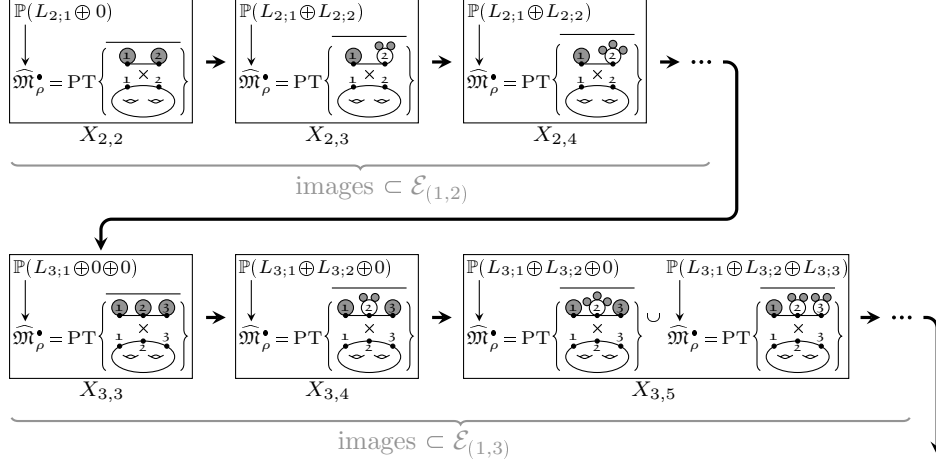


FIGURE 2. The second round of the modular blowups

The third round of blowups. This round is divided into four phases that are described in §4.5-4.8. The blowup loci of $r_3\mathfrak{p}_1$ (resp. $r_3\mathfrak{p}_2$) lie in the proper transforms of the exceptional divisors of $r_1\mathfrak{p}_1$ (resp. r_2). The description of these two phases is similar to that of r_2 , but slightly more complicated because of the conjugate points (resp. Weierstrass points).

In the last two phases of r_3 , we work on the relative Picard stack \mathfrak{P}_2 rather than $\mathfrak{M}_2^{\text{wt}}$, because there is no obvious morphism $\overline{\mathfrak{M}}_2(\mathbb{P}^n, d) \rightarrow \mathfrak{M}_2^{\text{div}}$ that (1.4) factors through. In $r_3\mathfrak{p}_3$, let \mathcal{H}_k ($k \geq 0$) be the closed substack of \mathfrak{P}_2 whose general points are pairs (C, L) with F denoting the core of C (i.e. the smallest genus 2 subcurve of C ; c.f. §2.2) such that $\deg L|_F = 2$, $h^0(L|_F) = 2$, and F has k rational tails attached to it. We blow up the stack $\mathfrak{P}_2 \times_{\mathfrak{M}_2^{\text{wt}}} \widetilde{\mathfrak{M}}^{r_3\mathfrak{p}_2}$ along the proper transforms of $\mathcal{H}_0 \times_{\mathfrak{M}_2^{\text{wt}}} \widetilde{\mathfrak{M}}^{r_3\mathfrak{p}_2}, \dots, \mathcal{H}_k \times_{\mathfrak{M}_2^{\text{wt}}} \widetilde{\mathfrak{M}}^{r_3\mathfrak{p}_2}, \dots$ and obtain $\widetilde{\mathfrak{P}}_2^{r_3\mathfrak{p}_3}$. The description of $r_3\mathfrak{p}_4$ is more sophisticated; it is in a mixed fashion of r_2 and $r_3\mathfrak{p}_3$. The key fact is that the pullback of φ becomes locally diagonalizable when r_3 terminates; c.f. Theorem 5.1.1.

We remark that the description of the modular blowups in §4 is *global*. However, it is crucial to understand how each round or phase of the modular blowups affects the moduli space *locally*, so as to study the local behavior of the structural homomorphism φ in §5. We thus introduce the notion of *locally tree-compatible* blowups in §3, which provides the desired connection between the globally defined the blowups and their local properties.

Once Theorem 5.1.1 is established in §5, we apply the technique in [5] to obtain the local structure of $\widetilde{\mathfrak{M}}_2(\mathbb{P}^n, d)$ in §6.

Throughout the paper, we work over an algebraically closed field of characteristic zero.

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2. THE STRUCTURES OF THE TAUTOLOGICAL DERIVED OBJECTS

We begin with the notion of the diagonalizable derived objects introduced in [6]. It is key to this article.

2.1. Diagonalizable homomorphism. Let $\varphi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ be a homomorphism of locally free sheaves of \mathcal{O}_U -modules on a DM stack U .

Definition 2.1.1. *We say φ is diagonalizable if there are integers r, l_1 and $l_2 \in \mathbb{Z}_{\geq 0}$, $p_i \in \Gamma(\mathcal{O}_U)$ such that the ideals $(p_i) \subset (p_{i+1})$, and isomorphisms $\mathcal{E}_i \cong \mathcal{O}_U^{\oplus r} \oplus \mathcal{O}_U^{\oplus l_i}$ such that*

$$\varphi = \text{diag}[p_1, \dots, p_r] \oplus 0 : \mathcal{O}_U^{\oplus r} \oplus \mathcal{O}_U^{\oplus l_1} \longrightarrow \mathcal{O}_U^{\oplus r} \oplus \mathcal{O}_U^{\oplus l_2},$$

where $\text{diag}[p_1, \dots, p_r] : \mathcal{O}_U^{\oplus r} \rightarrow \mathcal{O}_U^{\oplus r}$ is the diagonal homomorphism and $0 : \mathcal{O}_U^{\oplus l_1} \rightarrow \mathcal{O}_U^{\oplus l_2}$ is the zero homomorphism. We say φ is locally diagonalizable if there is an étale cover U_α of U such that φ is diagonalizable over U_α .

Definition 2.1.2. *Let \mathbf{E} be a perfect derived object on a DM stack M . We say \mathbf{E} is locally diagonalizable if there is an étale cover $\{U\}$ of M such that $\mathbf{E}|_U$ is quasi-isomorphic to $[\varphi : \mathcal{E}_1 \rightarrow \mathcal{E}_2]$, where \mathcal{E}_1 and \mathcal{E}_2 are locally free sheaves and φ is diagonalizable.*

Using [6, Proposition 3.2] (the universality of diagonalization), one checks directly that the definition does not depend on the local representation of the object \mathbf{E} . Here it is worth to point out that in Definition 2.1.1, if the requirement $(p_{i+1}) \subset (p_i)$ is removed, then the “diagonalization” thus defined is not a property of the derived object but merely a property of a presentation of the object \mathbf{E} .

The following result is a conclusion of [6, Corollary 4.1]. It is needed to deduce Corollary 3 from Theorem 1 in §1.

Proposition 2.1.3 ([6]). *Let M and \widetilde{M} be two DM stacks and \mathbf{E} be a perfect derived object on M . Assume there is a birational morphism $\psi : \widetilde{M} \rightarrow M$ such that the pullback of \mathbf{E} is locally diagonalizable in the sense of Definition 2.1.1. Then, for any irreducible component N of \widetilde{M} endowed with the reduced stack structure, $\mathcal{H}^0(L\psi^*\mathbf{E}|_N)$ is locally free.*

2.2. Nodal curves and their dual graphs. Let C be a nodal curve. We call a node q (resp. an irreducible component) of C **separating** if $C \setminus q$ (resp. $C \setminus \Sigma$) is disconnected. We call other nodes of C **non-separating**. A curve C is said to be **separable** (resp. **inseparable**) if it contains at least one separating node (resp. contains no separating node). An **inseparable component** of C is a smallest (via inclusion) inseparable subcurve of C .

If C is of positive (arithmetic) genus g , we call the smallest connected genus g subcurve F of C the **core** (of C). Removing F from C results in disjoint connected trees of rational curves, called the **tails** of C . Each tail is attached to the core F at a separating node, called a **pivotal node**.

A node q of C is said to be **principal** (resp. **non-principal**) if q is (resp. is not) a node of the core F of C . Every (not necessarily smooth) point $x \in C \setminus F$ belongs to a unique tail, whose pivotal node is denoted by $\langle x \rangle$. For $x \in F$, let $\langle x \rangle = x$.

A non-separating (resp. separating) **bridge** B of F is a chain of rational curves in F whose principal nodes are all non-separating (resp. separating). A bridge is **maximal** if it is not a strict subset of any other bridge. Every bridge B is connected to its complement in F at two distinct principal nodes p and q . We sometimes denote B by $B[p, q]$ to emphasize these nodes.

Let $\mathcal{W} \subset \overline{\mathcal{M}}_{2,1}$ be the locus of (stable) genus 2 curves with one marked Weierstrass point. Using admissible double covers (c.f. [9, §3G]), we can identify \mathcal{M}_2 with $\overline{\mathcal{M}}_{0,6}/S_6$, where S_6 is the symmetry group on 6 letters. Here, the 6 unordered markings on the target rational curve are the branch points of the hyperelliptic admissible double covers. With this identification, choosing a Weierstrass point in the domain curve is the same as specifying a distinguished marking out of the 6 unordered markings in the target rational curve. Hence, \mathcal{W} is isomorphic to $\overline{\mathcal{M}}_{0,6}/S_5$, where the last marking is declared to be distinguished from the others. In particular, \mathcal{W} is a smooth divisor in $\overline{\mathcal{M}}_{2,1}$.

For $1 \leq i \leq k$, let $\mathcal{W}_{k,i} \subset \overline{\mathcal{M}}_{2,k}$ be the locus of stable genus 2 curves with k marked points so that the i -th marked point is Weierstrass. Then, $\mathcal{W}_{k,i}$ is the pullback of \mathcal{W} in $\overline{\mathcal{M}}_{2,k}$ via the forgetful morphism that forgets all but the i -th marked point. Since \mathcal{W} is a Cartier divisor, so is every $\mathcal{W}_{k,i}$. We remark that the smoothness of \mathcal{W} does not imply that $\mathcal{W}_{k,i}$ is smooth.

Let $\mathcal{K} \subset \overline{\mathcal{M}}_{2,2}$ be the locus of (stable) genus 2 curves with two marked conjugate points. Similar to the above discussions, \mathcal{K} is isomorphic to $\overline{\mathcal{M}}_{0,7}/S_6$, where the 7-th marking corresponds to the image of a pair of conjugate points. In particular, \mathcal{K} is a smooth divisor in $\overline{\mathcal{M}}_{2,2}$. Similarly, let $\mathcal{K}_{k,i,j} \subset \overline{\mathcal{M}}_{2,k}$ be the locus of stable genus 2 curves with k marked points so that the i -th and the j -th marked points are conjugate. Then $\mathcal{K}_{k,i,j}$ is a Cartier divisor in $\overline{\mathcal{M}}_{2,k}$.

For a nodal curve C , we call a smooth point p of C a **Weierstrass point** if $[C, p] \in \mathcal{W}$. Similarly, we say a pair of smooth points p_1 and p_2 of C to be **conjugate** if $[C, p_1, p_2] \in \mathcal{K}$. The terminology can be generalized to all

non-principal node(s) by considering the connected component of \tilde{C} , the normalization of C at the given node(s), that contains the core F .

2.3. Basic setup. Let $\mathfrak{M}_2^{\text{wt}}$ be the Artin stack of the stable pairs (C, \mathbf{w}) of genus 2 nodal curves C with non-negative weights $\mathbf{w} \in H^2(C, \mathbb{Z})$, meaning that $\mathbf{w}(\Sigma) \geq 0$ for all irreducible components Σ of C . Here (C, \mathbf{w}) is said to be stable if every rational irreducible component of weight 0 contains at least three nodal points; c.f. [5, §2.1]. The stability requirements guarantees that each connected component of $\mathfrak{M}_2^{\text{wt}}$ is of finite type.

Similarly, let $\mathfrak{M}_2^{\text{div}}$ be the Artin stack of the stable pairs (C, D) where C are genus 2 nodal curves and D are effective divisors on C . Here (C, D) is said to be stable if every rational irreducible component disjoint from D contains at least three nodal points; c.f. [5, §2.3]. The stability requirements guarantees that each connected component of $\mathfrak{M}_2^{\text{div}}$ is also of finite type.

In studying $\overline{M}_2(\mathbb{P}^n, d)$ together with $\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k)$, we cover it by étale opens $U \rightarrow \overline{M}_2(\mathbb{P}^n, d)$ and pick degree k hypersurfaces $H \subset \mathbb{P}^n$ in general position so that for each $[u, C] \in U$, $u^{-1}(H) \subset C$ is a simple divisor lying in the smooth locus of C . The assignment $[u, C] \mapsto (C, u^{-1}(H))$ defines morphisms

$$(2.1) \quad \overline{M}_2(\mathbb{P}^n, d) \supset U \longrightarrow \mathfrak{M}_2^{\text{div}}.$$

We pick an affine étale $\mathcal{V} \rightarrow \mathfrak{M}_2^{\text{div}}$ so that $U \rightarrow \mathfrak{M}_2^{\text{div}}$ factor through $U \rightarrow \mathcal{V}$.

Using the (local) morphism (2.1), it suffices to study a parallel problem on $\mathfrak{M}_2^{\text{div}}$. For a fixed $(C, D) \in \mathfrak{M}_2^{\text{div}}$ lying over $(C, \mathbf{w}) \in \mathfrak{M}_2^{\text{wt}}$ with $\deg D = m$ for some positive integer m , we pick an affine smooth chart $\mathcal{V} \rightarrow \mathfrak{M}_2^{\text{div}}$ containing (C, D) . Let $\rho: \mathcal{C} \rightarrow \mathcal{V}$ and $\mathcal{D} \subset \mathcal{C}$ be the universal family on \mathcal{V} . We find disjoint sections $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}$ of \mathcal{C}/\mathcal{V} , disjoint from \mathcal{D} , called the auxiliary divisors. We list this package as

$$(2.2) \quad \rho: \mathcal{C} \rightarrow \mathcal{V} (\rightarrow \mathfrak{M}_2^{\text{div}}), \quad \mathcal{D} \subset \mathcal{C}; \quad \text{plus } \mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2, \mathcal{B} \subset \mathcal{C}.$$

Let

$$\mathcal{L} = \mathcal{O}_{\mathcal{C}}(\mathcal{D}), \quad \mathcal{M} = \mathcal{O}_{\mathcal{C}}(\mathcal{A} - \mathcal{B}), \quad \text{and} \quad \mathcal{O}_{\mathcal{A}}(\mathcal{A}) = \mathcal{O}_{\mathcal{V}}(\mathcal{A}) \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{O}_{\mathcal{A}}.$$

We consider the two term complex

$$(2.3) \quad \rho_* \mathcal{L}(\mathcal{A}) \longrightarrow \rho_* \mathcal{O}_{\mathcal{A}}(\mathcal{A})$$

via the evaluation homomorphism. These data will be fixed throughout the paper.

Basic Assumptions. *Possibly after an étale base change and a shrinking of \mathcal{V} , we assume*

- (A) $a_i = \mathcal{A}_i \cap C$ and $b = \mathcal{B} \cap C$ lie in the core F of C and are in general position; in case F is separable, then a_1 and a_2 lie on different genus-one inseparable components of F ;
- (B) $\mathcal{D} = \mathcal{D}_1 + \cdots + \mathcal{D}_m$ is a union of m disjoint sections \mathcal{D}_i of $\mathcal{C} \rightarrow \mathcal{V}$;
- (C) Assumptions (A) and (B) hold for all fibers of $\mathcal{C} \rightarrow \mathcal{V}$.

Under the assumptions, we have $R^1\rho_*\mathcal{L}(\mathcal{A}) = 0$, $\rho_*\mathcal{L}(\mathcal{A})$ is locally free, and $\mathbf{R}\rho_*\mathcal{L}$ is quasi-isomorphic to the complex (2.3).

2.4. The first reduction. The standard inclusion $\mathcal{O}_{\mathcal{C}} \subset \mathcal{O}_{\mathcal{C}}(\mathcal{D})$ determines the obvious section $1 \in \Gamma(\rho_*\mathcal{L})$. For other sections, let $\mathcal{M} = \mathcal{O}_{\mathcal{C}}(\mathcal{A} - \mathcal{B})$ and consider the inclusion

$$\mathcal{M}(\mathcal{D}_i) = \mathcal{O}_{\mathcal{C}}(\mathcal{D}_i + \mathcal{A} - \mathcal{B}) \xrightarrow{\subset} \mathcal{M}(\mathcal{D}) = \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{A} - \mathcal{B}),$$

and the induced inclusions

$$\eta_i : \rho_*\mathcal{M}(\mathcal{D}_i) \xrightarrow{\subset} \rho_*\mathcal{M}(\mathcal{D}).$$

Because of the Basic Assumptions, the above terms are locally free. By Riemann-Roch Theorem, $\rho_*\mathcal{M}(\mathcal{D}_i)$ is invertible and $\rho_*\mathcal{M}(\mathcal{D})$ has rank m . Let

$$(2.4) \quad \varphi : \rho_*\mathcal{M}(\mathcal{D}) \longrightarrow \rho_*\mathcal{O}_{\mathcal{A}}(\mathcal{A}) \quad \text{and} \quad \varphi_i : \rho_*\mathcal{M}(\mathcal{D}_i) \longrightarrow \rho_*\mathcal{O}_{\mathcal{A}}(\mathcal{A})$$

be the evaluation homomorphisms; then $\varphi_i = \varphi \circ \eta_i$. We denote their sum by

$$\begin{aligned} (\eta_i) &: \bigoplus_{i=1}^m \rho_*\mathcal{M}(\mathcal{D}_i) \longrightarrow \rho_*\mathcal{M}(\mathcal{D}), \\ (\varphi_i) &: \bigoplus_{i=1}^m \rho_*\mathcal{M}(\mathcal{D}_i) \longrightarrow \rho_*\mathcal{O}_{\mathcal{A}}(\mathcal{A}). \end{aligned}$$

Lemma 2.4.1. *We have (1). $\rho_*\mathcal{L} \cong \mathcal{O}_{\mathcal{V}} \oplus \rho_*\mathcal{L}(-\mathcal{B})$; (2). $\rho_*\mathcal{L}(-\mathcal{B}) \cong \ker \varphi$; (3). (η_i) is an isomorphism, and (4). $(\varphi_i) = \varphi \circ (\eta_i)$. Consequently,*

$$\rho_*\mathcal{L} \cong \mathcal{O}_{\mathcal{V}} \oplus \ker\{(\varphi_i)\}.$$

Proof. The proofs of (1) and (2) are the same as for the genus 1 case, [5, Lemma 4.10]; we omit the details. We now prove (3). Since both $\rho_*\mathcal{M}(\mathcal{D}_i)$ and $\rho_*\mathcal{M}(\mathcal{D})$ are locally free, we only need to show that for any closed $z \in \mathcal{V}$, $\bigoplus_{i=1}^m \rho_*\mathcal{M}(\mathcal{D}_i)|_z \rightarrow \rho_*\mathcal{M}(\mathcal{D})|_z$ is an isomorphism. By base change, it suffices to show that the tautological homomorphism

$$\bigoplus_{i=1}^m \eta_i(z) : \bigoplus_{i=1}^m H^0(\mathcal{O}_{\mathcal{C}_z}(\mathcal{D}_i + \mathcal{A} - \mathcal{B})) \longrightarrow H^0(\mathcal{O}_{\mathcal{C}_z}(\mathcal{D} + \mathcal{A} - \mathcal{B}))$$

is an isomorphism. Because both sides are of equal dimensions, it suffices to show that it is injective. The injectivity follows from that $\mathcal{D}_1 \cap \mathcal{C}_z, \dots, \mathcal{D}_m \cap \mathcal{C}_z$ are distinct in \mathcal{C}_z . The item (4) is a tautology. This proves the lemma. \square

This proves that the homomorphism φ in (2.4) is equivalent to the homomorphism (φ_i) , via the isomorphism (η_i) .

We will call either φ or (φ_i) a *structural homomorphism*. Note that each φ_i has the form

$$(2.5) \quad \varphi_i = \varphi_i^1 \oplus \varphi_i^2 : \rho_*\mathcal{M}(\mathcal{D}_i) \longrightarrow \rho_*\mathcal{O}_{\mathcal{A}_1}(\mathcal{A}_1) \oplus \rho_*\mathcal{O}_{\mathcal{A}_2}(\mathcal{A}_2).$$

These homomorphisms will be our focus in this section.

2.5. The structural homomorphism. By the deformation theory of nodal curves, for each node $q \in C$ there is a regular function $\zeta_q \in \Gamma(\mathcal{O}_V)$ so that $\Sigma_q = \{\zeta_q = 0\}$ is the locus where the node q is not smoothed; the divisor Σ_q is a smooth divisor. We will refer to ζ_q as a **modular parameter**.

To proceed, we introduce the notation of a node lying between two points (or components). We say a separating node q separates (or lies between) two points x and y of C if either $x = q$, or $y = q$, or x and y lie in different connected components of $C \setminus \{q\}$. If we replace y by a connected subcurve C' of C , we say q lies between x and C' if either $x = q$ or x and $C' \setminus \{q\}$ lie in different connected components of $C \setminus \{q\}$. We denote by $N_{[x,y]}$ (resp. $N_{[x,C']}$) the set of the separating nodes lying between x and y (resp. C').

If $B = B[q_1, q_2]$ is a non-separating bridge on F , then for every $x \in B$, we define $N_{[x,q_1]}$ (resp. $N_{[x,q_2]}$) in the same fashion with C replaced by the normalization of C at q_2 (resp. at q_1).

Now back to the family $C \rightarrow \mathcal{V}$ with $C = \mathcal{C} \times_{\mathcal{V}} 0$ the central fiber mentioned before. For $1 \leq i \leq m$ and $1 \leq s \leq 2$, let

$$\delta_i = \mathcal{D}_i \cap C \quad \text{and} \quad a_s = \mathcal{A}_s \cap C.$$

For $x, y \in C$, we define

$$(2.6) \quad \zeta_{[x,y]} = \prod_{q \in N_{[x,y]}} \zeta_q, \quad \text{and} \quad \zeta_{[x]} = \prod_{q \in N_{[x,F]}} \zeta_q.$$

(Here as always, F is the core of C .) In case $N_{[x,a_s]} = \emptyset$, we set $\zeta_{[x,a_s]} = 1$.

If T is a tail and $x, y \in T$, let $x \wedge y$ be the (non-principal) node so that

$$\zeta_{[x \wedge y]} = \prod_{q \in N_{[x,F]} \cap N_{[y,F]}} \zeta_q.$$

If $x \wedge y$ does not exist, i.e. $N_{[x,F]} \cap N_{[y,F]} = \emptyset$, we simply set $\zeta_{[x \wedge y]} = 1$.

We fix trivializations

$$(2.7) \quad \rho_* \mathcal{M}(\mathcal{D}_i) \cong \mathcal{O}_{\mathcal{V}}, \quad \rho_* \mathcal{O}_{\mathcal{A}_s}(\mathcal{A}_s) \cong \mathcal{O}_{\mathcal{V}}; \quad 1 \leq i \leq m, \quad 1 \leq s \leq 2,$$

and the induced trivialization $\rho_* \mathcal{O}_{\mathcal{A}}(\mathcal{A}) \cong \mathcal{O}_{\mathcal{V}}^{\oplus 2}$ throughout this section.

Proposition 2.5.1. *After fixing (2.7), the evaluation homomorphism*

$$(2.8) \quad \varphi_{si} : \rho_* \mathcal{M}(\mathcal{D}_i) \longrightarrow \rho_* \mathcal{O}_{\mathcal{A}_s}(\mathcal{A}_s)$$

is given by the multiplication

$$\varphi_{si} = c_{si} \cdot \zeta_{[\delta_i, a_s]} : \mathcal{O}_{\mathcal{V}} \longrightarrow \mathcal{O}_{\mathcal{V}}, \quad c_{si} \in \Gamma(\mathcal{O}_{\mathcal{V}}^*).$$

Proof. The proof is parallel to that of [5, Prop. 4.13] and is thus omitted. \square

2.6. A technical proposition on rational curves. We now focus on a maximal chain $R \subset C$ of smooth rational curves $R = R_1 \cup \cdots \cup R_h$ of C attached to a smooth point of the core $F \subset C$.

We fix some notation. First, we index the irreducible components of R so that R_1 is attached to F , and R_{j+1} is attached to R_j . Let

$$q_1 = F \cap R_1, \quad q_{j+1} = R_j \cap R_{j+1} \quad \forall 1 \leq j \leq h-1.$$

We order

$$(2.9) \quad D \cap R = \{\delta_1, \delta_2, \dots, \delta_r\}$$

such that the function

$$(2.10) \quad e: \{1, \dots, r\} \longrightarrow \{1, \dots, h\} \quad \text{s.t.} \quad \delta_i \in R_{e(i)} \quad \forall 1 \leq i \leq r$$

is non-decreasing. We denote by

$$(2.11) \quad \varphi^R: \bigoplus_{\delta_i \in D \cap R} \rho_* \mathcal{M}(\mathcal{D}_i) \longrightarrow \rho_* \mathcal{O}_{\mathcal{A}}(\mathcal{A})$$

the restriction homomorphism. Recall the modular parameter associated with the node q_j is denoted by ζ_{q_j} .

Proposition 2.6.1. *Let R be a maximal chain of rational curves in (C, D) as stated above. Then after choosing suitable trivialization*

$$\bigoplus_{\delta_i \in D \cap R} \rho_* \mathcal{M}(\mathcal{D}_i) \cong \mathcal{O}_{\mathcal{V}}^{\oplus |D \cap R|}$$

and (2.7), the homomorphism φ^R takes the form

$$(2.12) \quad \varphi^R = \begin{bmatrix} \widehat{c}_{11} \zeta_{[\delta_1, a_1]} & \widehat{c}_{12} \zeta_{[\delta_1, a_1]} \zeta_{[\delta_2]} & \widehat{c}_{13} \zeta_{[\delta_1, a_1]} \zeta_{[\delta_2]} \zeta_{[\delta_3]} & \cdots \\ \widehat{c}_{21} \zeta_{[\delta_1, a_2]} & \widehat{c}_{22} \zeta_{[\delta_1, a_2]} \zeta_{[\delta_2]} & \widehat{c}_{23} \zeta_{[\delta_1, a_2]} \zeta_{[\delta_2]} \zeta_{[\delta_3]} & \cdots \end{bmatrix}$$

with $\widehat{c}_{11} = c_{11}$, $\widehat{c}_{21} = c_{21} \in \Gamma(\mathcal{O}_{\mathcal{V}}^*)$, $\widehat{c}_{ij} \in \Gamma(\mathcal{O}_{\mathcal{V}}) \quad \forall j > 1$,

where c_{11} and c_{21} are as in Proposition 2.5.1. Moreover, with

$$\theta_{ij} \equiv \det \begin{bmatrix} \widehat{c}_{1i} & \widehat{c}_{1j} \\ \widehat{c}_{2i} & \widehat{c}_{2j} \end{bmatrix} \in \Gamma(\mathcal{O}_{\mathcal{V}}) \quad \forall 1 \leq i < j \leq m,$$

- (1) if q_1 is not Weierstrass, then $\theta_{12} \in \Gamma(\mathcal{O}_{\mathcal{V}}^*)$;
- (2) if q_1 is Weierstrass, then θ_{12} vanishes to the first order along the locus on which \mathcal{D}_1 and \mathcal{D}_2 are conjugate;
- (3) if q_1 is Weierstrass and is not on any bridge of F , then $\theta_{13} \in \Gamma(\mathcal{O}_{\mathcal{V}}^*)$;
- (4) if q_1 is on a maximal non-separating bridge $B[p, r]$ of F , then for every $z \in \mathcal{V}$, if there exist $p_1 \in N_{[q_1, p]}$ and $r_1 \in N_{[q_1, r]}$ such that

$$\zeta_{q_1}(z) = \zeta_{p_1}(z) = \zeta_{r_1}(z) = 0,$$

then $\theta_{1j}(z) = 0$ for all $3 \leq j \leq r$; moreover, the zero loci $\{\theta_{1i} = 0\}$, $2 \leq i \leq r$, are transverse to each other.

Note that in Part (3) of Proposition 2.6.1, the node q_1 is automatically Weierstrass. In addition, if q_1 is on a separating bridge of F , then it cannot be Weierstrass. Thus, the last two parts of Proposition 2.6.1 cover all possibilities of the Weierstrass q_1 .

Proof. Let ζ_{q_j} be the modular function associated with q_j . Since q_j is separating,

$$\mathcal{C} \times_{\mathcal{V}} (\zeta_{q_j} = 0) = \mathcal{G}_j \cup \mathcal{R}_j$$

is a union of two families of nodal curves over $(\zeta_{q_j} = 0)$ intersecting (i.e. glued along) a section of nodes $\Sigma_j = \mathcal{G}_j \cap \mathcal{R}_j$, such that $\Sigma_j \cap C = q_j$ and $\mathcal{R}_j \cap C = R_{j+1} \cup \cdots \cup R_h$.

Let $\mathfrak{M}_{2,ss}$ be the Artin stack of semi-stable genus two nodal curves. (A nodal curve is semi-stable if its smooth rational subcurves contain at least two nodes of the nodal curve.) By (successively) contracting rational subcurves that contain only one node of the ambient curves, we obtain a semi-stabilization $\mathfrak{M}_2 \rightarrow \mathfrak{M}_{2,ss}$. Possibly after shrinking and base changing \mathcal{V} , we can assume that there is a smooth chart $\underline{\mathcal{V}} \rightarrow \mathfrak{M}_{2,ss}$ with $\mathcal{X} \rightarrow \underline{\mathcal{V}}$ its universal family so that the composite $\mathcal{V} \rightarrow \mathfrak{M}_2^{\text{div}} \rightarrow \mathfrak{M}_{2,ss}$ lifts to a g together with a semi-stable contraction \tilde{g} as shown

$$(2.13) \quad \begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathfrak{M}_2^{\text{div}} \\ \downarrow g & & \downarrow \\ \underline{\mathcal{V}} & \longrightarrow & \mathfrak{M}_{2,ss} \end{array}, \quad \text{and} \quad \tilde{g} : \mathcal{C} \longrightarrow \bar{\mathcal{C}} = \mathcal{X} \times_{\underline{\mathcal{V}}} \mathcal{V}.$$

(Note that \tilde{g} contracts all rational tails of the curves in the family $\mathcal{C} \rightarrow \mathcal{V}$.) Let $\bar{\mathcal{D}}_i = \tilde{g}(\mathcal{D}_i)$, which are family of smooth divisors of $\bar{\mathcal{C}} \rightarrow \mathcal{V}$, and let

$$\mathcal{D}_R = \sum_{i=1}^r \mathcal{D}_i \subset \mathcal{C}, \quad \bar{\mathcal{D}}_R = \sum_{i=1}^r \bar{\mathcal{D}}_i \subset \bar{\mathcal{C}}.$$

Let $\bar{\rho} : \bar{\mathcal{C}} \rightarrow \mathcal{V}$ be the second projection of (2.13). By construction, the divisors $\mathcal{R}_1, \dots, \mathcal{R}_h$ are contracted under the morphism \tilde{g} , and

$$(2.14) \quad \tilde{g}^{-1}(\bar{\mathcal{D}}_i) = \mathcal{R}_1 + \cdots + \mathcal{R}_{e(i)} + \mathcal{D}_i,$$

where e is the index function (2.10).

Let r_i be the number of points in $R_i \cap \mathcal{D}$; thus $\sum_{i=1}^h r_i = r$. We set $\mathcal{R} = \sum_{i=1}^h (r_i + \cdots + r_h) \mathcal{R}_i$; then

$$(2.15) \quad \tilde{g}^* \mathcal{O}_{\bar{\mathcal{C}}}(\bar{\mathcal{D}}_R + \bar{\mathcal{A}} - \bar{\mathcal{B}}) = \mathcal{O}_{\mathcal{C}}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B}).$$

Then we have

$$(2.16) \quad \rho_* \mathcal{O}_{\mathcal{C}}(\mathcal{D}_R + \mathcal{A} - \mathcal{B}) \xrightarrow{\subset} \rho_* \mathcal{O}_{\mathcal{C}}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B}) = \bar{\rho}_* \mathcal{O}_{\bar{\mathcal{C}}}(\bar{\mathcal{D}}_R + \bar{\mathcal{A}} - \bar{\mathcal{B}}).$$

Here the identity follows from combining (2.15) and the identities

$$\rho_* \tilde{g}^* \mathcal{O}_{\bar{\mathcal{C}}}(\bar{\mathcal{D}}_R + \bar{\mathcal{A}} - \bar{\mathcal{B}}) = \bar{\rho}_* \tilde{g}_* \tilde{g}^* \mathcal{O}_{\bar{\mathcal{C}}}(\bar{\mathcal{D}}_R + \bar{\mathcal{A}} - \bar{\mathcal{B}}) = \bar{\rho}_* \mathcal{O}_{\bar{\mathcal{C}}}(\bar{\mathcal{D}}_R + \bar{\mathcal{A}} - \bar{\mathcal{B}}),$$

where the last equality holds since both \mathcal{C} and $\bar{\mathcal{C}}$ are smooth, $\tilde{g} : \mathcal{C} \rightarrow \bar{\mathcal{C}}$ is a proper divisorial contraction.

As before, we write $a_i = F \cap \mathcal{A}_i$, $a = a_1 + a_2$, and $b = F \cap \mathcal{B}$. Since they are in general position (the basic assumptions), we have $H^1(\mathcal{O}_F(rp_1 + a - b)) = 0$. Thus $\bar{\rho}_* \mathcal{O}_{\bar{\mathcal{C}}}(\bar{\mathcal{D}}_R + \bar{\mathcal{A}} - \bar{\mathcal{B}})$ is a free $\Gamma(\mathcal{O}_{\mathcal{V}})$ -module. For convenience, we denote

$$\mathbf{M} := \Gamma(\rho_* \mathcal{O}_{\mathcal{C}}(\mathcal{D}_R + \mathcal{A} - \mathcal{B})) \subset \bar{\mathbf{M}} := \Gamma(\bar{\rho}_* \mathcal{O}_{\bar{\mathcal{C}}}(\bar{\mathcal{D}}_R + \bar{\mathcal{A}} - \bar{\mathcal{B}})).$$

We now pick a basis of $\overline{\mathbf{M}}$. Since a and b are in general positions, by a simple vanishing argument, we see that $H^j(\mathcal{O}_F(a-b)) = 0$ for all j . Hence, the restriction homomorphism

$$H^0(\mathcal{O}_F(rq_1 + a - b)) \longrightarrow H^0(\mathcal{O}_F(rq_1)|_{rq_1})$$

is an isomorphism. Therefore, we can pick a basis s_1, \dots, s_r of the vector space $H^0(\mathcal{O}_F(rq_1 + a - b))$ so that s_i has vanishing order $(r-i)$ at q_1 .

Next, let $\mathcal{H}_i = (\zeta_{q_i} = 0) \subset \mathcal{V}$, and let $\mathcal{H} = \cup_{i=1}^r \mathcal{H}_i$. Let

$$\bar{\rho}_{\mathcal{H}} : \bar{\mathcal{C}}_{\mathcal{H}} := \bar{\mathcal{C}} \times_{\mathcal{V}} \mathcal{H} \longrightarrow \mathcal{H}$$

be the projection; let $\bar{\Sigma}_i = \tilde{g}(\Sigma_i)$, where $\Sigma_i = \mathcal{G}_i \cap \mathcal{R}_i$. For the same reason, $\overline{\mathbf{M}}_{\mathcal{H}} := (\bar{\rho}_{\mathcal{H}})_*(\mathcal{O}_{\bar{\mathcal{C}}_{\mathcal{H}}}(\bar{\mathcal{D}}_R + \bar{\mathcal{A}} - \bar{\mathcal{B}}))$ is a free $\Gamma(\mathcal{O}_{\bar{\mathcal{C}}_{\mathcal{H}}})$ -module, and has a basis $\bar{S}_{\mathcal{H},1}, \dots, \bar{S}_{\mathcal{H},r}$ extending the basis s_1, \dots, s_r along general directions so that $\bar{S}_{\mathcal{H},i}$ lifts to $\bar{S}'_{\mathcal{H},i} \in \Gamma(\mathcal{O}_{\bar{\mathcal{C}}_{\mathcal{H}}}(\bar{\mathcal{D}}_R + \bar{\mathcal{A}} - \bar{\mathcal{B}})(-(r-i)\bar{\Sigma}_i))$ such that $\bar{S}'_{\mathcal{H},i}$ are non-vanishing along $\bar{\Sigma}_i$.

By shrinking \mathcal{V} if necessary, we can extend $\bar{S}_{\mathcal{H},i}$ to $\bar{S}_i \in \overline{\mathbf{M}}$ along general directions so that $\overline{\mathbf{M}}$ (as a free $\Gamma(\mathcal{O}_{\mathcal{V}})$ -module) is generated freely by $\bar{S}_1, \dots, \bar{S}_r$. Thus by (2.16), $\tilde{g}^*\bar{S}_1, \dots, \tilde{g}^*\bar{S}_r$ freely generate the free $\Gamma(\mathcal{O}_{\mathcal{V}})$ -module $\Gamma(\mathcal{O}_{\mathcal{C}}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B}))$.

Sublemma. Recall $\zeta_{[[q_i]]} = \prod_{j=1}^i \zeta_{[q_j]}$, and let $S_i = \zeta_{[[q_i]]} \cdot \tilde{g}^*\bar{S}_i$. Then the submodule $\mathbf{M} \subset \overline{\mathbf{M}}$ is a free rank r $\Gamma(\mathcal{O}_{\mathcal{V}})$ -module generated by S_1, \dots, S_r .

Proof of Sublemma. We begin with more convention. Let $\mathcal{U}_i \subset \mathcal{C}$ be an affine open that contains the generic point ξ_i of \mathcal{R}_i . Let $u_i \in \Gamma(\mathcal{O}_{\mathcal{U}_i})$ be so that $\mathcal{R}_i \cap \mathcal{U}_i = (u_i = 0) \subset \mathcal{U}_i$. Since $\mathcal{R}_i \rightarrow \mathcal{H}_i$ is a family of rational curves and has general fibers isomorphic to \mathbb{P}^1 , we can find a rational function v_i on $\mathcal{U}_i \cap \mathcal{R}_i$ so that v_i restricts to general fibers of $\mathcal{R}_i \cap \mathcal{U}_i \rightarrow \mathcal{H}_i$ are birational maps to \mathbb{A}^1 . We next let ξ'_i be the generic point of $\bar{\Sigma}_i$. Since $\tilde{g}(\xi_i) = \xi'_i$, the field $K_i = \mathcal{O}_{\xi_i}$ contains $K'_i = \mathcal{O}_{\xi'_i}$ as its subfield. Because of our choice of v_i , we have $K_i = K'_i(v_i)$.

Let $\hat{\mathcal{O}}_{\mathcal{U}_i}$ be the formal completion of $\mathcal{O}_{\mathcal{U}_i}$ along ξ_i . Then $\hat{\mathcal{O}}_{\mathcal{U}_i} = K_i[[\hat{u}_i]]$, where \hat{u}_i is the image of u_i in $\hat{\mathcal{O}}_{\mathcal{U}_i}$. (For any $f \in \Gamma(\mathcal{O}_{\mathcal{V}})$, we denote by $[f]$ its image in $\hat{\mathcal{O}}_{\mathcal{U}_i}$ via the pullback $\mathcal{O}_{\mathcal{V}} \rightarrow \mathcal{O}_{\mathcal{U}_i}$ and the completion map.)

We next pick an open $\mathcal{U}'_i \subset \bar{\mathcal{C}}$ that contains the generic point ξ'_i of $\bar{\Sigma}_i$. By shrinking \mathcal{U}_i if necessary, we can assume $\tilde{g}(\mathcal{U}_i) \subset \mathcal{U}'_i$. We then fix a trivialization $\mathcal{O}_{\mathcal{U}'_i}(\bar{\mathcal{D}}_R + \bar{\mathcal{A}} - \bar{\mathcal{B}}) \cong \mathcal{O}_{\mathcal{U}'_i}$, and form the induced trivialization

$$(2.17) \quad \mathcal{O}_{\mathcal{U}_i}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B}) \cong \tilde{g}^* \mathcal{O}_{\mathcal{U}'_i}(\bar{\mathcal{D}}_R + \bar{\mathcal{A}} - \bar{\mathcal{B}}) \cong \mathcal{O}_{\mathcal{U}_i}.$$

Using this trivialization, we can identify any section in $\Gamma(\mathcal{O}_{\mathcal{U}_i}(\mathcal{R} + \bar{\mathcal{D}}_R + \bar{\mathcal{A}} - \bar{\mathcal{B}}))$ with a regular function in $\Gamma(\mathcal{O}_{\mathcal{U}_i})$, thus obtaining its image in $\hat{\mathcal{O}}_{\mathcal{U}_i}$. We denote this process by

$$\gamma \in \Gamma(\mathcal{O}_{\mathcal{U}_i}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B})) \mapsto [\gamma] \in \mathcal{O}_{\mathcal{U}_i}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B}) \otimes_{\mathcal{O}_{\mathcal{U}_i}} \hat{\mathcal{O}}_{\mathcal{U}_i} \cong \hat{\mathcal{O}}_{\mathcal{U}_i}.$$

Note that by our choice of \bar{S}_j , (i.e. its vanishing along $\bar{\Sigma}_j$), $\tilde{g}^* \bar{S}_j|_{\mathcal{U}_i}$ has vanishing order $r - j$ along \mathcal{R}_i ; thus $[\tilde{g}^* \bar{S}_j] = e_j(\hat{u}_i)^{r-j}$, where $e_j \neq 0 \in \hat{\mathcal{O}}_{\mathcal{U}_i} = K_i[[\hat{u}_i]]$ such that \hat{u}_i does not divide e_j .

We now prove the Sublemma. Let $F' \in \Gamma(\rho_* \mathcal{O}_{\mathcal{C}}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B}))$. By (2.16), it lies in \mathbf{M} if and only if

$$(2.18) \quad F'|_{\mathcal{R}} = 0 \in \Gamma(\mathcal{O}_{\mathcal{R}}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B})).$$

Let $\bar{r}_i = r_i + \dots + r_h$; it is the multiplicity of \mathcal{R}_i in \mathcal{R} . Since $\tilde{g}^* \bar{S}_j$ has vanishing order $r - j$ along \mathcal{R}_i , and since $\zeta_{[[q_j]]}$ has vanishing order $\max(0, j - (r - \bar{r}_i))$ along \mathcal{R}_i , $S_j = \zeta_{[[q_j]]} \cdot \tilde{g}^* \bar{S}_j$ has vanishing order $(r - j) + (j - r + \bar{r}_i) = \bar{r}_i$ along \mathcal{R}_i . Thus S_j lies in \mathbf{M} .

We now prove that \mathbf{M} is generated by S_j 's. Let $F \in \mathbf{M}$. Using (2.16), we denote its image in $\Gamma(\rho_* \mathcal{O}_{\mathcal{C}}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B}))$ by F' . Using the isomorphism below (2.16), and that $\bar{\mathbf{M}}$ is a free $\Gamma(\mathcal{O}_{\mathcal{V}})$ -module generated freely by \bar{S}_i , F' can be uniquely expressed as

$$F' = f_1 \cdot \tilde{g}^* \bar{S}_1 + \dots + f_r \cdot \tilde{g}^* \bar{S}_r, \quad f_i \in \Gamma(\mathcal{O}_{\mathcal{V}}).$$

By (2.18), $F'|_{\mathcal{R}} = 0$. In particular, $(u_i)^{\bar{r}_i}$ divides $F'|_{\mathcal{U}_i}$.

We claim that for any positive integer α , u_i^α divides $F'|_{\mathcal{U}_i}$ if and only if for all $1 \leq j \leq r$, $f_j|_{\mathcal{U}_i}$ has vanishing order at least $\max(0, \alpha - r - j)$ along \mathcal{R}_i . As the if direction is trivial, we prove the only if part. Taking their respective images in $\hat{\mathcal{O}}_{\mathcal{U}_i}$, u_i^α dividing $F'|_{\mathcal{U}_i}$ implies that

$$(2.19) \quad (\hat{u}_i)^\alpha \text{ divides } [F'] = \hat{f}_1 \cdot [\tilde{g}^* \bar{S}_1] + \dots + \hat{f}_r \cdot [\tilde{g}^* \bar{S}_r] \in K_i[[\hat{u}_i]].$$

By replacing v_i by v_i^{-1} if necessary, we can assume that $\hat{\zeta}_{q_i} = c_i \hat{u}_i v_i$, where $c_i \neq 0 \in K_i'$. Let α_j be the order of which $f_j|_{\mathcal{H}_i}$ is divisible by ζ_{q_i} , then $\hat{f}_j = e_j'(\hat{\zeta}_{q_i})^{\alpha_j}$, where $e_j' \neq 0 \in K_i'$. (The case $f_j = 0$ is trivially true and is ignored.)

Adding that $[\tilde{g}^* \bar{S}_j] = e_j(\hat{u}_i)^{r-j}$ mentioned earlier, (2.19) translates to that $(\hat{u}_i)^\alpha$ divides

$$\sum_{j=1}^r \hat{f}_j \cdot [\tilde{g}^* \bar{S}_j] = \sum_{j=1}^r e_j' (c_i \hat{u}_i v_i)^{\alpha_j} \cdot e_j(\hat{u}_i)^{r-j} = \sum_{k \geq 0} v_i^k \sum_{\alpha_j=k} (e_j' c_i^k e_j) (\hat{u}_i)^{k+r-j}$$

in $K_i[[\hat{u}_i]] = K_i'(v_i)[[u_i]]$. As \hat{u}_i does not divide $e_j' c_i^{\alpha_j} e_j$, this divisibility holds if and only if $(\hat{u}_i)^\alpha$ divides each individual $v_i^k (\hat{u}_i)^{k+r-j}$, which holds if and only if $\alpha \leq \alpha_j + r - j$ for each j where $f_j \neq 0$. This proves the claim.

Applying the claim to $\alpha = \bar{r}_i$, we conclude that $(u_i)^{\bar{r}_i}$ dividing $F'|_{\mathcal{U}_i}$ implies that $\alpha_j \geq \bar{r}_i - r + j$. This proves that $f_j/\zeta_{[[q_j]]}$, which is a meromorphic function on \mathcal{V} , is regular over an open subset of \mathcal{V} containing the generic point of \mathcal{H}_i , for each $1 \leq i \leq r$. Since $f_j/\zeta_{[[q_j]]}$ is regular over $\mathcal{V} - \cup_i \mathcal{H}_i$, by Hartogs Lemma, it is regular over \mathcal{V} .

This proves that F lies in the $\Gamma(\mathcal{O}_{\mathcal{V}})$ span of S_1, \dots, S_r . Since \mathbf{M} is a rank r free $\Gamma(\mathcal{O}_{\mathcal{V}})$ -modules, it is freely generated by S_1, \dots, S_r . This proves the Sublemma. \square

We continue the proof of Proposition 2.6.1. Let

$$b_{ij} = \bar{S}_j|_{\bar{\mathcal{A}}_i} \in \Gamma(\rho_* \mathcal{O}_{\mathcal{A}_i}(\mathcal{A}_i) \cong \Gamma(\mathcal{O}_{\mathcal{V}})).$$

Then under $\{S_1, \dots, S_r\}$ and (2.7), φ^R takes the form

$$(2.20) \quad B = \begin{bmatrix} b_{11}\zeta_{[\delta_1]} & b_{12}\zeta_{[\delta_1]}\zeta_{[\delta_2]} & b_{13}\zeta_{[\delta_1]}\zeta_{[\delta_2]}\zeta_{[\delta_3]} & \cdots \\ b_{21}\zeta_{[\delta_1]} & b_{22}\zeta_{[\delta_1]}\zeta_{[\delta_2]} & b_{23}\zeta_{[\delta_1]}\zeta_{[\delta_2]}\zeta_{[\delta_3]} & \cdots \end{bmatrix}.$$

By Proposition 2.5.1, under a basis change of \mathbf{M} , φ^R also takes the form

$$(2.21) \quad A = \begin{bmatrix} a_{11}\zeta_{[\delta_1, a_1]} & a_{12}\zeta_{[\delta_2, a_1]} & a_{13}\zeta_{[\delta_3, a_1]} & \cdots \\ a_{21}\zeta_{[\delta_1, a_2]} & a_{22}\zeta_{[\delta_2, a_2]} & a_{23}\zeta_{[\delta_3, a_2]} & \cdots \end{bmatrix},$$

where $a_{ij} \in \Gamma(\mathcal{O}_{\mathcal{V}}^*)$. Therefore, $B = A(\mu_{ij})$ for a $(\mu_{ij}) \in GL(r, \mathcal{O}_{\mathcal{V}})$. Since $\zeta_{[q_1, a_1]}$ divides the first row of (2.21), it divides the first row of (2.20) as well. Since $\zeta_{[q_1, a_1]}$ are coprime with $\zeta_{[\delta_j]}$ for all $1 \leq j \leq r$, we conclude $\zeta_{[q_1, a_1]}$ divides b_{1j} . Hence we can write

$$(2.22) \quad b_{1j} = \hat{c}_{1j}\zeta_{[q_1, a_1]} \quad \forall 1 \leq j \leq r.$$

Let $(\nu_{ij}) = (\mu_{ij})^{-1}$. Then,

$$a_{11}\zeta_{[\delta_1, a_1]} = \nu_{11}\hat{c}_{11}\zeta_{[\delta_1, a_1]} + \nu_{21}\hat{c}_{12}\zeta_{[\delta_1, a_1]}\zeta_{[\delta_2]} + \cdots.$$

Hence, $a_{11} = \nu_{11}\hat{c}_{11} + \nu_{21}\hat{c}_{12}\zeta_{[\delta_2]} + \cdots$. Thus $0 \neq a_{11}(0) = \nu_{11}(0)\hat{c}_{11}(0)$. Shrinking \mathcal{V} if necessary, we can assume $\hat{c}_{11} \in \Gamma(\mathcal{O}_{\mathcal{V}}^*)$. This brings the first row of (2.20) to the desired form. Similar arguments can be applied to the second row. This proves the form in (2.12).

Since $s_1, s_2 \in H^0(\mathcal{O}_F(rq_1 + a - b))$ have vanishing order $r - 1$ and $r - 2$ respectively at q_1 , one verifies directly that the image of $H^0(\mathcal{O}_F(2q_1 + a - b))$ in $H^0(\mathcal{O}_F(rq_1 + a - b))$ (via the inclusion) is the subspace spanned by s_1 and s_2 . By identifying the image of $H^0(\mathcal{O}_F(2q_1 + a - b))$ with this subspace, we obtain a homomorphism

$$H^0(\mathcal{O}_F(2q_1 + a - b)) \xrightarrow{s_{12}} \mathbf{k}^2,$$

where $s_{12} = \begin{bmatrix} s_1(a_1) & s_2(a_1) \\ s_1(a_2) & s_2(a_2) \end{bmatrix} = \begin{bmatrix} \hat{c}_{11} & \hat{c}_{12} \\ \hat{c}_{21} & \hat{c}_{22} \end{bmatrix} (0)$. Thus, $\text{rank} \begin{bmatrix} \hat{c}_{11} & \hat{c}_{12} \\ \hat{c}_{21} & \hat{c}_{22} \end{bmatrix} (0) = 1$ if and only if $h^0(\mathcal{O}_F(2q_1 - b)) = 1$. Since b is a general point, the evaluation homomorphism at the point b , $H^0(\mathcal{O}_F(2q_1)) \rightarrow \mathbf{k}$, is surjective. By the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_F(2q_1 - b)) \longrightarrow H^0(\mathcal{O}_F(2q_1)) \longrightarrow \mathbf{k} \longrightarrow 0,$$

$h^0(\mathcal{O}_F(2q_1 - b)) = 1$ is equivalent to $h^0(\mathcal{O}_F(2q_1)) = 2$, which is further equivalent to that q_1 is a Weierstrass point. Thus, $\theta_{12}(0) = 0$ if and only if q_1 is Weierstrass, which implies δ_1 and δ_2 are conjugate. We may further require \bar{S}_j extending s_j , $j = 1, 2$ so that θ_{12} vanishes along the the locus on which \mathcal{D}_1 and \mathcal{D}_2 are conjugate. Mimicking the proof of Proposition 4.13 of [5], one can show that the vanishing order of θ_{12} is 1. This proves Parts (1) and (2). We omit further details.

Likewise, for each $2 \leq \ell \leq r$, the natural image of $H^0(\mathcal{O}_F(\ell q_1 + a - b))$ in $H^0(\mathcal{O}_F(rq_1 + a - b))$ is the subspace spanned by s_1, \dots, s_ℓ . Assume that $\text{rank} \begin{bmatrix} \hat{c}_{11} & \cdots & \hat{c}_{1\ell} \\ \hat{c}_{21} & \cdots & \hat{c}_{2\ell} \end{bmatrix} (0) \leq 1$. The argument in the previous paragraph implies $h^0(\mathcal{O}_F(\ell q_1 - b)) = \ell - 1$, and consequently $h^0(\mathcal{O}_F(\ell q_1)) = \ell$. However, by the Riemann-Roch Theorem, $h^0(\mathcal{O}_F(\ell q_1)) \leq \ell$ with the equality holds if and only if q_1 is on a non-separating bridge. This shows Parts (3).

Obviously the argument in the above paragraph applies not only to the center $0 \in \mathcal{V}$, but also to any $z \in \mathcal{V}$ satisfying the displayed equalities in Part (4). This implies the former statement of Part (4). Parallel to the proof of Part (2), for any $3 \leq j \leq r$, if θ_{1j} vanishes at 0, it vanishes at the first order. The latter statement of Part (4) then follows from the assumption that \bar{S}_i extends s_i , $3 \leq i \leq r$, in general directions. This completes the proof of Proposition 2.6.1. \square

2.7. Further relations. In this section, assume $\delta_u, \delta_v, \delta_w \in D$ so that there does not exist any maximal tail rational chain containing a pair of the three points. Each of them may belong to the core F or a tail component. If δ_u lies on a tail, we make the following assumptions.

- (D1) We assume δ_u is the *first* marked point in D on one (and thus all) maximal tail rational chain containing δ_u , with respect to the order (2.9).
- (D2) If a maximal tail rational chain contains marked points in D other than δ_u , we denote by $\delta_{u'}$ the *second* marked point of the chain with respect to the order (2.9).

We make analogous assumptions to δ_v and δ_w if they are not on the core.

By Proposition 2.5.1, under the trivializations

$$\rho_* \mathcal{M}(\mathcal{D}_u) \cong \mathcal{O}_{\mathcal{V}}, \quad \rho_* \mathcal{M}(\mathcal{D}_v) \cong \mathcal{O}_{\mathcal{V}} \quad \text{and} \quad \rho_* \mathcal{O}_{\mathcal{A}_s}(\mathcal{A}_s) \cong \mathcal{O}_{\mathcal{V}}, \quad s = 1, 2,$$

the restriction homomorphism

$$\varphi_{uv} : \rho_* \mathcal{M}(\mathcal{D}_u) \oplus \rho_* \mathcal{M}(\mathcal{D}_v) \longrightarrow \rho_* \mathcal{O}_{\mathcal{A}}(\mathcal{A})$$

can be represented by a matrix

$$\begin{bmatrix} c_{1u} \zeta_{[\delta_u, a_1]} & c_{1v} \zeta_{[\delta_v, a_1]} \\ c_{2u} \zeta_{[\delta_u, a_2]} & c_{2v} \zeta_{[\delta_v, a_2]} \end{bmatrix},$$

where c_{iu} and c_{iv} are as in Proposition 2.5.1. Set

$$(2.23) \quad \lambda_{uv} \equiv \det \begin{bmatrix} c_{1u} & c_{1v} \\ c_{2u} & c_{2v} \end{bmatrix} \in \Gamma(\mathcal{O}_{\mathcal{V}}).$$

Corollary 2.7.1. *Assume that $\langle \delta_u \rangle = \langle \delta_v \rangle \equiv q_1$ but δ_u, δ_v do not belong to any common maximal tail rational chain.*

- (1) *If q_1 is not Weierstrass, then $\lambda_{uv} \sim \zeta_{[\delta_u \wedge \delta_v]}$.*
- (2) *If q_1 is Weierstrass, there then exists $\theta_{uv} \in \Gamma(\mathcal{O}_{\mathcal{V}})$ vanishing to the first order along the locus on which \mathcal{D}_u and \mathcal{D}_v are conjugate such that $\lambda_{uv} = \theta_{uv} \zeta_{[\delta_u \wedge \delta_v]}$. Moreover, with $\delta_{u'}, \delta_{v'}$ as in (D2) and $\theta_{u,u'}, \theta_{v,v'}$ as in Proposition 2.6.1, the zero loci of θ_{uv} , $\theta_{u,u'}$, and $\theta_{v,v'}$ are transverse.*

- (3) If q_1 is on a maximal non-separating bridge $B[p, r]$ of F , then the function θ_{uv} in Part (2) satisfies the following property: for every $z \in \mathcal{V}$, if there exist $p_1 \in N_{[q_1, p]}$ and $r_1 \in N_{[q_1, r]}$ such that $\zeta_{p_1}(z) = \zeta_{r_1}(z) = 0$, then $\theta_{uv}(z) = 0$.

Proof. We follow the notation in the proof of Proposition 2.6.1.

Let R_u and R_v be respectively maximal tail rational chains containing δ_u and δ_v , and R'_u and R'_v be respectively the subchains as the closures of $R_u \setminus (R_u \cap R_v)$ and $R_v \setminus (R_u \cap R_v)$. We take $\bar{\mathcal{V}}$ to be obtained from \mathcal{V} by collapsing the subchains R'_u and R'_v , and call the resulting chain \bar{R}_{uv} . The image of $D \cap (R_u \cup R_v)$ in \bar{R}_{uv} is denoted by \bar{D}_{uv} . For each $\delta \in D \cap (R_u \cup R_v)$, its image in \bar{D}_{uv} is denoted by $\bar{\delta}$. Note that the node

$$q_{uv} \equiv \delta_u \wedge \delta_v$$

lies on the smooth rational curve C_{uv} that contains \bar{D}_{uv} and separates it from the core F .

By Proposition 2.5.1, over the chart $\bar{\mathcal{V}}$,

$$\varphi^{\bar{R}_{uv}} = \begin{bmatrix} c_{1u}\zeta_{[\bar{\delta}_u, a_1]} & c_{1v}\zeta_{[\bar{\delta}_v, a_1]} & \cdots \\ c_{2u}\zeta_{[\bar{\delta}_u, a_2]} & c_{2v}\zeta_{[\bar{\delta}_v, a_2]} & \cdots \end{bmatrix}.$$

If q_1 is not Weierstrass, by Part (1) of Proposition 2.6.1, we observe that

$$\det \begin{bmatrix} c_{1u} & c_{1v} \\ c_{2u} & c_{2v} \end{bmatrix} \sim \zeta_{[q_{uv}]} \in \Gamma(\mathcal{O}_{\bar{\mathcal{V}}}).$$

Pulling $\varphi^{\bar{R}_{uv}}$ back to \mathcal{V} , we obtain

$$\varphi^{R_u \cup R_v} = \begin{bmatrix} c_{1u}\zeta_{[\delta_u, a_1]} & c_{1v}\zeta_{[\delta_v, a_1]} & \cdots \\ c_{2u}\zeta_{[\delta_u, a_2]} & c_{2v}\zeta_{[\delta_v, a_2]} & \cdots \end{bmatrix}, \quad \lambda_{uv} \sim \zeta_{[q_{uv}]} = \zeta_{[\delta_u \wedge \delta_v]},$$

where λ_{uv} is as in (2.23). This completes Part (1). Part (2) follows from Proposition 2.6.1 and Part (1). Part (3) holds because for every z satisfying the assumption of (3), $\langle \mathcal{D}_u(z) \rangle$ and $\langle \mathcal{D}_v(z) \rangle$ are on the same non-separating bridge of the core of z , hence are conjugate to each other. \square

Corollary 2.7.2. *Assume $\langle \delta_u \rangle = \langle \delta_v \rangle = \langle \delta_w \rangle \equiv q_1$ so that each pair among δ_u , δ_v , and δ_w do not belong to any common maximal tail rational chain. If q_1 is Weierstrass, then with θ_{uv}, θ_{vw} as in Part (2) of Corollary 2.7.1, there exist $f_{uvw}, g_{uvw} \in \Gamma(\mathcal{O}_{\mathcal{V}})$ such that*

$$\theta_{uw} \sim f_{uvw}\theta_{uv} + g_{uvw}\zeta_{[\delta_v \wedge \delta_w]}.$$

Moreover, if q_1 does not lie on any bridge of the core F , then $g_{uvw} \in \Gamma(\mathcal{O}_{\mathcal{V}}^)$; if q_1 lies on a non-separating bridge of F , the zero loci of θ_{uv} and g_{uvw} are then transverse.*

Proof. Let R_u , R_v , and R_w be maximal tail rational chains containing δ_u , δ_v , and δ_w , respectively. We denote by R'_u , R'_v , and R'_w respectively the subchains as the closure of $R_u \setminus (R_u \cap R_v \cap R_w)$, $R_v \setminus (R_u \cap R_v \cap R_w)$, and

$R_w \setminus (R_u \cap R_v \cap R_w)$. We take $\overline{\mathcal{V}}$ to be obtained from \mathcal{V} by collapsing the subchains R'_u , R'_v , and R'_w , and call the resulting chain \overline{R}_{uvw} .

The rest of proof is analogous to that of Corollary 2.7.1, but in addition to the application of Parts (1) and (2) of Proposition 2.6.1, Parts (3) and (4) of the same proposition are also needed. We omit further details. \square

Corollary 2.7.3. *Assume that $\langle \delta_u \rangle \neq \langle \delta_v \rangle$, i.e. no tail contains both δ_u and δ_v .*

- (1) *If δ_u is not conjugate to δ_v , then $\lambda_{uv} \in \Gamma(\mathcal{O}_{\mathcal{V}}^*)$.*
- (2) *If δ_u is conjugate to δ_v , then λ_{uv} vanishes to the first order along the locus on which \mathcal{D}_u and \mathcal{D}_v are conjugate. Moreover, with $\delta_{u'}$, $\delta_{v'}$ as in (D2) and $\theta_{u,u'}$, $\theta_{v,v'}$ as in Proposition 2.6.1, the zero loci of λ_{uv} , $\theta_{u,u'}$, and $\theta_{v,v'}$ are transverse.*
- (3) *If δ_u is conjugate to δ_v , and $\langle \delta_u \rangle$ is on a maximal non-separating bridge $B[p, r]$ of F , then so is $\langle \delta_v \rangle$. Moreover, for every $z \in \mathcal{V}$, if there exist $p_1 \in N_{[q_1, p]}$ and $r_1 \in N_{[q_1, r]}$ such that $\zeta_{p_1}(z) = \zeta_{r_1}(z) = 0$, then $\lambda_{uv}(z) = 0$. Furthermore,*
 - *if $\langle \delta_u \rangle = \langle \delta_w \rangle$, and δ_u, δ_w do not belong to any common maximal tail rational chain, then with θ_{uw} as in Corollary 2.7.1, the zero loci of λ_{uv} and θ_{uw} are transverse;*
 - *if $\langle \delta_u \rangle, \langle \delta_v \rangle$, and $\langle \delta_w \rangle$ are pairwise distinct and belong to the same non-separating bridge, then the zero loci of λ_{uv} and λ_{uw} are transverse.*

Proof. The statements follow from a semi-stable reduction argument parallel to the proof of Proposition 2.6.1. We omit further details. \square

2.8. Weierstrass and conjugate loci vs. modular parameters. Let $\mathcal{W}_{k;i}$ and $\mathcal{K}_{k;i,j}$ be the Weierstrass and the conjugate loci in $\overline{\mathcal{M}}_{2,k}$, respectively; c.f. §2.2. We have shown that they are Cartier divisors, and for $k=1$ (resp. $k=2$), $\mathcal{W}_{1;1} = \mathcal{W}$ (resp. $\mathcal{K}_{2;1,2} = \mathcal{K}$) is also smooth.

For $k \geq 1$, the locus $\mathcal{W}_{k;i}$ is the pullback of $\mathcal{W}_{1;1}$ via the forgetful map $\pi_{k;i} : \overline{\mathcal{M}}_{2,k} \rightarrow \overline{\mathcal{M}}_{2,1}$. Each $\mathcal{W}_{k;i}$ is possibly singular along the closed locus $\mathcal{W}_{k;i}^{\text{n-s}}$ where the i -th marked point is on a non-separating bridge; it is smooth on $\mathcal{W}_{k;i} \setminus \mathcal{W}_{k;i}^{\text{n-s}}$. More precisely, for

$$x = (C; t_1, \dots, t_n) \in \mathcal{W}_{k;i}^{\text{n-s}} \quad (\subset \mathcal{W}_{k;i}),$$

let $B = B[\mathbf{p}, \mathbf{q}] \subset C$ be the maximal non-separating bridge containing the marked point t_i . The complement of B in C is an elliptic curve C_1 . We denote by $x' = (C'; t'_i)$ the image of x in $\mathcal{W}_{1;1}$. Then, C' consists of an inseparable elliptic curve C'_1 and a smooth rational curve B' containing t'_i such that $C'_1 \cap B' = \{\mathbf{p}', \mathbf{q}'\}$.

Since $\mathcal{W}_{1;1}$ is smooth, there exist a smooth chart $\mathcal{V}' \subset \overline{\mathcal{M}}_{2,1}$ containing x' and a local parameter λ' on \mathcal{V}' such that

$$\mathcal{W}_{1;1} \cap \mathcal{V}' = \{\lambda' = 0\}.$$

Let $\{\zeta'_q\}$ be a set of modular parameters on \mathcal{V}' as in §2.5, centered at x' . Then,

$$\mathcal{W}_{1;1}^{\text{n-s}} \cap \mathcal{V}' = \{\zeta'_{\mathfrak{p}'} = \zeta'_{\mathfrak{q}'} = 0\}.$$

This implies $\lambda' = 0$ whenever $\zeta'_{\mathfrak{p}'} = \zeta'_{\mathfrak{q}'} = 0$, so there exist $f, g \in \Gamma(\mathcal{O}_{\mathcal{V}'})$ so that

$$\lambda' = f\zeta'_{\mathfrak{p}'} + g\zeta'_{\mathfrak{q}'}.$$

Since λ' is a local parameter, by symmetry we have

$$f|_{\mathcal{W}_{1;1}^{\text{n-s}} \cap \mathcal{V}'}, g|_{\mathcal{W}_{1;1}^{\text{n-s}} \cap \mathcal{V}'} \in \Gamma(\mathcal{O}_{\mathcal{W}_{1;1}^{\text{n-s}} \cap \mathcal{V}'})^*.$$

Thus, shrinking \mathcal{V}' if necessary, we can use $f\zeta'_{\mathfrak{p}'}$ and $g\zeta'_{\mathfrak{q}'}$ as new modular parameters instead of $\zeta'_{\mathfrak{p}'}$ and $\zeta'_{\mathfrak{q}'}$, respectively, and obtain

$$(2.24) \quad \lambda' = \zeta'_{\mathfrak{p}'} + \zeta'_{\mathfrak{q}'}$$

Lemma 2.8.1. *With notation as above, there exist a smooth chart \mathcal{V} containing x and a set of modular parameters $\{\zeta_q\}$ on \mathcal{V} centered at x so that*

$$\mathcal{W}_{k;i} \cap \mathcal{V} = \{\zeta_{[t_i, \mathfrak{p}]} + \zeta_{[t_i, \mathfrak{q}]} = 0\}.$$

Proof. Notice that up to a nowhere vanishing function, the modular parameter $\zeta'_{\mathfrak{p}'}$ (resp. $\zeta'_{\mathfrak{q}'}$) pulls back to the product $\zeta_{[t_i, \mathfrak{p}]}$ (resp. $\zeta_{[t_i, \mathfrak{q}]}$) on \mathcal{V} . This, along with (2.24), establishes Lemma 2.8.1. \square

Analogously, when $k > 2$, the locus $\mathcal{K}_{k;i,j}$ is the pullback of $\mathcal{W}_{2;1,2}$ via the forgetful map $\pi_{k;i,j} : \overline{\mathcal{M}}_{2,k} \rightarrow \overline{\mathcal{M}}_{2,2}$. Each $\mathcal{K}_{k;i,j}$ is possibly singular along the closed locus $\mathcal{K}_{k;i,j}^{\text{n-s}} \subset \mathcal{K}_{k;i,j}$ where the i, j -th marked points are on a same non-separating bridge; it is smooth on $\mathcal{K}_{k;i,j} \setminus \mathcal{K}_{k;i,j}^{\text{n-s}}$. The following statement is parallel to Lemma 2.8.1, so is its proof.

Lemma 2.8.2. *Given $x = (C; t_1, \dots, t_k) \in \mathcal{K}_{k;i,j}^{\text{n-s}}$, let $B = B[\mathfrak{p}, \mathfrak{q}] \subset C$ be the maximal non-separating bridge containing the marked points t_i and t_j . W.l.o.g. assume that $N_{[t_i, \mathfrak{p}]} \subset N_{[t_j, \mathfrak{p}]}$. There then exist a smooth chart \mathcal{V} containing x and a set of modular parameters $\{\zeta_q\}$ on \mathcal{V} centered at x so that*

$$\mathcal{K}_{k;i,j} \cap \mathcal{V} = \{\zeta_{[t_i, \mathfrak{p}]} + \zeta_{[t_j, \mathfrak{q}]} = 0\}.$$

3. LOCAL PROPERTIES OF SEQUENCES OF BLOWUPS

In this section, we describe the local models of the globally defined modular blowups in §4.

The following notation will be assumed throughout §3. For all $j, k \in \mathbb{Z}_{>0}$, we write

$$(3.1) \quad [j, k] \equiv \{j, j+1, \dots, k\}, \quad \llbracket k \rrbracket \equiv [1, k]$$

for conciseness. Given an affine smooth chart \mathcal{V} of a stack \mathfrak{M} and $f, g \in \Gamma(\mathcal{O}_{\mathcal{V}})$, we write

$$f \sim g \quad \text{if} \quad f/g \in \Gamma(\mathcal{O}_{\mathcal{V}})^*.$$

3.1. Local parameters. Let \mathcal{V} be an affine smooth chart of a stack \mathfrak{M} with local parameters z_1, \dots, z_r . Assume $Z \subset \mathfrak{M}$ is a closed codimension ℓ substack locally given by

$$z_1 = \dots = z_\ell = 0.$$

Let $\widetilde{\mathfrak{M}}$ be the blowup of \mathfrak{M} along Z , $\pi: \widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ be the projection, and \mathcal{E} be the exceptional divisor. If \mathfrak{N} is a substack of \mathfrak{M} , we denote by

$$\check{\mathfrak{N}} \equiv \text{PT}(\mathfrak{N}) \equiv \overline{\pi^{-1}(\mathfrak{N} \setminus Z)} \quad \text{and} \quad \check{\mathfrak{N}} \equiv \pi^{-1}(\mathfrak{N})$$

the proper transform and the pullback of \mathfrak{N} in $\widetilde{\mathfrak{M}}$, respectively. Set

$$(3.2) \quad \check{z}_i \equiv z_i \circ \pi \in \Gamma(\mathcal{O}_{\pi^{-1}(\mathcal{V})}) \quad \forall i \in \llbracket r \rrbracket.$$

Every $\check{x} \in \mathcal{E} \cap \pi^{-1}(\mathcal{V})$ is in the form

$$\check{x} = (0, \dots, 0, \check{z}_{\ell+1}, \dots, \check{z}_r; [x_1, \dots, x_\ell]).$$

Let

$$(3.3) \quad \Delta_{\check{x}} = \{k \in \llbracket \ell \rrbracket : x_k \neq 0\} \subset \llbracket \ell \rrbracket.$$

Since $\check{x} \in \mathcal{E}$, the entries x_1, \dots, x_ℓ cannot be zero simultaneously. Thus, $\Delta_{\check{x}} \neq \emptyset$.

For each nonempty $S \subset \llbracket \ell \rrbracket$, let

$$(3.4) \quad \mathcal{E}(S) = \{\check{x} \in \mathcal{E} \cap \pi^{-1}(\mathcal{V}) : \Delta_{\check{x}} \subset S\} \subset \mathcal{E} \cap \pi^{-1}(\mathcal{V}).$$

Such defined $\mathcal{E}(S)$ is smooth, closed, and of codimension $\ell - |S| + 1$ in $\pi^{-1}(\mathcal{V})$.

Lemma 3.1.1. *For every $\check{x} \in \mathcal{E} \cap \pi^{-1}(\mathcal{V})$, there is an affine smooth chart $\check{\mathcal{V}}_{\check{x}} \subset \pi^{-1}(\mathcal{V})$ containing \check{x} so that*

$$\check{z}_i \mid \check{z}_j, \quad \text{PT}\{z_j = 0\} \cap \check{\mathcal{V}}_{\check{x}} = \left\{ \frac{\check{z}_j}{\check{z}_i} = 0 \right\} \quad \forall i \in \Delta_{\check{x}}, j \in \llbracket \ell \rrbracket.$$

Moreover, if $i \in \Delta_{\check{x}}$ and $j \in \llbracket \ell \rrbracket \setminus \Delta_{\check{x}}$, then $\check{z}_j / \check{z}_i$ is a regular function on $\check{\mathcal{V}}_{\check{x}}$.

Proof. Every point in $\check{\mathcal{V}}_{\check{x}}$ is in the form $(\check{z}_1, \dots, \check{z}_r; [y_1, \dots, y_\ell])$ so that

$$\check{z}_i y_j = \check{z}_j y_i \quad \forall i, j \in \llbracket \ell \rrbracket.$$

Shrinking $\check{\mathcal{V}}_{\check{x}}$ if necessary, we may assume y_i does not vanish on $\check{\mathcal{V}}_{\check{x}}$ for all $i \in \Delta_{\check{x}}$. Thus, $\check{z}_j = (y_j / y_i) \check{z}_i$ for all $j \in \llbracket \ell \rrbracket$. This leads to the two statements in the display. The last statement follows from a direct check. \square

Let $\check{\mathcal{V}}_{\check{x}}$ be as in Lemma 3.1.1. For each $i \in \Delta_{\check{x}}$, the zero locus of \check{z}_i on $\check{\mathcal{V}}_{\check{x}}$ is the same as $\mathcal{E} \cap \check{\mathcal{V}}_{\check{x}}$. We select an arbitrary $i \in \Delta_{\check{x}}$ and set

$$\varepsilon = \check{z}_i \mid_{\check{\mathcal{V}}_{\check{x}}} \in \Gamma(\mathcal{O}_{\check{\mathcal{V}}_{\check{x}}}).$$

Corollary 3.1.2. *Let $\check{\mathcal{V}}_{\check{x}}$ be the affine smooth chart in Lemma 3.1.1. Then,*

$$(3.5) \quad \varepsilon; \quad \check{z}_j \equiv \frac{\check{z}_j}{\varepsilon}, \quad j \in \llbracket \ell \rrbracket \setminus \Delta_{\check{x}}; \quad \check{z}_k, \quad k \in \llbracket \ell + 1, r \rrbracket,$$

form a subset of a family of local parameters on $\check{\mathcal{V}}_{\check{x}}$.

Proof. By Lemma 3.1.1, ε and \check{z}_j , $j \in \llbracket \ell \rrbracket \setminus \Delta_{\check{x}}$ can be considered as local parameters. In addition, it follows from a direct check that on $\pi^{-1}(\mathcal{V})$,

$$\{\check{z}_k = 0\} = \text{PT}\{z_k = 0\} \quad \forall k \in \llbracket \ell + 1, r \rrbracket. \quad \square$$

We point out that $\check{\mathcal{V}}_{\check{x}}$ contains other local parameters describing the positions of points in

$$\{p \in \mathcal{E} \cap \check{\mathcal{V}}_{\check{x}} : \Delta_p = \Delta_{\check{x}}\} \subset \mathcal{E} \cap \check{\mathcal{V}}_{\check{x}}.$$

We will always denote by \check{z}_j , $j \in \llbracket r \rrbracket$, the pullback of z_j as in (3.2), and by \check{z}_j , $j \in \llbracket \ell \rrbracket \setminus \Delta_{\check{x}}$, the proper transform of z_j as in (3.5).

3.2. Locally tree-compatible blowups. Given a connected graph γ , we denote by

$$\text{Ver}(\gamma) \quad \text{and} \quad \text{Edg}(\gamma)$$

the collections of the vertices and the edges of γ , respectively. If $\text{Ver}(\gamma) = \emptyset$, we denote the empty graph by \emptyset .

If in addition γ is a rooted tree with the root o , then $\text{Ver}(\gamma)$ is endowed with a natural partial order $<$, called the *tree order*, so that $v < v'$ if and only if $v \neq v'$ and v belongs to a path between o and v' . Each edge $e \in \text{Edg}(\gamma)$ has two endpoints v_e^- and v_e^+ in $\text{Ver}(\gamma)$ so that

$$v_e^- < v_e^+.$$

This induces a partial order $<$ on $\text{Edg}(\gamma)$, also called the *tree order*, so that

$$e < e' \iff v_e^+ \leq v_{e'}^-.$$

For each $E \subset \text{Edg}(\gamma)$, we write

$$(3.6) \quad \begin{aligned} E^< &= \bigcup_{e \in E} \{e' \in \text{Edg}(\gamma) : e' < e\}, & E^{\leq} &= E^< \sqcup E, \\ E^> &= \bigcup_{e \in E} \{e' \in \text{Edg}(\gamma) : e' > e\}, & E^{\geq} &= E^> \sqcup E. \end{aligned}$$

The root o is the unique minimal element of $\text{Ver}(\gamma)$; each of the minimal elements of $\text{Edg}(\gamma)$, called a **pivotal edge**, contains o as an endpoint. We denote by $E_o(\gamma)$ the collection of the pivotal edges. When the context is clear, we write

$$E_o = E_o(\gamma).$$

For each $e \in \text{Edg}(\gamma)$, let $\langle e \rangle \in E_o$ be such that

$$\langle e \rangle \leq e.$$

The maximal elements of $\text{Ver}(\gamma)$ and $\text{Edg}(\gamma)$ are respectively called **terminal vertices** and **terminal edges**; they form the subsets

$$\text{Ver}(\gamma)^t \subset \text{Ver}(\gamma) \quad \text{and} \quad \text{Edg}(\gamma)^t \subset \text{Edg}(\gamma),$$

respectively.

For each $e \in \text{Edg}(\gamma)$, there is a unique induced subgraph γ_e of γ satisfying

$$\text{Edg}(\gamma_e) = \{e' \in \text{Edg}(\gamma) : e' > e\} \subset \text{Edg}(\gamma).$$

Such defined γ_e is a rooted tree with the root v_e^+ ; we call γ_e a **branch** of γ . If in addition $e \in E_o$, we call γ_e a **pivotal branch**.

A subset $E \subset \text{Edg}(\gamma)$ is called a **traverse section** of γ if for each $v \in \text{Ver}(\gamma)^t$, each path between o and v contains exactly one element of E . The collection of all traverse sections of γ is denoted by $\text{ET}(\gamma)$; it is endowed with a partial order $<$ such that for any $E, E' \in \text{ET}(\gamma)$,

$$(3.7) \quad E < E' \iff (E \neq E') \text{ and } (\forall e' \in E', \exists e \in E \text{ s.t. } e \leq e').$$

The unique minimal element of $\text{ET}(\gamma)$ is E_o .

Let γ be a rooted tree and \mathcal{V} be an affine smooth chart of a stack \mathfrak{M} . If there exists a collection of local parameters on \mathcal{V} so that a subset of which can be written as

$$(3.8) \quad \{z_e \in \Gamma(\mathcal{O}_{\mathcal{V}}) : e \in \text{Edg}(\gamma)\},$$

then (3.8) is called a γ -**labeled subset** of local parameters on \mathcal{V} . We set

$$(3.9) \quad z_{[e]} \equiv \prod_{e' \leq e} z_{e'} \quad \forall e \in \text{Edg}(\gamma).$$

Every *nonempty* subset E of $\text{Edg}(\gamma)$ determines a closed locus \mathcal{Z}_E in \mathcal{V} by

$$\mathcal{Z}_E = \{x \in \mathcal{V} : z_e(x) = 0 \quad \forall e \in E\}.$$

In addition, we set

$$\mathcal{Z}_{\gamma} \equiv \mathcal{Z}_{\text{Edg}(\gamma)}, \quad \mathcal{Z}_{\emptyset} \equiv \emptyset.$$

With notation as above, let

$$(3.10) \quad \pi : \widetilde{\mathfrak{M}} \longrightarrow \mathfrak{M} \quad \text{and} \quad \pi_{(k)} : \widetilde{\mathfrak{M}}_{(k)} \longrightarrow \mathfrak{M}, \quad k \in \llbracket \ell \rrbracket,$$

be the blowup of \mathfrak{M} successively along the proper transforms of a sequence of closed substacks

$$Z_1, \dots, Z_{\ell} \subset \mathfrak{M}$$

and the same blowup of \mathfrak{M} after the k -th step, respectively.

Definition 3.2.1. *The blowup π in (3.10) is said to be γ -compatible on \mathcal{V} if there exist a partition of $\text{ET}(\gamma)$:*

$$\text{ET}(\gamma) = \bigsqcup_{k=1}^{\ell} \text{ET}(\gamma)_k \equiv \bigsqcup_{k=1}^{\ell} \{E_k^1, \dots, E_k^{n_k}\}$$

and a γ -labeled subset of local parameters on \mathcal{V} such that

(C1) for each $k \in \llbracket \ell \rrbracket$,

$$Z_k \cap \mathcal{V} = \mathcal{Z}_{E_k^1} \cup \dots \cup \mathcal{Z}_{E_k^{n_k}};$$

(C2) if $E' \in \text{ET}(\gamma)_{k'}$, $E'' \in \text{ET}(\gamma)_{k''}$, and $E' < E''$, then $k' < k''$.

Note that (C1) implies that if $\text{ET}(\gamma)_k = \emptyset$ for some k (i.e. $n_k = 0$), then Z_k is disjoint from \mathcal{V} because $\mathcal{Z}_{\emptyset} = \emptyset$. Therefore, adding (resp. deleting) arbitrarily many blowup loci that are disjoint from \mathcal{V} to (resp. from) an existing sequence of blowups does not affect the γ -compatibility on \mathcal{V} .

The condition (C2) has two immediate conclusions. First, a sequence of blowups γ -compatible on \mathcal{V} must begin with \mathcal{Z}_{E_o} . The second conclusion is the next lemma.

Lemma 3.2.2. *Let $\pi: \widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ be γ -compatible on \mathcal{V} as in Definition 3.2.1. Then for each $k \in \llbracket \ell \rrbracket$, the proper transforms of $\mathcal{Z}_{E_k^1}, \dots, \mathcal{Z}_{E_k^{n_k}}$ after the first $k-1$ steps are pairwise disjoint and smooth. Therefore, the proper transform of $Z_k \cap \mathcal{V}$ after the first $k-1$ steps consists of one or more smooth connected component.*

Proof. For distinct $i, j \in \llbracket n_k \rrbracket$, notice that E_k^i and E_k^j are not comparable. With $E \in \text{ET}(\gamma)$ denoting the greatest common predecessor of E_k^i and E_k^j with respect to the order (3.7), we have

$$E < E_k^i, \quad E < E_k^j, \quad \mathcal{Z}_{E_k^i} \cap \mathcal{Z}_{E_k^j} \subset \mathcal{Z}_E.$$

By (C2), there is a unique $h \in \llbracket k-1 \rrbracket$ such that $E \in \text{ET}(\gamma)_h$. It is then a direct check that the proper transforms of $\mathcal{Z}_{E_k^i}$ and $\mathcal{Z}_{E_k^j}$ are smooth after every step, and they become disjoint after step h ; see [14, Lemma 2.3 & Corollary 2.4] for more details. \square

By Lemma 3.2.2, the effect of a blowup that is γ -compatible on \mathcal{V} is the same as blowing up \mathcal{V} successively along the proper transforms of

$$\mathcal{Z}_E, \quad E \in \text{ET}(\gamma)$$

with respect to *any* total order on $\text{ET}(\gamma)$ extending the partial order $<$ in (3.7). Hence the exceptional divisor \mathcal{E}_{Z_k} obtained in the k -th step has n_k connected components

$$(3.11) \quad \mathcal{E}_{E_k^1}, \dots, \mathcal{E}_{E_k^{n_k}}$$

on $\pi_{(k)}^{-1}(\mathcal{V})$. Each $\mathcal{E}_{E_k^i}$ is obtained by blowing up $\pi_{(k-1)}^{-1}(\mathcal{V})$ along the proper transform of $\mathcal{Z}_{E_k^i}$. Thus for each $E \in \text{ET}(\gamma)$, (3.11) determines the exceptional divisor

$$(3.12) \quad \mathcal{E}_E \subset \pi_{(k)}^{-1}(\mathcal{V}) \quad \text{if } E \in \text{ET}(\gamma)_k.$$

Moreover, it is a direct check that \mathcal{E}_E is transverse to the proper transforms of $\mathcal{Z}_{E'}$ for all $E' \in \text{ET}(\gamma)_{k'}, k' > k$. Thus, the proper transform of \mathcal{E}_E in $\pi^{-1}(\mathcal{V})$ is the same as its pullback, i.e.

$$(3.13) \quad \check{\mathcal{E}}_E = \tilde{\mathcal{E}}_E \left(\subset \pi^{-1}(\mathcal{V}) \right) \quad \forall E \in \text{ET}(\gamma).$$

Finally, we comment that the notions of \mathcal{E}_E and $\check{\mathcal{E}}_E$ are intrinsic to \mathcal{V} and γ . They are independent of the choice of the γ -compatible blowup on \mathcal{V} .

Remark 3.2.3. Definition 3.2.1 applies to the empty graph $\gamma = \emptyset$ trivially. A sequence of blowups \emptyset -compatible on \mathcal{V} is equivalent to that the blowup loci do not intersect \mathcal{V} .

Definition 3.2.4. The blowup $\pi : \widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ in (3.10) is said to be *locally tree-compatible* if \mathfrak{M} has an étale cover of affine smooth charts $\{\mathcal{V}\}$ so that each \mathcal{V} is associated with a graph $\gamma_{\mathcal{V}}$ that is either a rooted tree or the empty graph, and π is $\gamma_{\mathcal{V}}$ -compatible on \mathcal{V} for all $\mathcal{V} \in \{\mathcal{V}\}$.

Lemma 3.2.2 gives rise to the following smoothness statement.

Corollary 3.2.5. If the blowup $\pi : \widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ in (3.10) is locally tree-compatible, then for each $k \in \llbracket \ell \rrbracket$, the proper transform of Z_k after the first $k-1$ steps is smooth.

Remark 3.2.6. An example of locally tree-compatible blowups is the desingularization of $\mathfrak{M} = \overline{\mathfrak{M}}_1(\mathbb{P}^n, d)$, via either the symplectic approach in [14] or the algebraic approach in [5]. The subtle difference is the (idealized) blowup in [14] is on $\overline{\mathfrak{M}}_1(\mathbb{P}^n, d)$ directly, whereas in [5], the blowup is applied to the Artin stack $\mathfrak{M}_1^{\text{wt}}$ and a fiber product is taken at the end. Nonetheless the blowup in either approach is locally tree-compatible.

Consider the approach in [5] for example. The blowup loci Θ_k described in [5, §1] can be illustrated by the substacks

$$Z_k = \overline{\mathfrak{M}}_{(1, k+1)}$$

in Figure 1, but with the genus-2 core replaced by a genus-1 core. For every $x \in \mathfrak{M}_1^{\text{wt}}$, let γ be its terminally weighted tree in the sense of [5, Paragraph 3.16]; also see §4.1. Then there exists an affine smooth chart \mathcal{V} containing x so that the collection of modular parameters $\{\zeta_e\}_{e \in \text{Edg}(\gamma)}$ (corresponding to node-smoothing) is a γ -labeled subset of local parameters on \mathcal{V} . Each Z_k as above satisfies

$$Z_k \cap \mathcal{V} = \bigcup_{\substack{E \in \text{ET}(\gamma) \\ |E|=k}} \mathcal{Z}_E.$$

It is a direct check that the blowup successively along the proper transforms of Z_1, \dots, Z_d is γ -compatible on \mathcal{V} , which indicates the sequential blowup in [5] is locally tree-compatible.

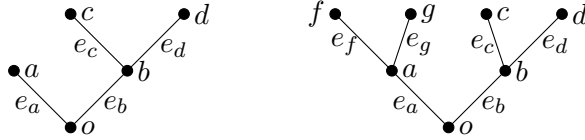


FIGURE 3. Rooted trees

For a concrete example, consider the tree γ illustrated in the second diagram of Figure 3. The sequential blowup described in the previous paragraph determines

$$\begin{aligned} \text{ET}(\gamma) &= \{E_1^1 \equiv \{e_a, e_b\}\} \sqcup \{E_2^1 \equiv \{e_a, e_c, e_d\}, E_2^2 \equiv \{e_b, e_f, e_g\}\} \\ &\quad \sqcup \{E_3^1 \equiv \{e_f, e_g, e_c, e_d\}\} \end{aligned}$$

such that

$$Z_1 \cap \mathcal{V} = \mathcal{Z}_{E_1^1}, \quad Z_2 \cap \mathcal{V} = \mathcal{Z}_{E_2^1} \cup \mathcal{Z}_{E_2^2}, \quad Z_3 \cap \mathcal{V} = \mathcal{Z}_{E_3^1}, \quad E_1^1 < E_2^1, E_2^2 < E_3^1,$$

thus (C1) and (C2) in Definition 3.2.1 are satisfied. We remark here that $Z_2 \cap \mathcal{V}$ is singular: the two smooth loci $\mathcal{Z}_{E_2^1}$ and $\mathcal{Z}_{E_2^2}$ meet at $\mathcal{Z}_\gamma \subset \mathcal{Z}_{E_1^1}$. Nonetheless, after blowing up along Z_1 , the proper transforms of $\mathcal{Z}_{E_2^1}$ and $\mathcal{Z}_{E_2^2}$ are smooth and meet the exceptional divisor $\mathcal{E}_{Z_1} \cap \tilde{\mathcal{V}} = \mathcal{E}_{E_1^1}$ at the loci $\{[\zeta_a, \zeta_b] = [0, 1]\}$ and $\{[\zeta_a, \zeta_b] = [1, 0]\}$, respectively, hence are disjoint.

At the end of this subsection, we describe a sequential blowup induced from a γ -compatible sequential blowup on \mathcal{V} . This is related to the modular blowups in r_3 .

Given a rooted tree γ with the root o , let γ_+ be the rooted tree with the same root o and an extra vertex v_+ directed connected to o , i.e.

$$\text{Ver}(\gamma_+) = \text{Ver}(\gamma) \sqcup \{v_+\}, \quad \text{Edg}(\gamma_+) = \text{Edg}(\gamma) \sqcup \{e_+\}, \quad v_{e_+}^+ = v_+, \quad v_{e_+}^- = o.$$

Assume that \mathcal{V} contains a γ_+ -labeled subset of local parameters.

Let $\pi: \tilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ be γ -compatible on \mathcal{V} . A blowup

$$\pi_+ : \tilde{\mathfrak{M}}_+ \longrightarrow \mathfrak{M}$$

successively along the proper transforms of closed substacks $Z_{1,+}, \dots, Z_{\ell,+}$ of \mathfrak{M} is said to be **supplementary** to π on \mathcal{V} if

$$Z_{k,+} \cap \mathcal{V} = \mathcal{Z}_{E_k^1 \sqcup \{e_+\}} \cup \dots \cup \mathcal{Z}_{E_k^{n_k} \sqcup \{e_+\}} \quad \forall k \in \llbracket \ell \rrbracket.$$

A blowup \emptyset -compatible on \mathcal{V} is supplementary to any blowup \emptyset -compatible on \mathcal{V} .

Over the entire stack \mathfrak{M} instead of locally on \mathcal{V} , π_+ is said to be **supplementary** to π if π is locally tree-compatible with an étale cover $\{\mathcal{V}\}$ as in Definition 3.2.4 so that π_+ is supplementary to π on all $\mathcal{V} \in \{\mathcal{V}\}$.

The construction of γ_+ leads to the following statement immediately.

Lemma 3.2.7. *Let $\pi: \tilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ and $\pi_+: \tilde{\mathfrak{M}}_+ \rightarrow \mathfrak{M}$ be two sequential blowups. If π is γ -compatible on \mathcal{V} and π_+ is supplementary to π on \mathcal{V} , then π_+ is γ_+ -compatible on \mathcal{V} . If π_+ is supplementary to π on \mathfrak{M} , then π_+ is locally tree-compatible.*

3.3. Ascending sequences and dominant edges. We continue with the notation from §3.2. Assume the blowup $\pi: \tilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ is γ -compatible on \mathcal{V} as in Definition 3.2.1. For each $E \in \text{ET}(\gamma)$, we still use $\tilde{\mathcal{E}}_E$ to denote the proper transform of the exceptional divisor \mathcal{E}_E in (3.12). By (3.13), it is the same as the pullback of \mathcal{E}_E .

By Lemma 3.2.2, given

$$\tilde{x} \in \pi^{-1}(\mathcal{Z}_\gamma) \subset \pi^{-1}(\mathcal{V}),$$

there exists a unique *maximal* sequence $\underline{E} = (E_1, \dots, E_t)$ of elements of $\text{ET}(\gamma)$ such that

$$(3.14) \quad E_1 < \dots < E_t, \quad \tilde{x} \in \check{\mathcal{E}}_{E_1} \cap \dots \cap \check{\mathcal{E}}_{E_t}.$$

By Definition 3.2.1 and the maximality of \underline{E} ,

$$\bigcup_{i=1}^k E_i = E_k^{\leq} \quad \forall k \in \llbracket t \rrbracket;$$

c.f. (3.6) for notation. We call \underline{E} the *ascending sequence* of \tilde{x} .

Recall that $\pi_{(k)}: \widetilde{\mathfrak{M}}_{(k)} \rightarrow \mathfrak{M}$ denotes the blowup after the k -th step. Given $i \in \llbracket t \rrbracket$, there exists a unique step $k = k(i)$ of the sequential blowup in which

$$\mathcal{E}_{E_i} \subset \mathcal{E}_{Z_k} \cap \pi_{(k)}^{-1}(\mathcal{V}).$$

Let $\tilde{x}_{(i)}$ be the image of \tilde{x} in $\widetilde{\mathfrak{M}}_{(k)}$. The latter condition in (3.14) implies that

$$\tilde{x}_{(i)} \in \mathcal{E}_{E_i}.$$

We can thus define $\Delta_{\tilde{x}, i} \subset E_i$ analogously to (3.3), with \tilde{x} and $\llbracket \ell \rrbracket$ in (3.3) replaced by $\tilde{x}_{(i)}$ and E_i , respectively. By (3.14),

$$\Delta_{\tilde{x}, i} = E_i \setminus E_{i+1} \quad \forall i \in \llbracket t-1 \rrbracket.$$

When $i = t$, the maximality of \underline{E} implies

$$\Delta_{\tilde{x}, t} \cap \text{Edg}(\gamma)^{\text{t}} \neq \emptyset \quad (\implies E_t \cap \text{Edg}(\gamma)^{\text{t}} \neq \emptyset).$$

Definition 3.3.1. *The edges in*

$$\text{Dom}(\tilde{x}) \equiv \Delta_{\tilde{x}, t} \cap \text{Edg}(\gamma)^{\text{t}}$$

are called dominant edges of \tilde{x} . A branch of γ that contains at least one dominant edge is called a dominant branch of \tilde{x} . If a dominant branch is also pivotal, it is called a dominant pivotal branch.

With notation as above, we set

$$(3.15) \quad \begin{aligned} \text{DP}(\tilde{x}) &= \text{DP}_{\gamma}(\tilde{x}) = \{\langle e \rangle : e \in \text{Dom}(\tilde{x})\} \subset E_o(\gamma), \\ \text{DPN}(\tilde{x}) &= \text{DPN}_{\gamma}(\tilde{x}) = \{q_e : e \in \text{DP}(\tilde{x})\}. \end{aligned}$$

The latter is the collection of the (pivotal) nodes of $\pi(\tilde{x})$ labeled by $\text{DP}(\tilde{x})$. The dominant pivotal branches of \tilde{x} are indexed by $\text{DP}(\tilde{x})$.

By (3.14), each $e \in E_t^{\leq}$ is contained in at least one E_i with $i \in \llbracket t \rrbracket$; let

$$(3.16) \quad s(e) = \max \{ i \in \llbracket t \rrbracket : e \in E_i \}.$$

This implies $e \in \Delta_{\tilde{x}, s}$.

For each $i \in \llbracket t \rrbracket$, the exceptional divisor \mathcal{E}_{E_i} is the zero locus of a local parameter ε_i on an affine smooth chart $\check{\mathcal{V}}_{\tilde{x}, i}$ containing $\tilde{x}_{(i)}$. The proper transform $\check{\mathcal{E}}_{E_i}$ is thus the zero locus of a local parameter $\check{\varepsilon}_i$ on an affine smooth chart $\check{\mathcal{V}}_{\tilde{x}}$ containing \tilde{x} .

Lemma 3.3.2. *Let γ, \mathcal{V} , and $\pi: \widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ be as in Lemma 3.2.2, $\tilde{x} \in \pi^{-1}(\mathcal{Z}_\gamma)$, and $\underline{E} = (E_1, \dots, E_t)$ be the ascending sequence of \tilde{x} . Then,*

- (1) *for each $i \in \llbracket t-1 \rrbracket$, ε_i can be taken to be the pullback $\tilde{z}_e \in \Gamma(\mathcal{O}_{\tilde{\mathcal{V}}_{\tilde{x},i}})$ of $z_e \in \Gamma(\mathcal{O}_\mathcal{V})$ with an arbitrary $e \in E_i - E_{i+1}$; the function ε_t can be taken to be any \tilde{z}_e with $e \in \Delta_{\tilde{x},t}$;*
- (2) *for each $i \in \llbracket t \rrbracket$, $\tilde{\varepsilon}_i$ can be taken as the pullback $\tilde{\varepsilon}_i$ of ε_i ;*
- (3) *there is an affine smooth chart $\tilde{\mathcal{V}}_{\tilde{x}} \subset \pi^{-1}(\mathcal{V})$ containing \tilde{x} such that*

$$\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_t; \quad \check{z}_e, e \in E_t \setminus \Delta_{\tilde{x},t}; \quad \tilde{z}_e, e \in E_t^\succ$$

are part of a family of local parameters on $\tilde{\mathcal{V}}_{\tilde{x}}$.

Proof. The statements (1) and (3) can be deduced from Corollary 3.1.2. The statement (2) follows from (3.13). \square

The next statement plays a crucial role in analyzing the change of the structural homomorphism φ throughout §5. Let $\gamma, \mathcal{V}, \pi, \tilde{x}, \underline{E}$, and $\tilde{\mathcal{V}}_{\tilde{x}}$ be as in Lemma 3.3.2. For each $e \in E_t^\succ$ and each $i \in \llbracket t \rrbracket$, we denote by $\langle e \rangle_i$ the unique edge satisfying

$$(3.17) \quad \langle e \rangle_i \in E_i \quad \text{and} \quad \langle e \rangle_i \leq e.$$

With the functions $z_{[e]}$ as in (3.9), we set

$$\tilde{z}_{[e]} = z_{[e]} \circ \pi \in \Gamma(\mathcal{O}_{\tilde{\mathcal{V}}_{\tilde{x}}}) \quad \forall e \in \text{Edg}(\gamma).$$

Proposition 3.3.3. *For every $\tilde{x} \in \pi^{-1}(\mathcal{Z}_\gamma)$ with the ascending sequence $\underline{E} = (E_1, \dots, E_t)$ as in (3.14), there exists an affine smooth chart $\tilde{\mathcal{V}}_{\tilde{x}} \subset \pi^{-1}(\mathcal{V})$ containing \tilde{x} so that*

$$\tilde{z}_{[d]} \sim \tilde{\varepsilon}_1 \cdots \tilde{\varepsilon}_t, \quad \tilde{z}_{[e]} \sim \tilde{\varepsilon}_1 \cdots \tilde{\varepsilon}_t \cdot \check{z}_{\langle e \rangle_t} \cdot \prod_{\langle e \rangle_t < e' \leq e} \tilde{z}_{e'}$$

for all $d \in \text{Dom}(\tilde{x})$ and all $e \in \text{Edg}(\gamma)^t \setminus \Delta_{\tilde{x},t}$.

Proof. Applying Lemma 3.3.2 and the middle equation in (3.5) repeatedly along the ascending sequence \underline{E} , we obtain

$$\tilde{z}_{[e]} \sim \tilde{\varepsilon}_1 \cdots \tilde{\varepsilon}_t \quad \forall e \in \Delta_{\tilde{x},t}, \quad \tilde{z}_{[e]} \sim \tilde{\varepsilon}_1 \cdots \tilde{\varepsilon}_t \cdot \check{z}_e \quad \forall e \in E_t \setminus \Delta_{\tilde{x},t}.$$

The former leads to the first displayed formula in Proposition 3.3.3. The latter implies

$$\tilde{z}_{[e]} \sim \tilde{\varepsilon}_1 \cdots \tilde{\varepsilon}_t \cdot \check{z}_{\langle e \rangle_t} \cdot \prod_{\langle e \rangle_t < e' \leq e} \tilde{z}_{e'} \quad \forall e \in \text{Edg}(\gamma)^t \setminus \Delta_{\tilde{x},t}.$$

This gives rise to the second displayed formula in Proposition 3.3.3. \square

Remark 3.3.4. In Remark 3.2.6, we observe that the desingularization of $\overline{\mathfrak{M}}_1(\mathbb{P}^n, d)$ is locally tree-compatible. The statements of Proposition 3.3.3 are consistent with the displayed formulae in the paragraph after [5, (5.23)], but are more precise. In [5], it suffices to know $\text{Dom}(\tilde{x}) \neq \emptyset$ and $\tilde{z}_{[d]} | \tilde{z}_{[e]}$ for some $d \in \text{Dom}(\tilde{x})$ and all $e \in \text{Edg}(\gamma)^t$.

3.4. Terminally fused multi-rooted (TFMR) trees. We continue with the notation from §3.2 and §3.3. Notice that in all the statements of §3.2 and §3.3, the relevant aspects of the rooted tree γ are $(\text{Edg}(\gamma), <)$. Thus, the same statements may still hold for other types of (directed) graphs, hence may play an important role in the desingularization of the higher genera moduli spaces. In this subsection, we focus on the generalization of the notion of the rooted trees that is necessary for the genus 2 case.

Let $\{\gamma_i\}_{i \in I}$ be a nonempty collection of rooted trees and $\{V_j\}_{j \in J}$ be a possibly empty collection of vertices satisfying

$$V_j \subset \bigsqcup_{i \in I} \text{Ver}(\gamma_i)^t \quad \forall j \in J, \quad V_{j_1} \cap V_{j_2} = \emptyset \quad \forall j_1 \neq j_2.$$

Note that each V_j may contain no vertex or several vertices of a given γ_i . Let γ be the graph obtained by identifying the vertices in V_j for each $j \in J$; this gives rise to a natural surjective graph homomorphism

$$\bigsqcup_{i \in I} \gamma_i \longrightarrow \gamma.$$

The collections $\text{Ver}(\gamma)$ and $\text{Edg}(\gamma)$ carry partial orders, both denoted by $<$, that are inherited from the tree orders $<_i$'s of γ_i 's. The minimal elements of $\text{Ver}(\gamma)$ with respect to $<$ are the roots o_i 's of γ_i 's. We call such defined γ a terminally fused multi-rooted (TFMR) tree with the roots $o_i, i \in I$, and call $<$ the TFMR tree order on γ . When the context is clear, we may simply call $<$ the tree order. Every rooted tree itself is a TFMR tree. In Figure 4, some examples and counterexamples of TFMR trees are illustrated.

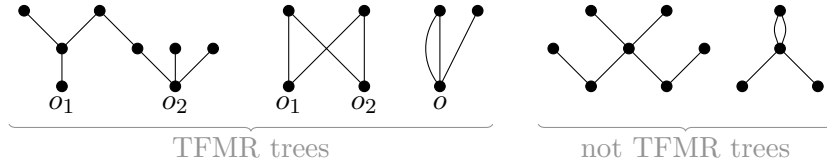


FIGURE 4. Examples and Counterexamples of TFMR trees

It is straightforward to verify that all notions for rooted trees in §3.2 and §3.3 hold verbatim for TFMR trees. The following operations on TFMR trees will appear in §3.5.

Lemma 3.4.1. *Let γ be a TFMR tree and e be an edge of γ . The graph obtained from γ by contracting all the edges e' with $e' > e$ is a TFMR tree. Moreover, if a non-terminal edge e' satisfies $e' > e$, the graph obtained from γ by contracting all the edges e'' with $e \leq e'' \leq e'$ is a TFMR tree.*

Analogous to Definitions 3.2.1 and 3.2.4, we can define γ -compatible blowups on a local chart \mathcal{V} and define locally TFMR tree-compatible blowups on \mathfrak{M} .

Proposition 3.4.2. *Lemmas 3.2.2 and 3.3.2, Corollary 3.2.5, and Proposition 3.3.3 remain valid if γ is a TFMR tree.*

Proof. Notice that the sets of vertices are irrelevant in the rooted trees of the above statements. Thus given a TFMR tree γ , let $\bar{\gamma}$ be the rooted tree obtained by identifying all the roots of γ , removing all the terminal vertices of γ , and then assigning each terminal edge of γ with a distinct terminal vertex. Obviously

$$(\text{Edg}(\bar{\gamma}), \langle \bar{\cdot} \rangle) = (\text{Edg}(\gamma), \langle \cdot \rangle).$$

Since the statements in Proposition 3.4.2 hold for $\bar{\gamma}$, so do they for γ . \square

3.5. Compatible sequences of blowups. In this subsection, we analyze how one TFMR tree-compatible blowup affects others locally.

Let γ_1 and γ_2 be two TFMR trees. We say γ_1 and γ_2 have a collection of common branches indexed by E_* if there exists $E_* \subset \text{Edg}(\gamma_1) \cap \text{Edg}(\gamma_2)$ such that the elements of E_* are not comparable, and the intersection of γ_1 and γ_2 consists of branches $\gamma_e, e \in E_*$. This implies

$$\text{Edg}(\gamma_1) \cap \text{Edg}(\gamma_2) = E_*^{\geq};$$

c.f. (3.6) for notation. Let

$$E_*^t = E_*^{\geq} \cap \text{Edg}(\gamma_1)^t \quad (= E_*^{\geq} \cap \text{Edg}(\gamma_2)^t).$$

For each $e \in E_*^{\geq}$, let $\langle e \rangle_* \in E_*$ be such that $\langle e \rangle_* \leq e$.

Let \mathcal{V} be an affine smooth chart with a subset of local parameters

$$\{z_e : e \in \text{Edg}(\gamma_1) \cup \text{Edg}(\gamma_2)\}.$$

Assume there are two families of closed substacks

$$Z_{1;1}, \dots, Z_{1;\ell_1} \subset \mathfrak{M} \quad \text{and} \quad Z_{2;1}, \dots, Z_{2;\ell_2} \subset \mathfrak{M}$$

such that the sequential blowups

$$(3.18) \quad \pi_1 : \widetilde{\mathfrak{M}}_1 \longrightarrow \mathfrak{M} \quad \text{and} \quad \pi_2 : \widetilde{\mathfrak{M}}_2 \longrightarrow \mathfrak{M}$$

successively along the proper transforms of the first and the second families are γ_1 - and γ_2 -compatible on \mathcal{V} , respectively.

Definition 3.5.1. *We say π_1 and π_2 in (3.18) are compatible on \mathcal{V} if γ_1 and γ_2 have a (possibly empty) collection of common branches.*

We first apply the sequential blowup π_1 to \mathfrak{M} and obtain the proper transforms $\check{Z}_{2;i}, i \in [\ell_2]$, of the blowup loci of π_2 in $\widetilde{\mathfrak{M}}_1$. Let

$$(3.19) \quad \check{\pi}_2 : \widetilde{\mathfrak{M}}_{12} \longrightarrow \widetilde{\mathfrak{M}}_1$$

be the sequential blowup of $\widetilde{\mathfrak{M}}_1$ successively along the proper transforms of $\check{Z}_{2;i}, i \in [\ell_2]$, called the proper transform of π_2 . In the remainder of this subsection, we analyze $\check{\pi}_2$ locally.

Given

$$\hat{x} \in \pi_1^{-1}(\mathcal{Z}_{\gamma_1} \cap \mathcal{Z}_{\gamma_2}) \subset \pi_1^{-1}(\mathcal{V}),$$

let $\underline{E} = (E_1, \dots, E_t)$ be the ascending sequence of \tilde{x} and $\tilde{\mathcal{V}}_{\tilde{x}} \subset \pi_1^{-1}(\mathcal{V})$ be the smaller chart containing \tilde{x} as in Lemma 3.3.2. We emphasize that

$$E_i \in \text{ET}(\gamma_1) \quad \forall i \in \llbracket t \rrbracket.$$

A priori $\text{Dom}(\tilde{x}) \subset \text{Edg}(\gamma_1)^{\text{t}}$. Set

$$\begin{aligned} \dot{E}_* &= \{ \langle e \rangle_* : e \in \text{Dom}(\tilde{x}) \cap E_*^{\geq} \} && (\subset E_*), \\ \ddot{E}_* &= \bigcup_{e \in \dot{E}_*} \{ e' \in \text{Edg}(\gamma_2) : v_{e'}^- = v_e^- \} && (\subset \text{Edg}(\gamma_2)), \\ \hat{E}_* &= E_* \setminus \ddot{E}_* && (\subset E_* \setminus \dot{E}_*). \end{aligned}$$

The definition of \ddot{E}_* implies $\dot{E}_* \subset \ddot{E}_*$ and $\dot{E}_* \cap \hat{E}_* = \emptyset$.

Recall $E_o(\gamma_2)$ denotes the collection of the pivotal edges in γ_2 . If

$$(3.20) \quad \dot{E}_* \cap E_o(\gamma_2) = \emptyset,$$

we construct a TFMR tree $(\gamma_2)_{\tilde{x}}$ from γ_2 as follows.

(a) For each

$$e \in (E_t \setminus \Delta_{\tilde{x}, t}) \cap \dot{E}_*^{\geq},$$

we contract all the edges $e' \in \text{Edg}(\gamma_2)$ satisfying $\langle e \rangle_* \leq e' < e$.

(b) For each

$$e \in \Delta_{\tilde{x}, t} \cap \dot{E}_*^{\geq} \quad (= (E_t \cap \Delta_{\tilde{x}, t}) \cap \dot{E}_*^{\geq}),$$

notice that e is not a terminal edge, for otherwise it would be in $\text{Dom}(\tilde{x})$ and hence in \dot{E}_*^{\geq} . We contract all the edges $e' \in \text{Edg}(\gamma_2)$ satisfying $\langle e \rangle_* \leq e' \leq e$.

(c) For each $e \in \ddot{E}_*$, we contract all the edges $e' \in \text{Edg}(\gamma_2)$ satisfying $e' \geq e$.

The assumption (3.20) implies $\ddot{E}_* \cap E_o(\gamma_2) = \emptyset$, thus each $e \in \ddot{E}_*$ has an immediate predecessor $e_{-1} \in \text{Edg}(\gamma_2)$ with respect to $<$. Hence Step (c) is equivalent to contracting all $e' \in \text{Edg}(\gamma_2)$ satisfying $e' > e_{-1}$. By Lemma 3.4.1, $(\gamma_2)_{\tilde{x}}$ is a TFMR tree. In particular, if $\dot{E}_*^{\geq} \cap E_t = \emptyset$, the above construction implies $(\gamma_2)_{\tilde{x}} = \gamma_2$.

Keeping track of the edges of γ_2 that are contracted in Steps (a)-(c), we obtain

$$\begin{aligned} \text{Edg}((\gamma_2)_{\tilde{x}}) &= (\text{Edg}(\gamma_2) \setminus (E_* \cup \ddot{E}_*)^{\geq}) \sqcup ((E_t \setminus \Delta_{\tilde{x}, t})^{\geq} \cap \dot{E}_*^{\geq}) \sqcup (\Delta_{\tilde{x}, t}^{\geq} \cap \dot{E}_*^{\geq}) \\ &= (\text{Edg}(\gamma_2) \setminus (E_* \cup \ddot{E}_*)^{\geq}) \sqcup ((E_t \setminus \Delta_{\tilde{x}, t}) \cap \dot{E}_*^{\geq}) \sqcup (E_t^{\geq} \cap \dot{E}_*^{\geq}). \end{aligned}$$

This implies

$$(3.21) \quad \text{Edg}((\gamma_2)_{\tilde{x}})^{\text{t}} = \text{Edg}(\gamma_2)^{\text{t}} \setminus \ddot{E}_*.$$

We define $\hat{E}_t \in \text{ET}(\gamma_1)$ to be

$$(3.22) \quad \hat{E}_t = \left(E_t \setminus (\Delta_{\tilde{x}, t} \setminus \text{Edg}(\gamma_1)^{\text{t}}) \right) \sqcup \bigcup_{e \in \Delta_{\tilde{x}, t} \setminus \text{Edg}(\gamma_1)^{\text{t}}} \{ e' \in \text{Edg}(\gamma_1) : v_{e'}^- = v_e^+ \}.$$

Then, $\text{Edg}((\gamma_2)_{\tilde{x}})$ can also be written as

$$(3.23) \quad \text{Edg}((\gamma_2)_{\tilde{x}}) = (\text{Edg}(\gamma_2) \setminus (E_* \cup \ddot{E}_*)^{\geq}) \sqcup (\hat{E}_t^{\geq} \cap \hat{E}_*^{\geq}).$$

Proposition 3.5.2. *With notation as above,*

- (1) *if (3.20) does not hold, then $\check{\pi}_2 : \widetilde{\mathfrak{M}}_{12} \rightarrow \widetilde{\mathfrak{M}}_1$ is \emptyset -compatible on $\widetilde{\mathcal{V}}_{\tilde{x}}$ (i.e. $\check{\pi}_2$ does not affect $\widetilde{\mathcal{V}}_{\tilde{x}}$);*
- (2) *if (3.20) holds, then there exists a $(\gamma_2)_{\tilde{x}}$ -labeled subset of local parameters on $\widetilde{\mathcal{V}}_{\tilde{x}}$ given by*

$$z_e^{(\gamma_2)_{\tilde{x}}} \equiv \check{z}_e, \quad e \in (E_t \setminus \Delta_{\tilde{x},t}) \cap \hat{E}_*^{\geq}; \quad z_e^{(\gamma_2)_{\tilde{x}}} \equiv \check{z}_e, \quad e \in E_t^{\geq} \cap \hat{E}_*^{\geq}$$

as in Lemma 3.3.2, along with

$$z_e^{(\gamma_2)_{\tilde{x}}} \equiv \check{z}_e, \quad e \in \text{Edg}(\gamma_2) \setminus (E_* \cup \ddot{E}_*)^{\geq}.$$

In addition, $\tilde{x} \in \mathcal{Z}_{(\gamma_2)_{\tilde{x}}}$. Furthermore, $\check{\pi}_2$ is $(\gamma_2)_{\tilde{x}}$ -compatible on $\widetilde{\mathcal{V}}_{\tilde{x}}$.

Proof. For each $E \in \text{ET}(\gamma_2)$, keeping track of the proper transforms of the zero locus \mathcal{Z}_E along the sequential blowups π_1 , we observe that \tilde{x} lies in the final proper transform $\check{\mathcal{Z}}_E$ of \mathcal{Z}_E when π_1 terminates if and only if

$$(3.24) \quad E \cap \Delta_{\tilde{x},i} = \emptyset \quad \forall i \in \llbracket t \rrbracket.$$

If (3.20) does not hold, then there exists $e_2 \in \text{Edg}(\gamma_2)^{\dagger} \cap \text{Dom}(\tilde{x})$ such that each $e' \leq e_2$ (in γ_2) belongs to some E_i of the ascending sequence of \tilde{x} . Since each $E \in \text{ET}(\gamma_2)$ contains an edge $e' \leq e_2$, we conclude that \tilde{x} is disjoint from all blowup loci of $\check{\pi}_2$. This shows Part (1).

In Part (2), the first two statements follow directly from Lemma 3.3.2 and Corollary 3.1.2. The last statement is obtained by the comparison of the traverse sections of γ_2 satisfying (3.24) and all the traverse sections of $(\gamma_2)_{\tilde{x}}$ (c.f. (3.23)). \square

We say the sequential blowups π_1 and π_2 in (3.18) are *compatible on \mathfrak{M}* if \mathfrak{M} has an étale cover of affine smooth charts $\{\mathcal{V}\}$ so that π_1 and π_2 are compatible over each $\mathcal{V} \in \{\mathcal{V}\}$. The assumption of π_1 and π_2 being compatible automatically implies they are both locally TFMR tree-compatible.

Proposition 3.5.2 leads to the following conclusion.

Corollary 3.5.3. *Let π_1 and π_2 be as in (3.18) and $\check{\pi}_2$ be the proper transform of π_2 as in (3.19). If π_1 and π_2 are compatible, then $\check{\pi}_2$ is locally TMFR tree-compatible. If in addition there is another sequential blowup $\pi_3 : \widetilde{\mathfrak{M}}_3 \rightarrow \mathfrak{M}$ that is compatible with both π_1 and π_2 , then when π_1 terminates, the proper transform $\check{\pi}_3 : \widetilde{\mathfrak{M}}_{13} \rightarrow \widetilde{\mathfrak{M}}_1$ of π_3 is compatible with $\check{\pi}_2$.*

Example 3.5.4. Let γ_1 and γ_2 be the first graph in Figure 3 and the first graph in Figure 5, respectively. They share a common branch γ_{e_b} . In this case,

$$E_* = \{e_b\}, \quad E_*^{\geq} = \{e_b, e_c, e_d\} = \text{Edg}(\gamma_1) \cap \text{Edg}(\gamma_2).$$

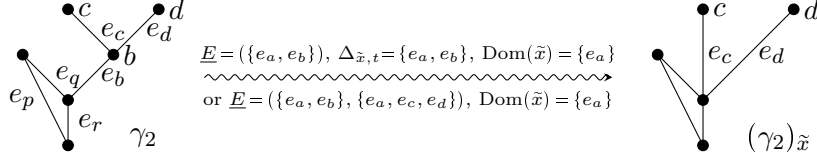


FIGURE 5. A possible change of a TFMR tree

Let \tilde{x} be in the pullback of

$$\mathcal{Z}_{\gamma_1} \cap \mathcal{Z}_{\gamma_2} = \{z_a = z_b = z_c = z_d = z_p = z_q = z_r = 0\}$$

after the sequential blowup π_1 as in Remark 3.2.6.

(1) If $\underline{E} = (\{e_a, e_b\})$ and $\Delta_{\tilde{x}, t} = \{e_a\}$, then

$$t = 1, \quad E_1 = \{e_a, e_b\}, \quad \text{Dom}(\tilde{x}) = \{e_a\}, \quad \dot{E}_* = \ddot{E}_* = \emptyset, \quad \mathring{E}_* = \{e_b\}.$$

The TFMR tree $(\gamma_2)_{\tilde{x}}$ is isomorphic to γ_2 .

(2) If $\underline{E} = (\{e_a, e_b\})$, $\Delta_{\tilde{x}, t} = \{e_a, e_b\}$, and $\text{Dom}(\tilde{x}) = \{e_a\}$, then

$$t = 1, \quad E_1 = \{e_a, e_b\}, \quad \dot{E}_* = \ddot{E}_* = \emptyset, \quad \mathring{E}_* = \{e_b\}.$$

By Step (b), the edge e_b is contracted, hence the TFMR tree $(\gamma_2)_{\tilde{x}}$ is given by the right graph in Figure 5.

(3) If $\underline{E} = (\{e_a, e_b\}, \{e_a, e_c, e_d\})$ and $\text{Dom}(\tilde{x}) = \{e_a\}$, then

$$t = 2, \quad E_2 = \{e_a, e_c, e_d\}, \quad \Delta_{\tilde{x}, t} = \{e_a\}, \quad \dot{E}_* = \ddot{E}_* = \emptyset, \quad \mathring{E}_* = \{e_b\}.$$

By Step (a), the edge e_b is contracted, hence the TFMR tree $(\gamma_2)_{\tilde{x}}$ is given by the right graph in Figure 5.

(4) If $\text{Dom}(\tilde{x}) \cap \{e_c, e_d\} \neq \emptyset$, then

$$t = 2, \quad E_1 = \{e_a, e_b\}, \quad E_2 = \{e_a, e_c, e_d\}, \\ \Delta_{\tilde{x}, t} = \text{Dom}(\tilde{x}), \quad \dot{E}_* = \{e_b\}, \quad \ddot{E}_* = \{e_q, e_b\}, \quad \mathring{E}_* = \emptyset.$$

By Step (c), $(\gamma_2)_{\tilde{x}}$ consists of two vertices: the root o and another vertex v , as well as two edges connecting o and v .

3.6. Derived blowups. In this subsection, we construct a new sequential blowup on the pullback of \mathcal{V} after a special type of sequential blowup that is TFMR-tree compatible on \mathcal{V} terminates. Such blowup (or its supplementary blowup) provides a local model of the modular blowups in r_2 , $r_3\mathfrak{p}_2$, and $r_3\mathfrak{p}_4$.

We continue with the notation from the previous subsections. Let γ be a TFMR tree, \mathcal{V} be an affine smooth chart with a collection of coordinate functions labeled by γ as in (3.8), and $\pi: \tilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ be the blowup successively along the proper transforms of Z_1, \dots, Z_ℓ that is γ -compatible on \mathcal{V} .

Consider the following subset of $\text{Edg}(\gamma) \times \text{ET}(\gamma) \times \text{Edg}(\gamma)$:

$$\mathfrak{D}_1(\gamma) = \{\mathfrak{E} \equiv (E, e, E') \in \text{ET}(\gamma) \times \text{Edg}(\gamma) \times \text{ET}(\gamma) : E \leq E', e \in E \cap E'\}.$$

We define a partial order $<$ on $\mathfrak{D}_1(\gamma)$ so that $(E_1, e_1, E'_1) < (E_2, e_2, E'_2)$ if and only if

$$(3.25) \quad \text{either } (E_1 < E_2) \text{ or } (E_1 = E_2, e_1 = e_2, E'_1 < E'_2).$$

For a terminal edge $\mathfrak{e} \in \text{Edg}(\gamma)^\dagger$, set

$$\mathfrak{D}_1(\gamma; \mathfrak{e}) = \{ (E, e, E') \in \mathfrak{D}_1(\gamma) : e \leq \mathfrak{e} \}.$$

The order (3.25) induces a partial order $<$ on $\mathfrak{D}_1(\gamma; \mathfrak{e})$ for every $\mathfrak{e} \in \text{Edg}(\gamma)^\dagger$.

For each $E \in \text{ET}(\gamma)$, let $k \in \llbracket \ell \rrbracket$ be such that $E \in \text{ET}(\gamma)_k$; c.f. Definition 3.2.1. For each nonempty subset E_0 of E , we define the locus

$$\mathcal{E}(E_0) \subset \mathcal{E}_E \quad (\subset \pi_{(k)}^{-1}(\mathcal{V}))$$

analogous to (3.4), where $\pi_{(k)}: \widetilde{\mathfrak{M}}_{(k)} \rightarrow \mathfrak{M}$ denotes the blowup after the k -th step as always. For each $\mathfrak{E} \in \mathfrak{D}_1(\gamma)$, let

$$Y_{\mathfrak{E}} = \mathcal{E}(\{e\} \sqcup (E \setminus E')) \cap \bigcap_{e' \in E' \setminus E} \{ \tilde{z}_{e'} = 0 \} \quad (\subset \mathcal{E}_E).$$

Lemma 3.6.1. *With notation as above, $Y_{\mathfrak{E}}$ is closed, smooth, and of codimension $|E'|$ in the corresponding pullback $\pi_{(k)}^{-1}(\mathcal{V})$ of \mathcal{V} .*

Proof. For every $y \in Y_{\mathfrak{E}}$, there is a chart $\mathcal{V}_y \subset \pi_{(k)}^{-1}(\mathcal{V})$ containing y such that

$$Y_{\mathfrak{E}} \cap \mathcal{V}_y = \{ \varepsilon_k = 0 \} \cap \bigcap_{e' \in E' \setminus E} \{ \tilde{z}_{e'} = 0 \} \cap \bigcap_{e'' \in (E' \cap E) \setminus \{e\}} \{ \tilde{z}_{e''} = 0 \}. \quad \square$$

For each $\mathfrak{E} \in \mathfrak{D}_1(\gamma)$, we denote by

$$\check{Y}_{\mathfrak{E}} \subset \pi^{-1}(\mathcal{V}) \subset \widetilde{\mathfrak{M}}$$

the proper transform of $Y_{\mathfrak{E}}$.

Remark 3.6.2. We point out that if $\gamma = \emptyset$, then $\mathfrak{D}_1(\gamma), Y_{(\cdot)}, \check{Y}_{(\cdot)} = \emptyset$.

Let $\pi': \widetilde{\mathfrak{M}}' \rightarrow \widetilde{\mathfrak{M}}$ be the blowup of $\widetilde{\mathfrak{M}}$ successively along the proper transforms of a family of closed substacks

$$\Upsilon_1, \dots, \Upsilon_r \subset \widetilde{\mathfrak{M}}.$$

Definition 3.6.3. *Assume $\pi: \widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ is γ -compatible on \mathcal{V} and $\pi': \widetilde{\mathfrak{M}}' \rightarrow \widetilde{\mathfrak{M}}$ is as above. We say π' is derived from π on $\pi^{-1}(\mathcal{V}) \subset \widetilde{\mathfrak{M}}$ if $\mathfrak{D}_1(\gamma)$ can be written as*

$$\mathfrak{D}_1(\gamma) = \bigsqcup_{k=1}^r \mathfrak{D}_1(\gamma)_k \quad \text{s.t.}$$

(C1) for each $k \in \llbracket r \rrbracket$,

$$\Upsilon_k \cap \pi^{-1}(\mathcal{V}) = \bigcup_{\mathfrak{E} \in \mathfrak{D}_1(\gamma)_k} \check{Y}_{\mathfrak{E}};$$

(C2) if $\mathfrak{E}_1 \in \mathfrak{D}_1(\gamma)_{k_1}$, $\mathfrak{E}_2 \in \mathfrak{D}_1(\gamma)_{k_2}$, and $\mathfrak{E}_1 < \mathfrak{E}_2$, then $k_1 < k_2$.

For a fixed $\epsilon \in \text{Edg}(\gamma)^t$, we say π' is *partially derived from π (along ϵ)* on $\pi^{-1}(\mathcal{V})$ if in the above definition, $\mathcal{D}_1(\gamma)$ is replaced by $\mathcal{D}_1(\gamma; \epsilon)$.

Given $\tilde{x} \in \pi^{-1}(\mathcal{Z}_\gamma)$, let $\underline{E} = (E_1, \dots, E_t)$ be the ascending sequence of \tilde{x} . Assume that \tilde{x} has a unique dominant pivotal branch, i.e.

$$(3.26) \quad \langle e \rangle = \langle e' \rangle \quad \forall e, e' \in \text{Dom}(\tilde{x}).$$

We then set

$$(3.27) \quad e_\bullet = \max\{e \in \text{Edg}(\gamma) : e \leq e' \ \forall e' \in \text{Dom}(\tilde{x})\} \in \text{Edg}(\gamma).$$

Since $\text{Dom}(\tilde{x}) \subset E_t$, we have $e_\bullet \in E_t^{\leq}$. Let

$$s \equiv s(e_\bullet) \in \llbracket t \rrbracket;$$

c.f. (3.16). This implies $e_\bullet \in \Delta_{\tilde{x}, s}$.

Recall that for each edge e satisfying $\langle e_\bullet \rangle = \langle e \rangle$, we denote by

$$e_\bullet \wedge e$$

the greatest (w.r.t. $<$) common predecessor of e_\bullet and e . This in turn determines an index $s(e_\bullet \wedge e) \in \llbracket s \rrbracket$ by (3.16). If $\langle e \rangle \neq \langle e_\bullet \rangle$, we set $s(e_\bullet \wedge e) = 0$.

Next, we construct a TFMR tree γ'_x for \tilde{x} satisfying (3.26).

(a) Let $[o, \mathbf{v}_s]$ be the simple path with vertices and edges

$$\mathbf{v}_0 \equiv o < \mathbf{v}_1 < \dots < \mathbf{v}_s \quad \text{and} \quad \mathbf{e}_1 < \dots < \mathbf{e}_s,$$

respectively.

(b) For each

$$e \in (E_t \setminus \Delta_{\tilde{x}, t}) \setminus \{e_\bullet\}^{\geq},$$

we connect the root v_e^+ of the branch γ_e and the vertex $\mathbf{v}_{s(e_\bullet \wedge e)}$ of the path $[o, \mathbf{v}_s]$ by an edge.

(c) For each

$$e \in \Delta_{\tilde{x}, t} \setminus \{e_\bullet\}^{\geq},$$

the definition of e_\bullet implies e is not a terminal edge. We connect the root v_e^+ of the branch γ_e and the vertex $\mathbf{v}_{s(e_\bullet \wedge e)}$ of the path $[o, \mathbf{v}_s]$ by an edge, and then contract this new edge.

By Lemma 3.4.1, γ'_x is still a TFMR tree with the root o . We call it the *derived TFMR tree* (or simply the *derived tree* when the context is clear) of γ at \tilde{x} . Notice that

$$\begin{aligned} \text{Edg}(\gamma'_x) &= \{\mathbf{e}_1, \dots, \mathbf{e}_s\} \sqcup ((E_t \setminus \Delta_{\tilde{x}, t})^{\geq} \setminus \{e_\bullet\}^{\geq}) \sqcup (\Delta_{\tilde{x}, t}^{\geq} \setminus \{e_\bullet\}^{\geq}) \\ &= \{\mathbf{e}_1, \dots, \mathbf{e}_s\} \sqcup ((E_t \setminus \Delta_{\tilde{x}, t}) \setminus \{e_\bullet\}^{\geq}) \sqcup (E_t^{\geq} \setminus \{e_\bullet\}^{\geq}). \end{aligned}$$

With \hat{E}_t as in (3.22), we can also write

$$(3.28) \quad \text{Edg}(\gamma'_x) = \{\mathbf{e}_1, \dots, \mathbf{e}_s\} \sqcup (\hat{E}_t^{\geq} \setminus \{e_\bullet\}^{\geq}).$$

The following statement follows from Lemma 3.3.2 and Corollary 3.1.2 directly.

Lemma 3.6.4. *Given $\tilde{x} \in \pi^{-1}(\mathcal{Z}_\gamma)$ satisfying (3.26), there exists an affine smooth chart $\tilde{\mathcal{V}}_{\tilde{x}} \subset \pi^{-1}(\mathcal{V})$ such that*

$$z_{\mathbf{e}_i}^{\gamma_{\tilde{x}}} \equiv \tilde{\varepsilon}_i, \quad i \in \llbracket s \rrbracket; \quad z_e^{\gamma_{\tilde{x}}} \equiv \tilde{z}_e, \quad e \in (E_t \setminus \Delta_{\tilde{x}, t}) \setminus \{e_\bullet\}^{\geq}; \quad z_e^{\gamma_{\tilde{x}}} \equiv \tilde{z}_e, \quad e \in E_t^{\geq} \setminus \{e_\bullet\}^{\geq}$$

as in Lemma 3.3.2 form a $\gamma_{\tilde{x}}'$ -labeled subset of local parameters on $\tilde{\mathcal{V}}_{\tilde{x}}$. Moreover, $\tilde{x} \in \mathcal{Z}_{\gamma_{\tilde{x}}}'$.

Remark 3.6.5. Given \tilde{x} satisfying (3.26) and $\mathbf{e} \in \text{Edg}(\gamma)^t$, we replace e_\bullet and $s = s(e_\bullet)$ in the construction of $\gamma_{\tilde{x}}'$ by

$$\mathbf{e}_\bullet \equiv \mathbf{e} \wedge e_\bullet \quad \text{and} \quad \mathfrak{s} = s(\mathbf{e}_\bullet),$$

respectively and call the resulting TFMR tree $\gamma_{\mathbf{e}, \tilde{x}}'$ the partially derived (TFMR) tree of γ along \mathbf{e} at \tilde{x} . It is a direct check that $\text{Edg}(\gamma_{\mathbf{e}, \tilde{x}}') \subset \text{Edg}(\gamma_{\tilde{x}}')$, hence the analogue of Lemma 3.6.4 for $\gamma_{\mathbf{e}, \tilde{x}}'$ naturally holds.

For \hat{E}_t in (3.22), we notice that

$$((E' \setminus \{e\}) \sqcup (\hat{E}_t \cap \{e\}^{\geq})) \in \text{ET}(\gamma) \quad \forall E' \in \text{ET}(\gamma), \quad e \in E'.$$

Lemma 3.6.6. *Given $\tilde{x} \in \pi^{-1}(\mathcal{Z}_\gamma)$, there exists an affine smooth chart $\tilde{\mathcal{V}}_{\tilde{x}} \subset \pi^{-1}(\mathcal{V})$ such that for each $\mathfrak{E} = (E, e, E') \in \mathfrak{D}_1(\gamma)$,*

(1) *if \tilde{x} satisfies (3.26), $E = E_i \in \{E_1, \dots, E_s\}$, $e \leq e_\bullet$, and*

$$((E' \setminus \{e\}) \sqcup (\hat{E}_t \cap \{e\}^{\geq})) \geq \hat{E}_t,$$

then $\check{Y}_{\mathfrak{E}} \cap \tilde{\mathcal{V}}_{\tilde{x}}$ is given by

$$\{\tilde{\varepsilon}_i = 0\} \cap \bigcap_{e' \in (E' \setminus E) \cap E_t^{\geq}} \{\tilde{z}_{e'} = 0\} \cap \bigcap_{e' \in (E' \setminus E) \cap (E_t \setminus \Delta_{\tilde{x}, t})} \{\tilde{z}_{e'} = 0\} \cap \bigcap_{e'' \in (E' \cap E) \setminus \{e\}} \{\tilde{z}_{e''} = 0\};$$

(2) *otherwise, $\check{Y}_{\mathfrak{E}} \cap \tilde{\mathcal{V}}_{\tilde{x}} = \emptyset$.*

Proof. For each $\mathfrak{E} \in \mathfrak{D}_1(\gamma)$, the explicit local expression of $Y_{\mathfrak{E}}$ is given in the proof of Lemma 3.6.1. Keeping track of the proper transforms of $Y_{\mathfrak{E}}$, we observe that $\tilde{x} \in \check{Y}_{\mathfrak{E}}$ if and only if the assumptions in Part (1) hold. \square

Lemma 3.6.7. *Let $\pi'_1: \tilde{\mathfrak{M}}'_1 \rightarrow \tilde{\mathfrak{M}}$ and $\pi'_2: \tilde{\mathfrak{M}}'_2 \rightarrow \tilde{\mathfrak{M}}$ be two sequential blowups that are both derived (or both partially derived along the same $\mathbf{e} \in \text{Edg}(\gamma)^t$) from the γ -compatible blowup $\pi: \tilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ on $\pi^{-1}(\mathcal{V})$. Then the pullbacks of $\pi^{-1}(\mathcal{V})$ in $\tilde{\mathfrak{M}}'_1$ and $\tilde{\mathfrak{M}}'_2$ are isomorphic.*

Proof. The proof is parallel to Lemma 3.2.2 and the paragraph after it. \square

Proposition 3.6.8. *Let $\pi: \tilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ and $\pi': \tilde{\mathfrak{M}}' \rightarrow \tilde{\mathfrak{M}}$ be two sequential blowups. Assume π is γ -compatible on \mathcal{V} .*

(1) *If π' is derived from π on $\pi^{-1}(\mathcal{V})$, then for every $\tilde{x} \in \pi^{-1}(\mathcal{Z}_\gamma)$, there exists an affine smooth chart $\tilde{\mathcal{V}}_{\tilde{x}} \subset \pi^{-1}(\mathcal{V})$ containing \tilde{x} such that π' is $\gamma_{\tilde{x}}'$ -compatible on $\tilde{\mathcal{V}}_{\tilde{x}}$ if \tilde{x} satisfies (3.26) and \emptyset -compatible otherwise.*

- (2) If π' is partially derived from π along \mathfrak{e} on $\pi^{-1}(\mathcal{V})$, then for every $\tilde{x} \in \pi^{-1}(\mathcal{Z}_\gamma)$, there exists an affine smooth chart $\tilde{\mathcal{V}}_{\tilde{x}} \subset \pi^{-1}(\mathcal{V})$ containing \tilde{x} such that π' is $\gamma'_{\mathfrak{e}, \tilde{x}}$ -compatible on $\tilde{\mathcal{V}}_{\tilde{x}}$ if \tilde{x} satisfies (3.26) and \emptyset -compatible otherwise.

Proof. The statements follow from Definition 3.6.3 and the comparison of the local expression of $\check{Y}_{\mathfrak{e}}$ in Lemma 3.6.6 and the traverse sections of $\gamma'_{\tilde{x}}$ (c.f. (3.28)). \square

If $\pi: \tilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ is locally TFMR tree-compatible, let $\{\mathcal{V}\}$ be the étale cover of affine smooth charts associated with π as in Definition 3.2.4. A sequential blowup $\pi': \tilde{\mathfrak{M}}' \rightarrow \tilde{\mathfrak{M}}$ is said to be derived from π if for each $\mathcal{V} \in \{\mathcal{V}\}$, π' is derived from π on $\pi^{-1}(\mathcal{V})$. It is said to be partially derived from π if for each $\mathcal{V} \in \{\mathcal{V}\}$, π' is partially derived from π on $\pi^{-1}(\mathcal{V})$.

The following statement follows immediately from Proposition 3.6.8.

Corollary 3.6.9. *Let $\pi: \tilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ be locally TFMR tree-compatible and $\pi': \tilde{\mathfrak{M}}' \rightarrow \tilde{\mathfrak{M}}$ be a sequential blowup derived (or partially derived) from π . Then, π' is locally TFMR tree-compatible.*

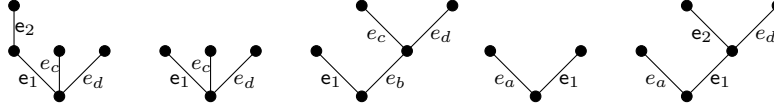


FIGURE 6. Some possible (TFMR) trees derived from the first diagram in Figure 3

Example 3.6.10. Let γ be the first graph in Figure 3 and $\tilde{x} \in \pi^{-1}(\mathcal{Z}_\gamma)$ be fixed. Recall in Remark 3.2.6, we see that γ has two traverse sections:

$$E_1 = \{e_a, e_b\} < E_2 = \{e_a, e_c, e_d\}.$$

It is straightforward that $\mathfrak{D}_1(\gamma) = \{\mathfrak{E}_1, \dots, \mathfrak{E}_6\}$, where

$$\begin{aligned} \mathfrak{E}_1 &= (E_1, e_a, E_1), & \mathfrak{E}_2 &= (E_1, e_a, E_2), & \mathfrak{E}_3 &= (E_2, e_a, E_2), \\ \mathfrak{E}_4 &= (E_1, e_b, E_1), & \mathfrak{E}_5 &= (E_2, e_c, E_2), & \mathfrak{E}_6 &= (E_2, e_d, E_2). \end{aligned}$$

The partial order on $\mathfrak{D}_1(\gamma)$ is given by

$$\mathfrak{E}_1 < \mathfrak{E}_2 < \mathfrak{E}_3, \quad \mathfrak{E}_4 < \mathfrak{E}_5, \quad \mathfrak{E}_4 < \mathfrak{E}_6.$$

- (1) If $\text{Dom}(\tilde{x}) = \{e_a\}$ and $\underline{E} = (E_1, E_2)$, then

$$e_s = e_a, \quad s = t = 2, \quad \Delta_{\tilde{x}, 1} = \{e_b\}, \quad \Delta_{\tilde{x}, 2} = \{e_a\}, \quad \hat{E}_2 = E_2.$$

The derived tree $\gamma'_{\tilde{x}}$ is the first graph of Figure 6. Locally near \tilde{x} , the relevant $\check{Y}_{\mathfrak{E}}$ are

$$\check{Y}_{\mathfrak{E}_2} = \{\tilde{\varepsilon}_1 = \tilde{z}_c = \tilde{z}_d = 0\} \quad \text{and} \quad \check{Y}_{\mathfrak{E}_3} = \{\tilde{\varepsilon}_2 = \tilde{z}_c = \tilde{z}_d = 0\}.$$

(2) If $\text{Dom}(\tilde{x}) = \{e_a\}$, $\underline{E} = (E_1)$, and $\Delta_{\tilde{x},1} = \{e_a, e_b\}$, then

$$e_\bullet = e_a, \quad s = t = 1, \quad \widehat{E}_2 = E_2.$$

The derived tree γ'_x is the second graph of Figure 6. Locally near \tilde{x} , the relevant $\check{Y}_{\mathfrak{E}}$ is

$$\check{Y}_{\mathfrak{E}_2} = \{\tilde{\varepsilon}_1 = \tilde{z}_c = \tilde{z}_d = 0\}.$$

(3) If $\text{Dom}(\tilde{x}) = \{e_a\}$, $\underline{E} = (E_1)$, and $\Delta_{\tilde{x},1} = \{e_a\}$, then

$$e_\bullet = e_a, \quad s = t = 1, \quad \widehat{E}_2 = E_1.$$

The derived tree γ'_x is the third graph of Figure 6. Locally near \tilde{x} , the relevant $\check{Y}_{\mathfrak{E}}$ are

$$\check{Y}_{\mathfrak{E}_1} = \{\tilde{\varepsilon}_1 = \tilde{z}_b = 0\} \quad \text{and} \quad \check{Y}_{\mathfrak{E}_2} = \{\tilde{\varepsilon}_2 = \tilde{z}_c = \tilde{z}_d = 0\}.$$

(4) If $\text{Dom}(\tilde{x}) = \{e_c, e_d\}$, then $\underline{E} = (E_1, E_2)$,

$$e_\bullet = e_b, \quad s = 1, \quad t = 2, \quad \Delta_{\tilde{x},1} = \{e_b\}, \quad \Delta_{\tilde{x},2} = \{e_c, e_d\}, \quad \widehat{E}_2 = E_2.$$

The derived tree γ'_x is the fourth graph of Figure 6. Locally near \tilde{x} , the relevant $\check{Y}_{\mathfrak{E}}$ is

$$\check{Y}_{\mathfrak{E}_4} = \{\tilde{\varepsilon}_1 = \tilde{z}_a = 0\}.$$

(5) If $\text{Dom}(\tilde{x}) = \{e_c\}$, then $\underline{E} = (E_1, E_2)$,

$$e_\bullet = e_c, \quad s = t = 2, \quad \Delta_{\tilde{x},1} = \{e_b\}, \quad \Delta_{\tilde{x},2} = \{e_c\}, \quad \widehat{E}_2 = E_2.$$

The derived tree γ'_x is the last graph of Figure 6. Locally near \tilde{x} , the relevant $\check{Y}_{\mathfrak{E}}$ are

$$\check{Y}_{\mathfrak{E}_4} = \{\tilde{\varepsilon}_1 = \tilde{z}_a = 0\} \quad \text{and} \quad \check{Y}_{\mathfrak{E}_5} = \{\tilde{\varepsilon}_2 = \tilde{z}_a = \tilde{z}_d = 0\}.$$

The case that $\text{Dom}(\tilde{x}) = \{e_d\}$ is parallel.

3.7. Bi-dominantly derived blowups. This subsection is similar to the previous subsection. We construct another sequential blowup on the pull-back of \mathcal{V} after a sequential blowup that is TFMR-tree compatible on \mathcal{V} terminates. It provides a local model of the modular blowups in $r_3\mathfrak{p}_1$.

Let γ , \mathcal{V} , and $\pi: \widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ be as in §3.6. Consider the following subset of $\text{ET}(\gamma) \times \text{Edg}(\gamma) \times \text{Edg}(\gamma) \times \text{ET}(\gamma)$:

$$\begin{aligned} \mathfrak{D}_2(\gamma) = \{ & \mathfrak{F} \equiv (E, e_1, e_2, E') \in \text{ET}(\gamma) \times \text{Edg}(\gamma) \times \text{Edg}(\gamma) \times \text{ET}(\gamma) : \\ & E \leq E', \quad e_1 \neq e_2, \quad \{e_1, e_2\} \subset E \cap E' \}. \end{aligned}$$

Thus, e_1 and e_2 are not comparable if $(E, e_1, e_2, E') \in \mathfrak{D}_2(\gamma)$. We define a partial order $<$ on $\mathfrak{D}_2(\gamma)$ so that $(E, e_1, e_2, E') < (\widehat{E}, \widehat{e}_1, \widehat{e}_2, \widehat{E}')$ if and only if

$$\text{either } (E < \widehat{E}) \quad \text{or} \quad (E = \widehat{E}, \quad (e_1, e_2) = (\widehat{e}_1, \widehat{e}_2), \quad E' < \widehat{E}').$$

For each $\mathfrak{F} \in \mathfrak{D}_2(\gamma)$, let k be such that $E \in \text{ET}(\gamma)_k$, and let

$$Y_{\mathfrak{F}} = \mathcal{E}(\{e_1, e_2\} \sqcup (E \setminus E')) \cap \bigcap_{e' \in E' \setminus E} \{\tilde{z}_{e'} = 0\} \subset \mathcal{E}_E.$$

The following statement is the analogue of Lemma 3.6.1 under the current setup; we omit its proof.

Lemma 3.7.1. *With notation as above, $Y_{\mathfrak{F}}$ is closed, smooth, and of codimension $|E'| - 1$ in the corresponding pullback $\pi_{(k)}^{-1}(\mathcal{V})$ of \mathcal{V} .*

For each $\mathfrak{F} \in \mathfrak{D}_2(\gamma)$, we denote by

$$\check{Y}_{\mathfrak{F}} \subset \pi^{-1}(\mathcal{V}) \subset \check{\mathfrak{M}}$$

the proper transform of $Y_{\mathfrak{F}}$.

Remark 3.7.2. We point out that if $\gamma = \emptyset$, then $\mathfrak{D}_2(\gamma), Y_{(\cdot)}, \check{Y}_{(\cdot)} = \emptyset$.

Let

$$\pi' : \check{\mathfrak{M}}' \longrightarrow \check{\mathfrak{M}} \quad \text{and} \quad \pi^{(2)} : \check{\mathfrak{M}}'' \longrightarrow \check{\mathfrak{M}}$$

be respectively the blowup derived from $\pi : \check{\mathfrak{M}} \rightarrow \mathfrak{M}$ and the blowup of $\check{\mathfrak{M}}$ successively along the proper transforms of a family of closed substacks

$$\Upsilon_1^\dagger, \dots, \Upsilon_r^\dagger \subset \check{\mathfrak{M}}.$$

We denote by

$$\check{\pi}^{(2)} : \check{\mathfrak{M}}^{(2)} \longrightarrow \check{\mathfrak{M}}'$$

blowup of $\check{\mathfrak{M}}'$ successively along the proper transforms of $\Upsilon_1^\dagger, \dots, \Upsilon_r^\dagger$ after π' and call it the proper transform of $\pi^{(2)}$.

Definition 3.7.3. *Assume $\pi : \check{\mathfrak{M}} \rightarrow \mathfrak{M}$ is γ -compatible on \mathcal{V} and $\check{\pi}^{(2)} : \check{\mathfrak{M}}^{(2)} \rightarrow \check{\mathfrak{M}}'$ is as above. We say $\check{\pi}^{(2)}$ is bi-dominantly derived from π on $(\pi')^{-1}(\pi^{-1}(\mathcal{V})) \subset \check{\mathfrak{M}}'$ if $\mathfrak{D}_2(\gamma)$ can be written as*

$$\mathfrak{D}_2(\gamma) = \bigsqcup_{k=1}^r \mathfrak{D}_2(\gamma)_k \quad \text{s.t.}$$

(C1) for each $k \in \llbracket r \rrbracket$,

$$\Upsilon_k^\dagger \cap \pi^{-1}(\mathcal{V}) = \bigcup_{\mathfrak{F} \in \mathfrak{D}_2(\gamma)_k} \check{Y}_{\mathfrak{F}};$$

(C2) if $\mathfrak{F}_1 \in \mathfrak{D}_2(\gamma)_{k_1}$, $\mathfrak{F}_2 \in \mathfrak{D}_2(\gamma)_{k_2}$, and $\mathfrak{F}_1 < \mathfrak{F}_2$, then $k_1 < k_2$.

Let \tilde{x} and \underline{E} be as in § 3.6. If there exist $e_1, e_2 \in \text{Edg}(\gamma)$ such that they are not comparable and

$$(3.29) \quad \text{Dom}(\tilde{x}) = D_1 \sqcup D_2, \quad \emptyset \neq D_i \subset \text{Edg}(\gamma_{e_i}), \quad i = 1, 2,$$

we then set

$$e_\bullet^i = \max\{e \in \text{Edg}(\gamma) : e \leq e' \ \forall e' \in D_i\} \in \text{Edg}(\gamma), \quad i = 1, 2.$$

Since $\text{Dom}(\tilde{x}) \subset E_t$, we observe that $e_\bullet^1, e_\bullet^2 \in E_t^\leq$, although $s(e_\bullet^1)$ may not equal $s(e_\bullet^2)$; c.f. (3.16). Set

$$\varsigma \equiv \min\{s(e_\bullet^1), s(e_\bullet^2)\} \in \llbracket t \rrbracket, \quad \varsigma_0 \equiv s(e_\bullet^1 \wedge e_\bullet^2) \in \{0\} \sqcup \llbracket t \rrbracket.$$

By definition, $\varsigma_0 < \varsigma$. Recall that $s(e \wedge e') = 0$ whenever $\langle e \rangle \neq \langle e' \rangle$. Hence if $\langle e_\bullet^1 \rangle \neq \langle e_\bullet^2 \rangle$, then $\varsigma_0 = 0$.

Let $e_\circ^1, e_\circ^2 \in E_\varsigma$ be such that

$$e_\circ^i \leq e_\bullet^i, \quad i = 1, 2.$$

For each $e \in \text{Edg}(\gamma)$, define

$$\sigma(e) \equiv \max \{ s(e_\circ^1 \wedge e), s(e_\circ^2 \wedge e), \varsigma_0 \}.$$

This implies $\varsigma_0 \leq \sigma(e) \leq \varsigma$ for all $e \in \text{Edg}(\gamma)$.

We construct a rooted tree $\tau_{\tilde{x}}$ following the steps below.

(a) Let $[o', v'_\varsigma]$ be the simple path with vertices and edges

$$v'_{\varsigma_0} \equiv o' < v'_{\varsigma_0+1} < \cdots < v'_\varsigma \quad \text{and} \quad e'_{\varsigma_0+1} < \cdots < e'_\varsigma,$$

respectively.

(b) For each

$$e \in (E_t \setminus \Delta_{\tilde{x}, t}) \setminus \{e_\circ^1, e_\circ^2\}^{\geq},$$

we connect the root v_e^+ of the branch γ_e and the vertex $v'_{\sigma(e)}$ of the path $[o', v'_\varsigma]$ by an edge.

(c) For each

$$e \in \Delta_{\tilde{x}, t} \setminus \{e_\circ^1, e_\circ^2\}^{\geq},$$

the definition of e_\bullet implies e is not a terminal edge. We connect the root v_e^+ of the branch γ_e and the vertex $v'_{\sigma(e)}$ of the path $[o', v'_\varsigma]$ by an edge, and then contract this new edge.

Analogous to the derived tree as in §3.6, the graph $\tau_{\tilde{x}}$ is still a TFMR tree.

If $\text{Dom}(\tilde{x})$ is contained in exactly two pivotal branches, i.e. the edges e_1, e_2 in (3.29) belong to $E_o(\gamma)$, then $\pi': \widetilde{\mathfrak{M}}' \rightarrow \widetilde{\mathfrak{M}}$ does not affect a neighborhood of \tilde{x} . Moreover, $\varsigma_0 = 0$. Let $\tilde{x}' = (\pi')^{-1}(\tilde{x})$. We set

$$\gamma_{\tilde{x}'}^{(2)} = \tau_{\tilde{x}}$$

and call it the bi-dominantly derived TFMR tree of γ at \tilde{x}' .

Next, we consider the case in which $\text{Dom}(\tilde{x})$ is contained in a unique pivotal branch, i.e. (3.26) holds. Let $\gamma'_{\tilde{x}}$ be the derived tree of γ at \tilde{x} . When the derived blowup $\pi': \widetilde{\mathfrak{M}}' \rightarrow \widetilde{\mathfrak{M}}$ terminates, for every lift \tilde{x}' of \tilde{x} , there are two possibilities when $\gamma_{\tilde{x}'}^{(2)}$ can be defined:

- (S1) $\text{Dom}(\tilde{x}') = \{e_s\}$ and meanwhile (3.29) holds for $\text{Dom}(\tilde{x})$;
- (S2) there exists $e \in \text{Edg}(\gamma'_{\tilde{x}}) \setminus \{e_1, \dots, e_s\}$ such that $e \leq e'$ for all $e' \in \text{Dom}(\tilde{x}')$ (thus $e_s \notin \text{Dom}(\tilde{x}')$).

For Possibility (S1), notice that $\tau_{\tilde{x}}$ is well-defined and $\varsigma_0 = s$. We set

$$\gamma_{\tilde{x}'}^{(2)} = (\tau_{\tilde{x}})_{\tilde{x}'}$$

and call it the bi-dominantly derived TFMR tree of γ at \tilde{x}' ; see §3.5 for notation.

For Possibility (S2), we construct the TFMR tree $\gamma_{\tilde{x}'}^{(2)}$ out of $\gamma'_{\tilde{x}}$, mimicking the construction of $\tau_{\tilde{x}}$ with $\text{Dom}(\tilde{x})$ in (3.29) replaced by $\text{Dom}(\tilde{x}') \sqcup \{e_s\}$. We still call $\gamma_{\tilde{x}'}^{(2)}$ the bi-dominantly derived TFMR tree of γ at \tilde{x}' .

The following statements are parallel to Lemma 3.6.7 and Proposition 3.6.8, respectively.

Lemma 3.7.4. *Let $\tilde{\pi}_1^{(2)} : \widetilde{\mathfrak{M}}_1^{(2)} \rightarrow \widetilde{\mathfrak{M}}'$ and $\tilde{\pi}_2^{(2)} : \widetilde{\mathfrak{M}}_2^{(2)} \rightarrow \widetilde{\mathfrak{M}}'$ be two sequential blowups that are both bi-dominantly derived from the γ -compatible blowup $\pi : \widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ on $\pi^{-1}(\mathcal{V})$. Then the pullbacks of $(\pi')^{-1}(\pi^{-1}(\mathcal{V}))$ in $\widetilde{\mathfrak{M}}_1^{(2)}$ and $\widetilde{\mathfrak{M}}_2^{(2)}$ are isomorphic.*

Proposition 3.7.5. *Let $\pi : \widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ be γ -compatible on \mathcal{V} , $\pi' : \widetilde{\mathfrak{M}}' \rightarrow \widetilde{\mathfrak{M}}$ be derived from π on $\pi^{-1}(\mathcal{V})$, and $\tilde{\pi}^{(2)} : \widetilde{\mathfrak{M}}^{(2)} \rightarrow \widetilde{\mathfrak{M}}'$ be bi-dominantly derived from π on $(\pi')^{-1}(\pi^{-1}(\mathcal{V}))$. Then, for every $\tilde{x}' \in (\pi')^{-1}(\pi^{-1}(\mathcal{Z}_\gamma))$, there exists an affine smooth chart $\tilde{\mathcal{V}}_{\tilde{x}'}$ containing \tilde{x}' such that $\tilde{\pi}^{(2)}$ is $\gamma_{\tilde{x}'}^{(2)}$ -compatible on $\tilde{\mathcal{V}}_{\tilde{x}'}$ if $\gamma_{\tilde{x}'}^{(2)}$ exists and is \emptyset -compatible on $\tilde{\mathcal{V}}_{\tilde{x}'}$ otherwise.*

If $\pi : \widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ is locally TFMR tree-compatible, let $\{\mathcal{V}\}$ be the étale cover of affine smooth charts associated with π as in Definition 3.2.4 and $\pi' : \widetilde{\mathfrak{M}}' \rightarrow \widetilde{\mathfrak{M}}$ be derived from π . A sequential blowup $\tilde{\pi}^{(2)} : \widetilde{\mathfrak{M}}^{(2)} \rightarrow \widetilde{\mathfrak{M}}'$ is said to be bi-dominantly derived from π if the for each $\mathcal{V} \in \{\mathcal{V}\}$, $\tilde{\pi}^{(2)}$ is bi-dominantly derived from π on $(\pi')^{-1}(\pi^{-1}(\mathcal{V}))$. The following statement is parallel to Corollary 3.6.9.

Corollary 3.7.6. *Let $\pi : \widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$ be locally TFMR tree-compatible, $\pi' : \widetilde{\mathfrak{M}}' \rightarrow \widetilde{\mathfrak{M}}$ be derived from π , and $\tilde{\pi}^{(2)} : \widetilde{\mathfrak{M}}^{(2)} \rightarrow \widetilde{\mathfrak{M}}'$ be bi-dominantly derived from π . Then, $\tilde{\pi}^{(2)}$ is locally TFMR tree-compatible.*

4. DESCRIPTION OF THE MODULAR BLOWUPS

In this section, we introduce the three-round modular blowups of $\mathfrak{M}_2^{\text{wt}}$. The first two rounds are respectively described in §4.3 and §4.4. The last round is more sophisticated (due to the existence of conjugate and Weierstrass points for genus 2 curves), hence is divided into four phases described in §4.5-4.8. Throughout this section, the proper transform of a substack along the way is decorated with the corresponding step as the *superscript*.

4.1. Weighted graphs. We summarize the combinatorial notation for several TFMR trees related to a weighted curve $(C, \mathbf{w}) \in \mathfrak{M}_2^{\text{wt}}$ in this subsection.

Let γ be an arbitrary graph. Contracting all the edges in a subset E of $\text{Edg}(\gamma)$ results in a new graph, which we denote by $\gamma_{[E]}$, along with a surjective map

$$\pi_{[E]} : \text{Ver}(\gamma) \longrightarrow \text{Ver}(\gamma_{[E]}).$$

A pair (γ, \mathbf{w}) consisting of a graph γ and a function

$$\mathbf{w} : \text{Ver}(\gamma) \longrightarrow \mathbb{Z}_{\geq 0}$$

is called a **weighted graph** with the weight function \mathbf{w} . For each $E \subset \text{Edg}(\gamma)$, the weight function \mathbf{w} on γ induces a function

$$\mathbf{w}_{[E]} : \text{Ver}(\gamma_{[E]}) \longrightarrow \mathbb{Z}_{\geq 0}, \quad \mathbf{w}_{[E]}(v) = \sum_{v' \in \pi_{[E]}^{-1}(v)} \mathbf{w}(v'),$$

which makes $(\gamma_{[E]}, \mathbf{w}_{[E]})$ a weighted graph. We call $\mathbf{w}_{[E]}$ the induced weight on $\gamma_{[E]}$.

If in addition γ is a TFMR tree with the tree order $<$, let (γ^b, \mathbf{w}^b) be the weighted TFMR tree so that

$$\gamma^b \equiv \gamma_{[E^b]}, \quad \text{where } E^b = \left\{ e \in \text{Edg}(\gamma) : \left(\sum_{e' < e} \mathbf{w}(v_{e'}^+) \right) > 0 \right\},$$

and \mathbf{w}^b be the induced weight on γ^b . We call (γ^b, \mathbf{w}^b) the **terminally weighted TFMR tree** of (γ, \mathbf{w}) , because every positively weighted vertex of γ^b is either a terminal vertex or a root.

Let C be a nodal curve. We denote by γ^* the dual graph of C . There is a natural partition of $\text{Edg}(\gamma^*)$ given by

$$\text{Edg}(\gamma^*) = \text{E}_{\text{cr}}(\gamma^*) \sqcup \text{E}_{\text{pv}}(\gamma^*) \sqcup \text{E}_{\text{tl}}(\gamma^*)$$

corresponding to the principal nodes, the pivotal nodes, and the nodes of the tails, respectively. Recall a weighted curve is a pair (C, \mathbf{w}) consisting of a nodal curve C and an element $\mathbf{w} \in H^2(C; \mathbb{Z})$, which can be considered as a function

$$\mathbf{w} : \text{Ver}(\gamma^*) \longrightarrow \mathbb{Z}_{\geq 0}.$$

Thus, (γ^*, \mathbf{w}) is a weighted graph.

With notation as above, we set

$$(4.1) \quad \hat{\gamma} \equiv \gamma_{[\text{E}_{\text{cr}}(\gamma^*)]}^*.$$

The graph $\hat{\gamma}$ is a rooted tree whose root o corresponds to the core F . By definition,

$$\text{Edg}(\hat{\gamma}) = \text{E}_{\text{pv}}(\gamma^*) \sqcup \text{E}_{\text{tl}}(\gamma^*) = E_o(\hat{\gamma}) \sqcup \text{E}_{\text{tl}}(\gamma^*),$$

where $E_o(\hat{\gamma})$ denotes the collection of the minimal edges of $\hat{\gamma}$. As in [5], $\hat{\gamma}$ is called the **reduced dual tree** of C and

$$(4.2) \quad \gamma \equiv \hat{\gamma}^b$$

the terminally weighted tree of C .

Each subcurve F' of F determines a subgraph $\gamma_{F'}$ of γ^* . We write

$$\text{Ver}(F') \equiv \text{Ver}(\gamma_{F'}), \quad \text{Edg}(F') \equiv \text{Edg}(\gamma_{F'})$$

for conciseness. Let

$$\text{E}_{\text{pv}; F'}(\gamma^*) \subset \text{E}_{\text{pv}}(\gamma^*)$$

be the subset of the pivotal edges that each has an endpoint in $\text{Ver}(F')$. Set

$$(4.3) \quad E_{F'}^{\geq} \equiv \text{Edg}(F') \sqcup \bigsqcup_{e \in \text{E}_{\text{pv}; F'}(\gamma^*)} \{e\}^{\geq},$$

where $<$ on the right-hand side denotes the tree order on $\hat{\gamma}$.

If the core F is separable, it contains two genus 1 inseparable components F_1 and F_2 . In the weighted graph $\gamma_{[\text{Edg}(F_2) \sqcup E_{\hat{F}_1}^{\geq}]}^*$, F_1 and F_2 respectively correspond to two vertices o_1 and o_2 . This weighted graph can be considered as a rooted tree with the root o_2 . With respect to the induced tree order, let

$$(4.4) \quad \gamma_{\vee} \equiv \gamma_{\vee, F_2} \equiv \left(\gamma_{[\text{Edg}(F_2) \sqcup E_{\hat{F}_1}^{\geq}]}^* \right)^b,$$

which is still a rooted tree with the root o_2 . It is obvious that

$$o_1 \in \text{Ver}(\gamma_{\vee})^t$$

with respect to the induced tree order.

If in addition F contains a (not necessarily maximal) separating bridge B , then in the weighted graph $\gamma_{[\text{Edg}(F_1) \sqcup \text{Edg}(F_2) \sqcup E_{\hat{B}}^{\geq}]}^*$, once again F_1 and F_2 respectively correspond to two vertices o_1 and o_2 . This weighted graph can be considered as a TFMR tree with o_1 and o_2 as the roots and the bridge B corresponding to a *terminal* vertex. With respect to the induced tree order, we set

$$(4.5) \quad \gamma_{\wedge} \equiv \gamma_{\wedge, B} = \left(\gamma_{[\text{Edg}(F_1) \sqcup \text{Edg}(F_2) \sqcup E_{\hat{B}}^{\geq}]}^* \right)^b,$$

which is still a TFMR tree with the roots o_1 and o_2 . The bridge B (along with the irreducible components of C in the tails attached to B) corresponds to the unique terminal vertex

$$v_B \in \text{Ver}(\gamma_{\wedge})^t$$

that is the common endpoint of two distinct terminal edges

$$\mathbf{e}_1, \mathbf{e}_2 \in \text{Edg}(\gamma_{\wedge})^t.$$

If F contains a non-separating bridge B , let B' be the *maximal* non-separating bridge of F containing B . We denote by C_2 the closure of $F \setminus B'$ in F , which is a genus 1 subcurve. In the weighted graph $\gamma_{[\text{Edg}(C_2) \sqcup E_{\hat{B}}^{\geq}]}^*$, C_2 corresponds to a vertex \bar{o}_2 . This weighted graph can be considered as a TFMR tree with \bar{o}_2 as the root and the bridge B corresponding to a *terminal* vertex. With respect to the induced tree order, we set

$$(4.6) \quad \gamma_{\diamond} \equiv \gamma_{\diamond, B} = \left(\gamma_{[\text{Edg}(C_2) \sqcup E_{\hat{B}}^{\geq}]}^* \right)^b,$$

which is still as a TFMR tree with the root \bar{o}_2 . The bridge B (along with the irreducible components of C in the tails attached to B) corresponds to the unique terminal vertex

$$v_B \in \text{Ver}(\gamma_{\diamond})^t$$

that is the common endpoint of two distinct terminal edges

$$\mathbf{e}_1, \mathbf{e}_2 \in \text{Edg}(\gamma_{\diamond})^t.$$

4.2. **A stratification of $\mathfrak{M}_2^{\text{wt}}$.** In this section, we describe a stratification of $\mathfrak{M}_2^{\text{wt}}$ consisting of the following strata

$$\mathfrak{M}_{(1,k)}, \mathfrak{M}_{(2,k)}, \mathfrak{M}_{(3,k)}, \mathfrak{M}_{(4,k)}, \mathfrak{M}_{(5,k)}, \quad k \in \mathbb{Z}_{>0},$$

of $\mathfrak{M}_2^{\text{wt}}$. Recall we denote by F the core of a curve C . For each k ,

- $(C, \mathbf{w}) \in \mathfrak{M}_{(1,k)}$ if F is a smooth genus 2 curve of weight 0 and the tails of C are k smooth rational curves of positive weights.
- $(C, \mathbf{w}) \in \mathfrak{M}_{(2,k)}$ if F consists of two smooth genus 1 curves of weight 0 and one smooth rational curve of positive weight, and the tails of C are $(k-1)$ smooth rational curves of positive weights;
- $(C, \mathbf{w}) \in \mathfrak{M}_{(3,k)}$ if F consists of one smooth genus 1 curve of weight 0 and one smooth rational curve of positive weight, and the tails of C are $(k-1)$ smooth rational curves of positive weights;
- $(C, \mathbf{w}) \in \mathfrak{M}_{(4,k)}$ if F consists of one smooth genus 1 curve of weight 0 and one smooth genus 1 curve of positive weight, and the tails of C are $(k-1)$ smooth rational curves of positive weights;
- $(C, \mathbf{w}) \in \mathfrak{M}_{(5,k)}$ if F is a smooth genus 2 curve of weight 1 and the tails of C are k smooth rational curves of positive weights.

For each pair (i, k) , let $\overline{\mathfrak{M}}_{(i,k)}$ be the closure of $\mathfrak{M}_{(i,k)}$ in $\mathfrak{M}_2^{\text{wt}}$ and $S_{(i,k)}$ be the symmetric group of the dual graph corresponding to $\mathfrak{M}_{(i,k)}$. In Figure 1, we illustrate $\mathfrak{M}_{(i,k)}$ with $k \leq 3$. An unshaded irreducible component indicates the weight is 0.

The strata $\overline{\mathfrak{M}}_{(i,k)}$ are the images of natural node-identifying surjective immersions from smooth domains as follows. Given a finite set S , let $\overline{\mathcal{M}}_{g,S}$ be the moduli space of genus g stable curves whose marked points are indexed by S . Let $\overline{\mathfrak{M}}_{g,S}^{\text{wt};+}$ be the Artin stack of stable pairs (C, \mathbf{w}) of genus g nodal curves C whose marked points are indexed by S and weights $\mathbf{w} \in H^2(C; \mathbb{Z})$ satisfying $\mathbf{w}(C) > 0$ and $\mathbf{w}(\Sigma) \geq 0$ for all irreducible $\Sigma \subset C$. Here (C, \mathbf{w}) is said to be stable if any smooth rational irreducible component of weight 0 contains at least three nodal and/or marked points. With $\llbracket k \rrbracket$ and $\llbracket j, k \rrbracket$ as in (3.1), we define

$$\begin{aligned} \widehat{\mathfrak{M}}_{(1,k)} &\equiv \overline{\mathcal{M}}_{2, \llbracket k \rrbracket} \times \prod_{i \in \llbracket k \rrbracket} \overline{\mathfrak{M}}_{0, \{i\}}^{\text{wt};+}, \\ \widehat{\mathfrak{M}}_{(2,k)} &\equiv \bigsqcup_{\ell=1}^k \left(\overline{\mathcal{M}}_{1, \llbracket \ell-1 \rrbracket \sqcup \{\ell_+\}} \times \overline{\mathcal{M}}_{1, \llbracket \ell+1, k \rrbracket \sqcup \{\ell_-\}} \times \overline{\mathfrak{M}}_{0, \{\ell_+, \ell_-\}}^{\text{wt};+} \times \prod_{i \in \llbracket k \rrbracket \setminus \{\ell\}} \overline{\mathfrak{M}}_{0, \{i\}}^{\text{wt};+} \right), \\ \widehat{\mathfrak{M}}_{(3,k)} &\equiv \overline{\mathcal{M}}_{1, \llbracket k-1 \rrbracket \sqcup \{k_+, k_-\}} \times \overline{\mathfrak{M}}_{0, \{k_+, k_-\}}^{\text{wt};+} \times \prod_{i \in \llbracket k-1 \rrbracket} \overline{\mathfrak{M}}_{0, \{i\}}^{\text{wt};+}, \\ \widehat{\mathfrak{M}}_{(4,k)} &\equiv \overline{\mathcal{M}}_{1, \llbracket k \rrbracket} \times \overline{\mathfrak{M}}_{1, \{k\}}^{\text{wt};+} \times \prod_{i \in \llbracket k-1 \rrbracket} \overline{\mathfrak{M}}_{0, \{i\}}^{\text{wt};+}, \\ \widehat{\mathfrak{M}}_{(5,k)} &\equiv \overline{\mathcal{M}}_{2, \llbracket k \rrbracket} \times \prod_{i \in \llbracket k \rrbracket} \overline{\mathfrak{M}}_{0, \{i\}}^{\text{wt};+}. \end{aligned}$$

For each $(i, k) \in \llbracket 5 \rrbracket \times \mathbb{Z}_{>0}$, there is a natural node-identifying immersion

$$\iota_{(i,k)} : \widehat{\mathfrak{M}}_{(i,k)} \longrightarrow \overline{\mathfrak{M}}_{(i,k)} \subset \mathfrak{M}_2^{\text{wt}}$$

obtained by identifying the marked points that share the same index. Such $\iota_{(i,k)}$ is surjective onto $\overline{\mathfrak{M}}_{(i,k)}$ and descends to the quotient

$$(4.7) \quad \bar{\iota}_{(i,k)} : \widehat{\mathfrak{M}}_{(i,k)}/S_{(i,k)} \longrightarrow \overline{\mathfrak{M}}_{(i,k)} \subset \mathfrak{M}_2^{\text{wt}},$$

where $S_{(i,k)}$ is the symmetric group of the dual graph of $\mathfrak{M}_{(i,k)}$. Notice that $\bar{\iota}_{(i,k)}$ is generally not an isomorphism to its image; c.f. [14, Figure 3] with the genus 1 component replaced by a genus 2 curve, as well as the example at the end of Remark 3.2.6. Thus $\overline{\mathfrak{M}}_{(i,k)}$ is generally not smooth.

By the topology of $\mathfrak{M}_2^{\text{wt}}$, $\overline{\mathfrak{M}}_{(i,k)} \cap \mathfrak{M}_{(j,\ell)} = \emptyset$ whenever $(j, \ell) > (i, k)$ (w.r.t. the lexicographical order). Let

$$(4.8) \quad \begin{aligned} \overline{\mathfrak{M}}_{(i)} &\equiv \bigcup_{k \geq 1} \overline{\mathfrak{M}}_{(i,k)}, & \mathfrak{M}_{(i)}^{\text{mn}} &\equiv \overline{\mathfrak{M}}_{(i)} \setminus \left(\bigcup_{h \in \llbracket i-1 \rrbracket} \overline{\mathfrak{M}}_{(h)} \right), \quad 1 \leq i \leq 4; \\ \mathfrak{M}^{\text{mn}} &\equiv \mathfrak{M}_2^{\text{wt}} \setminus \left(\bigcup_{i \in \llbracket 4 \rrbracket} \overline{\mathfrak{M}}_{(i)} \right). \end{aligned}$$

In particular, $\mathfrak{M}_{(1)}^{\text{mn}} = \overline{\mathfrak{M}}_{(1)}$. Then we obtain a partition of $\mathfrak{M}_2^{\text{wt}}$:

$$\mathfrak{M}_2^{\text{wt}} = \mathfrak{M}_{(1)}^{\text{mn}} \sqcup \mathfrak{M}_{(2)}^{\text{mn}} \sqcup \mathfrak{M}_{(3)}^{\text{mn}} \sqcup \mathfrak{M}_{(4)}^{\text{mn}} \sqcup \mathfrak{M}^{\text{mn}}.$$

Notice that $\overline{\mathfrak{M}}_{(2)} \cap \overline{\mathfrak{M}}_{(3)} \subset \overline{\mathfrak{M}}_{(1)}$, thus

$$(4.9) \quad \mathfrak{M}_{(3)}^{\text{mn}} \cap \mathfrak{M}_{(2)}^{\text{mn}} = \emptyset, \quad \mathfrak{M}_{(3)}^{\text{mn}} = \overline{\mathfrak{M}}_{(3)} \setminus \overline{\mathfrak{M}}_{(1)}.$$

4.3. The first round (r_1). In r_1 , $\mathfrak{M}_2^{\text{wt}}$ is blown up successively along the proper transforms of $\overline{\mathfrak{M}}_{(i,k)}$ with respect to the lexicographical order on (i, k) :

$$\overline{\mathfrak{M}}_{(i,k)} < \overline{\mathfrak{M}}_{(j,\ell)} \iff i < j \text{ or } (i = j, k < \ell),$$

as is illustrated in Figure 1. Since each connected component of $\mathfrak{M}_2^{\text{wt}}$ is of finite type, the sequential blowup r_1 on this component will terminate after finitely many steps.

Each row (resp. element) of Figure 1 is called a **phase** (resp. **step**) of r_1 . As in (1.3), the k -th step of the i -th phase of r_1 is denoted by $r_1 p_i s_k$. The proper transform of a substack \mathfrak{N} after $r_1 p_i s_k$ (resp. after r_1) is denoted by $\mathfrak{N}^{r_1 p_i s_k}$ (resp. \mathfrak{N}^{r_1}). We write

$$\widetilde{\mathfrak{M}}^{r_1 p_i s_k} \equiv (\mathfrak{M}_2^{\text{wt}})^{r_1 p_i s_k}, \quad \widetilde{\mathfrak{M}}^{r_1} \equiv (\mathfrak{M}_2^{\text{wt}})^{r_1}.$$

The exceptional divisor obtained in $r_1 p_i s_k$ is denoted by $\mathcal{E}_{(i,k)}$, which lies in $\widetilde{\mathfrak{M}}^{r_1 p_i s_k}$.

Lemma 4.3.1. $r_1 p_1$ is locally tree-compatible.

Proof. Let $x \in \mathfrak{M}_2^{\text{wt}}$.

If $x \in \mathfrak{M}_{(1)}^{\text{mn}}$, then x has a neighborhood \mathcal{V} such that $r_1\mathfrak{p}_1$ is γ -compatible on \mathcal{V} , where γ is the terminally weighted tree of x . This follows from the same reasoning as in the desingularization of $\overline{\mathfrak{M}}_1(\mathbb{P}^n, d)$ ([5, 14]); c.f. the second paragraph of Remark 3.2.6. If $x \notin \mathfrak{M}_{(1)}^{\text{mn}} (= \overline{\mathfrak{M}}_{(1)})$, then x has a neighborhood \mathcal{V} not affected by $r_1\mathfrak{p}_1$, thus $r_1\mathfrak{p}_1$ is \emptyset -compatible on \mathcal{V} . In sum, $r_1\mathfrak{p}_1$ is locally tree-compatible. \square

Lemma 4.3.2. $r_1\mathfrak{p}_2$ is locally TFMR tree-compatible.

Proof. Let $x \in \mathfrak{M}_2^{\text{wt}}$.

If $x \in \mathfrak{M}_{(2)}^{\text{mn}}$, then the core of x contains a minimal separating bridge B so that $\deg D \cap F = \deg D \cap B$. Let γ_\wedge be the TFMR tree as in (4.5) so that B corresponds to the fused vertex v_B . It is straightforward that there is an affine smooth chart \mathcal{V} containing x so that the collection of modular parameters $\{\zeta_e\}_{e \in \text{Edg}(\gamma_\wedge)}$ is a γ_\wedge -labeled subset of local parameters on \mathcal{V} and

$$(4.10) \quad \overline{\mathfrak{M}}_{(1,k)} \cap \mathcal{V} = \emptyset, \quad \overline{\mathfrak{M}}_{(2,k)} \cap \mathcal{V} = \bigcup_{\substack{E \in \text{ET}(\gamma_\wedge) \\ |E|=k}} \mathcal{Z}_E \quad \forall k \in \mathbb{Z}_{>0}.$$

Thus, $r_1\mathfrak{p}_1$ and $r_1\mathfrak{p}_2$ are \emptyset - and γ_\wedge -compatible on \mathcal{V} , respectively.

If $x \in \overline{\mathfrak{M}}_{(2)} \cap \overline{\mathfrak{M}}_{(1)}$, then the core of x contains a (non-degenerate) *maximal* separating bridge B_M . After $r_1\mathfrak{p}_1$, let y be a lift of x and $\text{DPN}(y)$ be as in (3.15). If $\text{DPN}(y) \not\subset B_M$, then y is not in the proper transform of any $\overline{\mathfrak{M}}_{(2,k)}$, hence $r_1\mathfrak{p}_2$ is \emptyset -compatible on \mathcal{V}_y . If $\text{DPN}(y) \subset B_M$, we denote by $B \subset B_M$ the *minimal* separating bridge containing $\text{DPN}(y)$. With γ_\wedge as in the previous paragraph, we observe that $r_1\mathfrak{p}_2$ is $(\gamma_\wedge)_y$ -compatible on an affine smooth \mathcal{V}_y containing y .

If $x \notin \overline{\mathfrak{M}}_{(2)}$, then $r_1\mathfrak{p}_2$ is \emptyset -compatible on the pullback of \mathcal{V} after $r_1\mathfrak{p}_1$. In sum, $r_1\mathfrak{p}_2$ is locally TFMR tree-compatible. \square

Lemma 4.3.3. $r_1\mathfrak{p}_3$ is locally TFMR tree-compatible.

Proof. By (4.9), the proof is almost identical to that of Lemma 4.3.2, but with the separating bridge B and the TFMR tree γ_\wedge replaced by the non-separating bridge B and the TFMR tree γ_\diamond of (4.6), respectively. \square

Lemma 4.3.4. $r_1\mathfrak{p}_4$ is locally tree-compatible.

Proof. The approach in the proof of Lemma 4.3.2 can as well be applied to the current case.

To be precise, if $x \in \mathfrak{M}_{(4)}^{\text{mn}}$, let γ_\vee be as in (4.4) so that $D \cap F = D \cap F_1$. Mimicking the paragraph containing (4.10), we conclude that $r_1\mathfrak{p}_1$ - $r_1\mathfrak{p}_3$ are all \emptyset -compatible on an affine smooth \mathcal{V} containing x , whereas $r_1\mathfrak{p}_4$ is γ_\vee -compatible on \mathcal{V} .

If $x \in \overline{\mathfrak{M}}_{(4)} \cap \mathfrak{M}_{(3)}^{\text{mn}}$, let y be a lift of x after $r_1\mathfrak{p}_3$, γ_\diamond be as in (4.6), and $\text{DPN}_{\gamma_\diamond}(y)$ be as in (3.15). The core F is still separable; w.l.o.g. assume $D \cap F = D \cap F_1$. If $\text{DPN}_{\gamma_\diamond}(y) \cap F_2 \neq \emptyset$, then y is not in the proper transform

of any $\overline{\mathfrak{M}}_{(4,k)}$, hence r_1p_4 is \emptyset -compatible on an affine smooth chart \mathcal{V}_y of y . If $\text{DPN}_{\gamma_\circ}(y) \cap F_2 = \emptyset$, then by Proposition 3.5.2, r_1p_4 is $(\gamma_\vee)_y$ -compatible on an affine smooth chart \mathcal{V}_y containing y .

If $x \in \overline{\mathfrak{M}}_{(4)} \cap \mathfrak{M}_{(2)}^{\text{mn}}$, let y be a lift of x after r_1p_2 (hence after r_1p_3 by (4.9)), γ_\wedge be as in (4.5), and $\text{DP}_{\gamma_\wedge}(y)$ be as in (3.15). If there exist edges in $\text{DP}_{\gamma_\wedge}(y)$ that are respectively connected to the roots o_1 and o_2 , then y is not in the proper transform of any $\overline{\mathfrak{M}}_{(4,k)}$, hence r_1p_4 is \emptyset -compatible on a chart \mathcal{V}_y of y . If all the edges in $\text{DP}_{\gamma_\wedge}(y)$ are over one root of γ_\wedge , say o_1 , then by Proposition 3.5.2, r_1p_4 is $(\gamma_\vee)_y$ -compatible on an affine smooth chart \mathcal{V}_y containing y .

If $x \in \overline{\mathfrak{M}}_{(4)} \cap \mathfrak{M}_{(1)}^{\text{mn}}$, let y be a lift of x after r_1p_1 and $\text{DPN}(y)$ be as in (3.15). If $\text{DPN}(y) \cap F_i \neq \emptyset$ for $i = 1, 2$, then r_1p_2 - r_1p_4 are all \emptyset -compatible on an affine smooth \mathcal{V}_y containing y . Otherwise, w.l.o.g. assume that $\text{DPN}(y) \cap F_2 = \emptyset$. If in addition $\text{DPN}(y)$ lies on a bridge of F , then the argument is similar to one of the above two paragraphs. If $\text{DPN}(y)$ is not contained in any bridge of F , then r_1p_2 and r_1p_3 do not affect a chart \mathcal{V}_y containing y , and r_1p_4 is $(\gamma_\vee)_y$ -compatible on \mathcal{V}_y .

If $x \notin \overline{\mathfrak{M}}_{(4)}$, then r_1p_4 is \emptyset -compatible on the pullback of \mathcal{V} after r_1p_3 . In sum, r_1p_4 is locally TFMR tree-compatible. \square

Lemma 4.3.5. r_1p_5 is locally tree-compatible.

Proof. The approach in the proofs of Lemmas 4.3.2-4.3.4 can as well be applied to the current case.

To be precise, let $x \in \mathfrak{M}_2^{\text{wt}}$. If $x \in \mathfrak{M}_{(5)}^{\text{mn}}$, then $\deg D \cap F = 1$, hence r_1p_1 - r_1p_4 are all \emptyset -compatible on an affine smooth \mathcal{V} containing x , whereas r_1p_5 is γ -compatible on \mathcal{V} .

If $x \in \overline{\mathfrak{M}}_{(5)} \cap \mathfrak{M}_{(4)}^{\text{mn}}$, let γ_\vee be as in (4.4) so that $D \cap F \subset F_1$ and $\mathbf{e}_1 \in \text{Edg}(\gamma_\vee)^t$ be such that $q_{\mathbf{e}_1}$ is the node on F_1 separating F_1 from F_2 . Let y be a lift of x after r_1p_4 . If $\text{Dom}(y) = \{\mathbf{e}_1\}$ (resp. $\text{Dom}(y) \neq \{\mathbf{e}_1\}$), then r_1p_5 is γ_y -compatible (resp. \emptyset -compatible) on an affine smooth \mathcal{V}_y containing y .

If $x \in \overline{\mathfrak{M}}_{(5)} \cap \mathfrak{M}_{(3)}^{\text{mn}}$, let B be the irreducible component containing the only point of $D \cap F$. Since $x \in \mathfrak{M}_{(3)}^{\text{mn}}$, B belongs to a non-separating bridge. Let γ_\circ be as in (4.6) and $\mathbf{e}_1, \mathbf{e}_2 \in \text{Edg}(\gamma_\circ)^t$ be such that $q_{\mathbf{e}_1}, q_{\mathbf{e}_2}$ are the two principal nodes on B . Let y be a lift of x after r_1p_3 . If $\text{Dom}(y) \not\subset \{\mathbf{e}_1, \mathbf{e}_2\}$, then r_1p_5 is \emptyset -compatible on the pullback after r_1p_4 of an affine smooth \mathcal{V}_y containing y . If $\text{Dom}(y) \subset \{\mathbf{e}_1, \mathbf{e}_2\}$ and F is inseparable, then r_1p_4 and r_1p_5 are \emptyset - and γ_y -compatible on \mathcal{V}_y , respectively. If $\text{Dom}(y) \subset \{\mathbf{e}_1, \mathbf{e}_2\}$ and F is separable, then the argument is analogous to the $x \in \overline{\mathfrak{M}}_{(5)} \cap \mathfrak{M}_{(4)}^{\text{mn}}$ case.

If $x \in \overline{\mathfrak{M}}_{(5)} \cap \mathfrak{M}_{(2)}^{\text{mn}}$, let B be the irreducible component containing the only point of $D \cap F$. Since $x \in \mathfrak{M}_{(2)}^{\text{mn}}$, B belongs to a separating bridge. Let γ_\wedge be as in (4.5) and $\mathbf{e}_1, \mathbf{e}_2 \in \text{Edg}(\gamma_\wedge)^t$ be such that $q_{\mathbf{e}_1}$ and $q_{\mathbf{e}_2}$ are the two principal nodes on B separating B from F_1 and F_2 , respectively. Let y be a lift of x after r_1p_2 . If $\text{Dom}(y) \not\subset \{\mathbf{e}_1, \mathbf{e}_2\}$, then r_1p_5 is \emptyset -compatible on the

pullback after $r_1\mathfrak{p}_4$ of an affine smooth \mathcal{V}_y containing y . If $\text{Dom}(y) = \{\mathfrak{e}_1, \mathfrak{e}_2\}$, then $r_1\mathfrak{p}_3$ and $r_1\mathfrak{p}_4$ are \emptyset -compatible on \mathcal{V}_y , whereas $r_1\mathfrak{p}_5$ is γ_y -compatible on \mathcal{V}_y , respectively. If $\text{Dom}(y) \subsetneq \{\mathfrak{e}_1, \mathfrak{e}_2\}$, say $\text{Dom}(y) = \{\mathfrak{e}_1\}$, then $r_1\mathfrak{p}_3$ is \emptyset -compatible on \mathcal{V}_y . The remaining argument is analogous to the $x \in \overline{\mathfrak{M}}_{(5)} \cap \mathfrak{M}_{(4)}^{\text{mn}}$ case.

If $x \in \overline{\mathfrak{M}}_{(5)} \cap \mathfrak{M}_{(1)}^{\text{mn}}$, let y be a lift of x after $r_1\mathfrak{p}_1$. If $|\text{Dom}(y)| > 1$ or $\text{Dom}(y) = e$ but $|D \cap C_{v_e^+}| > 1$, then $r_1\mathfrak{p}_4$ is \emptyset -compatible on the pullback after $r_1\mathfrak{p}_4$ of an affine smooth \mathcal{V}_y containing y . It remains to consider the case when $\text{Dom}(y) = e$ but $|D \cap C_{v_e^+}| = 1$. If in addition $\langle e \rangle$ lies on a bridge of F , then the argument is similar to one of the two preceding paragraphs. If $\langle e \rangle$ is not contained in any bridge of F , then $r_1\mathfrak{p}_2$ and $r_1\mathfrak{p}_3$ do not affect an affine smooth \mathcal{V}_y containing y . If F is inseparable, then $r_1\mathfrak{p}_5$ is $((\widehat{\gamma}_{[\{e\} \leq]})^b)_y$ -compatible on \mathcal{V}_y . If F is separable, the remaining argument is analogous to the $x \in \overline{\mathfrak{M}}_{(5)} \cap \mathfrak{M}_{(4)}^{\text{mn}}$ case.

If $x \notin \overline{\mathfrak{M}}_{(5)}$, then $r_1\mathfrak{p}_5$ is \emptyset -compatible on the pullback of \mathcal{V} after $r_1\mathfrak{p}_4$. In sum, $r_1\mathfrak{p}_5$ is locally TFMR tree-compatible. \square

Lemmas 4.3.1-4.3.5, along with Corollary 3.2.5 and Proposition 3.4.2, imply the following statement.

Corollary 4.3.6. *The blowup locus at every step of r_1 is closed and smooth.*

4.4. The second round (r_2). The blowup loci of r_2 lie in the proper transforms $\mathcal{E}_{(1,k)}^{r_1\mathfrak{p}_5}$ of the exceptional divisors $\mathcal{E}_{(1,k)}$ obtained in $r_1\mathfrak{p}_1$. We first construct the corresponding loci in $\mathcal{E}_{(1,k)}^{r_1\mathfrak{p}_1}$ and show the sequential blowup along these loci is derived from $r_1\mathfrak{p}_1$. We then take their proper transform after $r_1\mathfrak{p}_5$.

The following terminology is consistent with [14, Section 3.1]. For an arbitrary stack $\overline{\mathfrak{M}}$, denote by $T\overline{\mathfrak{M}}$ and $TC\overline{\mathfrak{M}}$ its Zariski tangent space and its tangent cone, respectively. If X is a smooth stack, a morphism $\iota_X : X \rightarrow \overline{\mathfrak{M}}$ is an immersion if its differential

$$d\iota_X : TX \rightarrow \iota_X^* TC\overline{\mathfrak{M}}$$

is injective at every point of X . Let

$$\mathcal{N}_{\iota_X} \equiv \iota_X^* TC\overline{\mathfrak{M}} / \text{Im} d\iota_X$$

be the normal cone of ι_X in $\overline{\mathfrak{M}}$.

For $k \geq 1$, the normal cone of the immersion $\iota_{(1,k)} : \widehat{\mathfrak{M}}_{(1,k)} \rightarrow \mathfrak{M}_2^{\text{wt}}$ is the direct sum of k line bundles:

$$(4.11) \quad \mathcal{N}_{\iota_{(1,k)}} = \bigoplus_{i \in [k]} \left(\text{pr}_1^* \mathcal{L}_{k;i} \otimes \text{pr}_2^* \mathcal{L} \right) \equiv \bigoplus_{i \in [k]} \widehat{L}_{k;i} \rightarrow \widehat{\mathfrak{M}}_{(1,k)},$$

where $\mathcal{L}_{k;i}$ and \mathcal{L} are respectively the universal tangent line bundle at the marked point of $\overline{\mathcal{M}}_{2, [k]}$ labeled by i and the marked point of $\overline{\mathfrak{M}}_{0, \{i\}}^{\text{wt};+}$, and

$\text{pr}_1 : \widehat{\mathfrak{M}}_{(1,k)} \rightarrow \overline{\mathcal{M}}_{2, \llbracket k \rrbracket}$ and $\text{pr}_{2;i} : \widehat{\mathfrak{M}}_{(1,k)} \rightarrow \overline{\mathfrak{M}}_{0, \{i\}}^{\text{wt};+}$, $i \in \llbracket k \rrbracket$, are the projections onto components.

The immersion $\bar{\iota}_{(i,k)}$ in (4.7) induced by $\iota_{(i,k)}$ is generally not an isomorphism to its image. For $j < k$, the immersion $\iota_{(1,k)}$ induces an immersion and the corresponding quotient:

$$(4.12) \quad \begin{aligned} \iota_{(1,k)}^{r_1 \text{P}1^{\text{S}j}} : \widehat{\mathfrak{M}}_{(1,k)}^{r_1 \text{P}1^{\text{S}j}} &\longrightarrow \overline{\mathfrak{M}}_{(1,k)}^{r_1 \text{P}1^{\text{S}j}} \subset \widetilde{\mathfrak{M}}^{r_1 \text{P}1^{\text{S}j}}, \\ \bar{\iota}_{(1,k)}^{r_1 \text{P}1^{\text{S}j}} : \widehat{\mathfrak{M}}_{(1,k)}^{r_1 \text{P}1^{\text{S}j}} / S_{(1,k)} &\longrightarrow \overline{\mathfrak{M}}_{(1,k)}^{r_1 \text{P}1^{\text{S}j}} \subset \widetilde{\mathfrak{M}}^{r_1 \text{P}1^{\text{S}j}}. \end{aligned}$$

The domain $\widehat{\mathfrak{M}}_{(1,k)}^{r_1 \text{P}1^{\text{S}j}}$ is constructed by applying [14, Lemma 3.3.(1)] repeatedly. It is analogous to the smooth variety introduced in [14, Section 2.3]. To be precise, for an arbitrary finite set S , let $\mathcal{M}_{2,S;j} \subset \overline{\mathcal{M}}_{2,S}$ be consisting of marked curves that has exactly j special points (pivotal nodes and marked points) on its core. We denote by $\overline{\mathcal{M}}_{2,S;j}$ its closure in $\overline{\mathcal{M}}_{2,S}$. Mimicking the proof of Lemma 4.3.1, we conclude that the sequential blowup of $\overline{\mathcal{M}}_{2,S}$ successively along the proper transforms of

$$\overline{\mathcal{M}}_{2,S;1}, \dots, \overline{\mathcal{M}}_{2,S;|S|-1}$$

is locally tree compatible. Let $\overline{\mathcal{M}}_{2,S}^{\text{S}j}$ be the blowup of $\overline{\mathcal{M}}_{2,S}$ after the j -th step. For $S = \llbracket k \rrbracket$ and $j \in \llbracket k-1 \rrbracket$, let

$$(4.13) \quad \widehat{\mathfrak{M}}_{(1,k)}^{r_1 \text{P}1^{\text{S}j}} \equiv \overline{\mathcal{M}}_{2, \llbracket k \rrbracket}^{\text{S}j} \times \prod_{i \in \llbracket k \rrbracket} \overline{\mathfrak{M}}_{0, \{i\}}^{\text{wt};+},$$

which is smooth by Corollary 3.2.5. For $j=0$, set $\iota_{(i,k)}^{r_1 \text{P}1^{\text{S}0}} \equiv \iota_{(i,k)}$.

The immersion $\bar{\iota}_{(1,k)}^{r_1 \text{P}1^{\text{S}k-1}$ in (4.12) is an embedding. This follows from [14, Lemma 3.3.(2)] repeatedly; c.f. [14, §4.3 (I13)]. The normal cone of $\iota_{(1,k)}^{r_1 \text{P}1^{\text{S}k-1}$ is still a direct sum of k line bundles:

$$(4.14) \quad \mathcal{N}_{\iota_{(1,k)}^{r_1 \text{P}1^{\text{S}k-1}}} = \bigoplus_{j \in \llbracket k \rrbracket} L_{k;j} \longrightarrow \widehat{\mathfrak{M}}_{(1,k)}^{r_1 \text{P}1^{\text{S}k-1}};$$

c.f. [14, Lemma 3.5]. Each line bundle $L_{k;\ell}$ above is the pullback of $\widehat{L}_{k;\ell}$ twisted by some exceptional divisors of previous steps. The immersion $\iota_{(1,k)}^{r_1 \text{P}1^{\text{S}k-1}$ induces an immersion

$$(4.15) \quad \iota_{(1,k)}^{\mathcal{E}} : \mathbb{P}\mathcal{N}_{\iota_{(1,k)}^{r_1 \text{P}1^{\text{S}k-1}}} \longrightarrow \mathcal{E}_{(1,k)} = \mathbb{P}\mathcal{N}_{\widetilde{\mathfrak{M}}^{r_1 \text{P}1^{\text{S}k-1}}} \overline{\mathfrak{M}}_{(1,k)}^{r_1 \text{P}1^{\text{S}k-1}} \subset \widetilde{\mathfrak{M}}^{r_1 \text{P}1^{\text{S}k}},$$

which in turn determines an isomorphism

$$(4.16) \quad \bar{\iota}_{(1,k)}^{\mathcal{E}} : \mathbb{P}\mathcal{N}_{\iota_{(1,k)}^{r_1 \text{P}1^{\text{S}k-1}}} \longrightarrow \mathcal{E}_{(1,k)}.$$

Next, consider the index sets

$$(4.17) \quad \begin{aligned} \widehat{\mathcal{D}}_{\ell}(m) &= \left\{ (J, J', S, I = \bigsqcup_{i \in S} I_i) : I \sqcup J \subset \llbracket m \rrbracket, |J'| = \ell, \right. \\ &\quad \left. J' \sqcup S \subset J, |I_i| \geq 2 \forall i \in S \right\}, \\ \mathcal{D}_1(m) &= \left\{ (\llbracket k \rrbracket, \{j\}, S, I) \in \widehat{\mathcal{D}}_1(m) : k \in \llbracket m \rrbracket \right\} \quad (\subset \widehat{\mathcal{D}}_1(m)). \end{aligned}$$

For each $\rho = (J, J', S, I) \in \widehat{\mathcal{D}}_\ell(m)$, set

$$(4.18) \quad \begin{aligned} \kappa(\rho) &= |J|, & \sigma(\rho) &= I \sqcup (J \setminus S), \\ \widehat{M}_\rho &= \overline{\mathcal{M}}_{2,J} \times \prod_{i \in S} \overline{\mathcal{M}}_{0, \{i\} \sqcup I_i} \times \prod_{h \in \sigma(\rho)} \overline{\mathfrak{M}}_{0, \{h\}}^{\text{wt};+}. \end{aligned}$$

For each $\rho \in \mathcal{D}_1(m)$, let

$$(4.19) \quad \widehat{\mathfrak{M}}_\rho \subset \widehat{\mathfrak{M}}_{(1, \kappa(\rho))}$$

be the image of \widehat{M}_ρ in $\widehat{\mathfrak{M}}_{(1, \kappa(\rho))}$ by identifying the marked points labeled by I . With $\widehat{\mathfrak{M}}_{(1, \kappa(\rho))}^{r_1 \mathbf{p}_1 \mathbf{S}_{\kappa(\rho)} - 1}$ as in (4.12), we denote by

$$\widehat{\mathfrak{M}}_\rho^{r_1 \mathbf{p}_1 \mathbf{S}_{\kappa(\rho)} - 1} \subset \widehat{\mathfrak{M}}_{(1, \kappa(\rho))}^{r_1 \mathbf{p}_1 \mathbf{S}_{\kappa(\rho)} - 1}$$

the proper transform of $\widehat{\mathfrak{M}}_\rho$ in $\widehat{\mathfrak{M}}_{(1, \kappa(\rho))}^{r_1 \mathbf{p}_1 \mathbf{S}_{\kappa(\rho)} - 1}$. Set

$$(4.20) \quad \begin{aligned} \widehat{X}_\rho &\equiv \mathbb{P} \left(\bigoplus_{i \in \{j\} \sqcup S} L_{\kappa(\rho); i} \right) \Big|_{\widehat{\mathfrak{M}}_\rho^{r_1 \mathbf{p}_1 \mathbf{S}_{\kappa(\rho)} - 1}} \subset \mathbb{P}\mathcal{N}_{\widehat{\mathfrak{M}}_\rho^{r_1 \mathbf{p}_1 \mathbf{S}_{\kappa(\rho)} - 1}, \\ X_\rho &\equiv \bar{l}_{(1, \kappa(\rho))}^\varepsilon(\widehat{X}_\rho) = \bar{l}_{(1, \kappa(\rho))}^\varepsilon(\widehat{X}_\rho / S_{(1, \kappa(\rho))}) \subset \mathcal{E}_{(1, \kappa(\rho))}. \end{aligned}$$

If $\rho \in \mathcal{D}_1(m)$ satisfies $S = \emptyset$, then \widehat{X}_ρ and X_ρ are isomorphic to $\widehat{\mathfrak{M}}_\rho^{r_1 \mathbf{p}_1 \mathbf{S}_{\kappa(\rho)} - 1}$ and $\overline{\mathfrak{M}}_\rho^{r_1 \mathbf{p}_1 \mathbf{S}_{\kappa(\rho)} - 1}$, respectively. Some of these loci are illustrated in Figure 2. In each diagram of Figure 2, the proper transform $\widehat{\mathfrak{M}}_\rho^{r_1 \mathbf{p}_1 \mathbf{S}_{\kappa(\rho)} - 1}$ is abbreviated as $\widehat{\mathfrak{M}}_\rho^\bullet$ (to save space).

We are ready to describe a sequential blowup derived from $r_1 \mathbf{p}_1$. Let

$$\text{Ad}_1(m) \equiv \{ (k, k') \in \mathbb{Z} \times \mathbb{Z} : 1 \leq k \leq k' \leq m \}$$

be endowed with the lexicographical order. For each $(k, k') \in \text{Ad}_1(m)$, set

$$(4.21) \quad \mathcal{D}_1(k, k') = \{ \rho \in \mathcal{D}_1(m) : \kappa(\rho) = k, |\sigma(\rho)| = k' \}, \quad X_{k, k'} = \bigcup_{\rho \in \mathcal{D}_1(k, k')} X_\rho.$$

Following the notation in §4.3, we denote by $X_{k, k'}^{r_1 \mathbf{p}_1}$ the proper transform of $X_{k, k'}$ after $r_1 \mathbf{p}_1$.

Let

$$(4.22) \quad \pi : \widetilde{\mathfrak{M}}^{r_1 \mathbf{p}_1} \longrightarrow \mathfrak{M}_2^{\text{wt}} \quad \text{and} \quad \pi' : \widetilde{\mathfrak{M}}' \longrightarrow \widetilde{\mathfrak{M}}^{r_1 \mathbf{p}_1}$$

be respectively the sequential blowup of $r_1 \mathbf{p}_1$ and the blowup of $\widetilde{\mathfrak{M}}^{r_1 \mathbf{p}_1}$ successively along the proper transforms of $X_{k, k'}^{r_1 \mathbf{p}_1}$, $1 \leq k \leq k'$, with respect to the lexicographical order. For each connected component of $\mathfrak{M}_2^{\text{wt}}$ with the total weight m , the relevant blowup loci $X_{k, k'}^{r_1 \mathbf{p}_1}$ are indexed by the finite set $\text{Ad}_1(m)$, hence π' on this component will terminate after finitely many steps.

Lemma 4.4.1. *With notation as above, π' is derived from π .*

Proof. Given $x \in \mathfrak{M}_2^{\text{wt}}$, if $x \notin \overline{\mathfrak{M}}_{(1)} = \mathfrak{M}_{(1)}^{\text{mn}}$, then π is \emptyset -compatible on a chart \mathcal{V} containing x . We may assume \mathcal{V} is sufficiently small so that every $\mathcal{E}_{(1,k)}^{r_1 p_1}$ is disjoint from \mathcal{V} . Hence π' is \emptyset -compatible on $\pi^{-1}(\mathcal{V})$. By Remark 3.6.2, this implies π' is derived from π on \mathcal{V} .

If $x \in \mathfrak{M}_{(1)}^{\text{mn}}$, let γ be its terminally weighted tree. The splitting of $\mathfrak{D}_1(m)$ into $\mathfrak{D}_1(k, k')$, $(k, k') \in \text{Ad}_1(m)$, induces a splitting of $\mathfrak{D}_1(\gamma)$ into $\mathfrak{D}_1(\gamma)_{k, k'}$, $(k, k') \in \text{Ad}_1(m)$, given by

$$\mathfrak{D}_1(\gamma)_{k, k'} = \{ \mathfrak{E} = (E, e, E') \in \mathfrak{D}_1(\gamma) : |E| = k, |E'| = k' \}.$$

To each $\mathfrak{E} \in \mathfrak{D}_1(\gamma)_{k, k'}$, we assign a bijection $\iota_{\mathfrak{E}} : E \rightarrow \llbracket k \rrbracket$. Then \mathfrak{E} uniquely determines

$$(\llbracket k \rrbracket, \{ \iota_{\mathfrak{E}}(e) \}, E \setminus E', E' \setminus E) \in \mathfrak{D}_1(k, k').$$

By (4.20) and (4.19),

$$\bigcup_{\rho \in \mathfrak{D}_1(k, k')} X_{\rho}^{r_1 p_1} \cap \pi^{-1}(\mathcal{V}) = \bigcup_{\mathfrak{E} \in \mathfrak{D}_1(\gamma)_{k, k'}} \check{Y}_{\mathfrak{E}} \quad \forall (k, k') \in \text{Ad}_1(m).$$

This shows π' satisfies (C1) of Definition 3.6.3.

Given $\mathfrak{E}_i \in \mathfrak{D}_1(\gamma)_{k_i, k'_i}$, $i = 1, 2$, with $\mathfrak{E}_1 < \mathfrak{E}_2$, by (3.25), either $E_1 < E_2$, which implies $k_1 < k_2$ and hence $(k_1, k'_1) < (k_2, k'_2)$, or $E_1 = E_2$, $e_1 = e_2$, and $E'_1 < E'_2$, which implies $k_1 = k_2$ and $k'_1 < k'_2$, and thus $(k_1, k'_1) < (k_2, k'_2)$. Therefore, the condition (C2) of Definition 3.6.3 is also satisfied. \square

In r_2 , the stack $\widetilde{\mathfrak{M}}^{r_1}$ is blown up successively along the proper transforms of $X_{k, k'}^{r_1}$, $1 \leq k \leq k'$, with respect to the lexicographical order; c.f. Figure 2. As explained before Lemma 4.4.1, the sequential blowup r_2 will terminate after finitely many steps on each connected component of $\widetilde{\mathfrak{M}}^{r_1}$. In each diagram of Figure 2, we illustrate (the preimage under (4.15) of) a typical element of $X_{k, k'}$. Note that the only non-empty $X_{1, k'}^{r_1}$ is the divisor $X_{1, 1}^{r_1}$; blowing up along it does not change anything. Thus in Figure 2, the first diagram is for $X_{2, 2}$.

The proper transform of a substack \mathfrak{N} (of either the original stack $\mathfrak{M}_2^{\text{wt}}$ or its blowup after any intermediate step) after the step $r_2 s_{k, k'}$ (resp. after r_2) is denoted by $\mathfrak{N}^{r_2 s_{k, k'}}$ (resp. \mathfrak{N}^{r_2}). We write

$$\widetilde{\mathfrak{M}}^{r_2 s_{k, k'}} \equiv (\mathfrak{M}_2^{\text{wt}})^{r_2 s_{k, k'}}, \quad \widetilde{\mathfrak{M}}^{r_2} \equiv (\mathfrak{M}_2^{\text{wt}})^{r_2}.$$

The exceptional divisor in the step $r_2 s_{k, k'}$ is denoted by $\mathcal{E}_{k, k'}$ ($\subset \widetilde{\mathfrak{M}}^{r_2 s_{k, k'}}$).

It is a direct check that (the proper transform of) π' in Lemma 4.4.1 is compatible with each sequential blowup of $r_1 p_j$, $2 \leq j \leq 5$. Applying Corollary 3.5.3 repeatedly, we thus obtain the following statement.

Lemma 4.4.2. r_2 is locally tree-compatible.

Lemma 4.4.2 and Corollary 3.2.5 lead to the following conclusion.

Corollary 4.4.3. The blowup locus at every step of r_2 is closed and smooth.

4.5. **The first phase of the third round** ($r_3\mathbf{p}_1$). The blowup loci of $r_3\mathbf{p}_1$ lie in the proper transforms $\mathcal{E}_{(1,k)}^{r_2}$ of the exceptional divisors $\mathcal{E}_{(1,k)}$ obtained in $r_1\mathbf{p}_1$. We first construct the corresponding loci in $\mathcal{E}_{(1,k)}^{r_1\mathbf{p}_1}$ and take its proper transform after r_2 . We then show the blowup along these loci is *supplementary* to the proper transform of a sequential blowup *bi-dominantly* derived from $r_1\mathbf{p}_1$ (c.f. §3.2 and §3.7 for terminology).

Given a finite set S and distinct $i_1, i_2 \in S$, we denote by $\mathcal{K}_{S; i_1, i_2}$ the locus in $\overline{\mathcal{M}}_{2,S}$ so that the marked points indexed by i_1 and i_2 are conjugate. As in §2.2, it is a Cartier divisor. Consider the index set

$$\mathfrak{D}_2(m) = \{ ([k], \{j_1, j_2\}, S, I) \in \widehat{\mathfrak{D}}_2(m) : k \in [2, m] \} \quad (\subset \widehat{\mathfrak{D}}_2(m)),$$

where $\widehat{\mathfrak{D}}_2(m)$ is as in (4.17). For each $\rho = ([k], \{j_1, j_2\}, S, I) \in \mathfrak{D}_2(m)$, let

$$\widehat{K}_\rho \equiv \mathcal{K}_{[k]; j_1, j_2} \times \prod_{i \in S} \overline{\mathcal{M}}_{0, \{i\} \sqcup I_i} \times \prod_{h \in \sigma(\rho)} \overline{\mathfrak{M}}_{0, \{h\}}^{\text{wt}; +} \quad (\subset \widehat{M}_\rho).$$

Then, \widehat{K}_ρ is a Cartier divisor of \widehat{M}_ρ ; see (4.18) for notation.

For $\rho \in \mathfrak{D}_2(m)$, we denote by

$$(4.23) \quad \widehat{\mathfrak{M}}_\rho \quad \text{and} \quad \widehat{\mathfrak{K}}_\rho \quad \subset \quad \widehat{\mathfrak{M}}_{(1, \kappa(\rho))}$$

the images of \widehat{M}_ρ and \widehat{K}_ρ in $\widehat{\mathfrak{M}}_{(1, \kappa(\rho))}$ by identifying the marked points labeled by I , respectively. With $\widehat{\mathfrak{M}}_{(1, \kappa(\rho))}^{r_1\mathbf{p}_1\mathfrak{S}_{\kappa(\rho)-1}}$ as in (4.12), we denote by

$$\widehat{\mathfrak{M}}_\rho^{r_1\mathbf{p}_1\mathfrak{S}_{\kappa(\rho)-1}} \quad \text{and} \quad \widehat{\mathfrak{K}}_\rho^{r_1\mathbf{p}_1\mathfrak{S}_{\kappa(\rho)-1}} \quad \subset \quad \widehat{\mathfrak{M}}_{(1, \kappa(\rho))}^{r_1\mathbf{p}_1\mathfrak{S}_{\kappa(\rho)-1}}$$

the proper transforms of $\widehat{\mathfrak{M}}_\rho$ and $\widehat{\mathfrak{K}}_\rho$ in $\widehat{\mathfrak{M}}_{(1, \kappa(\rho))}^{r_1\mathbf{p}_1\mathfrak{S}_{\kappa(\rho)-1}}$, respectively. Set

$$\begin{aligned} \widehat{K}_\rho &\equiv \mathbb{P} \left(\bigoplus_{i \in \{j_1, j_2\} \sqcup S} L_{\kappa(\rho); i} \right) \Big|_{\widehat{\mathfrak{K}}_\rho^{r_1\mathbf{p}_1\mathfrak{S}_{\kappa(\rho)-1}}} \\ &\subset \widehat{X}_\rho^{(2)} \equiv \mathbb{P} \left(\bigoplus_{i \in \{j_1, j_2\} \sqcup S} L_{\kappa(\rho); i} \right) \Big|_{\widehat{\mathfrak{M}}_\rho^{r_1\mathbf{p}_1\mathfrak{S}_{\kappa(\rho)-1}}} \quad (\subset \mathbb{P}\mathcal{N}_{\iota_{(1, \kappa(\rho))}^{r_1\mathbf{p}_1\mathfrak{S}_{\kappa(\rho)-1}}}), \\ K_\rho &\equiv \iota_{(1, \kappa(\rho))}^\varepsilon(\widehat{K}_\rho) \quad \subset \quad X_\rho^{(2)} \equiv \iota_{(1, \kappa(\rho))}^\varepsilon(\widehat{X}_\rho^{(2)}) \quad \subset \mathcal{E}_{(1, \kappa(\rho))}. \end{aligned}$$

Similarly, K_ρ is a Cartier divisor of $X_\rho^{(2)}$.

We next describe a sequence of blowups bi-dominantly derived from the modular blowups in $r_1\mathbf{p}_1$. Let

$$\text{Ad}_2(m) \equiv \{ (k, k') \in \mathbb{Z} \times \mathbb{Z} : 2 \leq k \leq k' + 1 \leq m \}$$

For each $(k, k') \in \text{Ad}_2(m)$, set

$$\begin{aligned} \mathfrak{D}_2(k, k') &= \{ \rho \in \mathfrak{D}_2(m) : \kappa(\rho) = k, |\sigma(\rho)| = k' \}, \\ X_{k, k'}^{(2)} &= \bigcup_{\rho \in \mathfrak{D}_2(k, k')} X_\rho^{(2)}, \quad K_{k, k'} = \bigcup_{\rho \in \mathfrak{D}_2(k, k')} K_\rho. \end{aligned}$$

Following the notation in §4.3, we denote by $X_{k,k'}^{(2);r_1\mathbf{p}_1}$ and $K_{k,k'}^{r_1\mathbf{p}_1}$ the proper transforms of $X_{k,k'}^{(2)}$ and $K_{k,k'}$ after $r_1\mathbf{p}_1$, respectively.

Let $\pi: \widetilde{\mathfrak{M}}^{r_1\mathbf{p}_1} \rightarrow \mathfrak{M}_2^{\text{wt}}$ and $\pi': \widetilde{\mathfrak{M}}' \rightarrow \widetilde{\mathfrak{M}}^{r_1\mathbf{p}_1}$ be as in (4.22), and

$$\check{\pi}^{(2)}: \widetilde{\mathfrak{M}}^{(2)} \longrightarrow \widetilde{\mathfrak{M}}'$$

be the blowup of $\widetilde{\mathfrak{M}}'$ successively along the proper transforms of $X_{k,k'}^{(2);r_1\mathbf{p}_1}$, $2 \leq k \leq k'$, with respect to the lexicographical order. For each connected component of $\mathfrak{M}_2^{\text{wt}}$ with the total weight m , the relevant blowup loci $X_{k,k'}^{(2);r_1\mathbf{p}_1}$ are indexed by the finite set $\text{Ad}_2(m)$, hence $\check{\pi}^{(2)}$ on this component will terminate after finitely many steps. The following statement is parallel to Lemma 4.4.1; we omit its proof.

Lemma 4.5.1. *With notation as above, $\check{\pi}^{(2)}$ is bi-dominantly derived from the blowup π , and the proper transform of $\check{\pi}^{(2)}$ is compatible with $r_1\mathbf{p}_2$ - r_2 .*

In $r_3\mathbf{p}_1$, we blow up $\widetilde{\mathfrak{M}}^{r_2}$ successively along the proper transforms of $K_{k,k'}^{r_1\mathbf{p}_1}$, $2 \leq k \leq k'$, with respect to the lexicographical order. As explained before Lemma 4.5.1, the sequential blowup $r_3\mathbf{p}_1$ will terminate after finitely many steps on each connected component of $\widetilde{\mathfrak{M}}^{r_2}$. The proper transform of a substack \mathfrak{N} (of either the original stack $\mathfrak{M}_2^{\text{wt}}$ or its blowup after any intermediate step) after the step $r_3\mathbf{p}_1\mathfrak{s}_{k,k'}$ (resp. after $r_3\mathbf{p}_1$) is denoted by $\mathfrak{N}^{r_3\mathbf{p}_1\mathfrak{s}_{k,k'}}$ (resp. $\mathfrak{N}^{r_3\mathbf{p}_1}$). We write

$$\widetilde{\mathfrak{M}}^{r_3\mathbf{p}_1\mathfrak{s}_{k,k'}} \equiv (\mathfrak{M}_2^{\text{wt}})^{r_3\mathbf{p}_1\mathfrak{s}_{k,k'}}, \quad \widetilde{\mathfrak{M}}^{r_3\mathbf{p}_1} \equiv (\mathfrak{M}_2^{\text{wt}})^{r_3\mathbf{p}_1}.$$

Lemma 4.5.2. *$r_3\mathbf{p}_1$ is locally tree-compatible.*

Proof. Let $\widehat{\mathfrak{K}}_\rho^{\text{n-s}}$ be the closed locus in $\widehat{\mathfrak{K}}_\rho$ where the nodes labeled by j_1 and j_2 are on a same non-separating bridge. Notice that $\widehat{\mathfrak{K}}_\rho$ is possibly singular only along $\widehat{\mathfrak{K}}_\rho^{\text{n-s}}$. Given $x \in \widehat{\mathfrak{K}}_\rho^{\text{n-s}}$, w.l.o.g. we assume the nodes q_{j_1} and q_{j_2} are on a maximal non-separating bridge $B[\mathfrak{p}, \mathfrak{q}]$ such that $N_{[q_{j_1}, \mathfrak{p}]} \subset N_{[q_{j_2}, \mathfrak{p}]}$. Lemma 2.8.2 then implies there exist a smooth chart $\mathcal{V} \subset \widehat{\mathfrak{M}}_\rho$ containing x and modular parameters $\{\zeta_q\}$ corresponding to the nodes of $B[\mathfrak{p}, \mathfrak{q}]$ so that

$$\widehat{\mathfrak{K}}_\rho \cap \mathcal{V} = \{\zeta_{[q_{j_1}, \mathfrak{p}]} + \zeta_{[q_{j_2}, \mathfrak{q}]} = 0\}.$$

For any lift \tilde{x} of x after $r_1\mathbf{p}_3$ that lies in the proper transform $\widehat{\mathfrak{K}}_\rho^{r_1\mathbf{p}_3}$, we see that $\zeta_{[q_{j_1}, \mathfrak{p}]}$ and $\zeta_{[q_{j_2}, \mathfrak{q}]}$ pull back to

$$\tilde{\varepsilon}_1 \cdots \tilde{\varepsilon}_t \cdot u_1 \cdots u_{h_1} \quad \text{and} \quad \tilde{\varepsilon}_1 \cdots \tilde{\varepsilon}_t \cdot v_1 \cdots v_{h_2},$$

respectively, where $\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_t$ are the local parameters as in Lemma 3.3.2 whose vanishing loci are the proper transforms of the exceptional divisors of $r_1\mathbf{p}_3$, and $u_1, \dots, u_{h_1}, v_1, \dots, v_{h_2}$ are distinct *nowhere vanishing* local parameters corresponding to the positions of the images of \tilde{x} in the relevant exceptional divisors. Thus, $\widehat{\mathfrak{K}}_\rho^{r_1\mathbf{p}_3}$ is locally given by

$$u_1 \cdots u_{h_1} + v_1 \cdots v_{h_2} = 0.$$

It is straightforward that $\widehat{\mathfrak{K}}_\rho^{r_1\mathfrak{p}_3}$ is smooth near \tilde{x} .

In sum, we have shown that the proper transform of $\widehat{\mathfrak{K}}_\rho$ is a smooth Cartier divisor of $\widehat{\mathfrak{M}}_\rho$ after $r_1\mathfrak{p}_3$. This, along with Lemma 4.5.1, implies that $r_3\mathfrak{p}_1$ is supplementary to the proper transform of $\tilde{\pi}^{(2)}$, hence is locally tree-compatible. \square

Lemma 4.5.2 and Corollary 3.2.5 lead to the following conclusion.

Corollary 4.5.3. *The blowup locus at every step of $r_3\mathfrak{p}_1$ is closed and smooth.*

4.6. The second phase of the third round ($r_3\mathfrak{p}_2$). The blowup loci of $r_3\mathfrak{p}_2$ lie in the proper transforms $\mathcal{E}_{k,k'}^{r_3\mathfrak{p}_1}$ of the exceptional divisors $\mathcal{E}_{k,k'}$, $(k, k') \in \text{Ad}_1(m)$, obtained in r_2 . We first construct the corresponding loci in $\mathcal{E}_{k,k'}^{r_2}$ and show the blowup along these loci is *supplementary* to a sequential blowup *partially* derived from r_2 (c.f. §3.2 and §3.6 for terminology). We then take its proper transform after $r_3\mathfrak{p}_1$.

Given a finite set J and $j \in J$, we denote by $\mathcal{W}_{J;j}$ the locus in $\overline{\mathcal{M}}_{2,J}$ so that the marked point labeled by j is Weierstrass. As shown in §2.2, it is a Cartier divisor.

For $\rho = (\llbracket k \rrbracket, \{j\}, S, I) \in \mathfrak{D}_1(m)$, let

$$\mathfrak{N}(\rho) = \{(\dot{J}, \{\dot{j}\}, \dot{S}, \dot{I}) \in \widehat{\mathfrak{D}}_1(m) : \dot{J} = \sigma(\rho)\} \subset \widehat{\mathfrak{D}}_1(m).$$

For each $\dot{\rho} = (\sigma(\rho), \{\dot{j}\}, \dot{S}, \dot{I}) \in \mathfrak{N}(\rho)$, set

$$\widehat{\mathcal{W}}_{\rho, \dot{\rho}}^\dagger = \mathcal{W}_{\llbracket k \rrbracket; j} \times \prod_{i \in S} \overline{\mathcal{M}}_{0, \{i\} \sqcup I_i} \times \prod_{\ell \in \dot{S}} \overline{\mathcal{M}}_{0, \{\ell\} \sqcup \dot{I}_\ell} \times \prod_{h \in \sigma(\dot{\rho})} \overline{\mathfrak{M}}_{0, \{h\}}^{\text{wt};+}.$$

Let $\widehat{\mathcal{W}}_{\rho, \dot{\rho}}$ be the image of $\widehat{\mathcal{W}}_{\rho, \dot{\rho}}^\dagger$ in $\widehat{\mathcal{M}}_\rho$ of (4.18) by identifying the marked points labeled by \dot{I} . As in (4.13), we set

$$\widehat{\mathcal{M}}_\rho^{r_1\mathfrak{p}_1\mathfrak{s}_{\kappa(\rho)-1}} = \overline{\mathcal{M}}_{2, \llbracket \kappa(\rho) \rrbracket}^{\mathfrak{s}_{\kappa(\rho)-1}} \times \prod_{i \in S} \overline{\mathcal{M}}_{0, \{i\} \sqcup I_i} \times \prod_{h \in \sigma(\rho)} \overline{\mathfrak{M}}_{0, \{h\}}^{\text{wt};+},$$

which in turn determines $\widehat{\mathcal{W}}_{\rho, \dot{\rho}}^{r_1\mathfrak{p}_1\mathfrak{s}_{\kappa(\rho)-1}} \subset \widehat{\mathcal{M}}_\rho^{r_1\mathfrak{p}_1\mathfrak{s}_{\kappa(\rho)-1}}$. With

$$\nu_\rho : \widehat{\mathcal{M}}_\rho \rightarrow \widehat{\mathfrak{M}}_\rho$$

denoting the node identifying immersion in (4.19) and $\widehat{\mathcal{X}}_\rho$ denoting the pull-back of \widehat{X}_ρ of (4.20) to $\widehat{\mathcal{M}}_\rho$, let

$$\widehat{\mathcal{X}}_{\mathfrak{w}, \rho, \dot{\rho}} = \widehat{\mathcal{X}}_\rho|_{\widehat{\mathcal{W}}_{\rho, \dot{\rho}}^{r_1\mathfrak{p}_1\mathfrak{s}_{\kappa(\rho)-1}}}.$$

We next analyze the exceptional divisors $\mathcal{E}_{k,k'}$ of r_2 . With the line bundles $L_{k;i}$ as in (4.14), for each non-empty $J \subset \llbracket k \rrbracket$, let

$$\gamma_{k,J} \longrightarrow \mathbb{P}\left(\bigoplus_{i \in J} L_{k;i}\right)$$

be the tautological line bundle. For each $\rho \in \mathfrak{D}_1(m)$, let

$$(4.24) \quad \begin{aligned} \iota_\rho &\equiv \iota_{(1,\kappa(\rho))}^\varepsilon \circ \nu_\rho : \widehat{X}_\rho \longrightarrow X_\rho \subset \widetilde{\mathfrak{M}}^{r_1 \mathbf{P}1\mathbf{S}_{\kappa(\rho)}}, \\ \bar{\iota}_\rho &\equiv \bar{\iota}_{(1,\kappa(\rho))}^\varepsilon \circ \nu_\rho : \widehat{X}_\rho / G_\rho \longrightarrow X_\rho \subset \widetilde{\mathfrak{M}}^{r_1 \mathbf{P}1\mathbf{S}_{\kappa(\rho)}}, \end{aligned}$$

where $\iota_{(1,\kappa(\rho))}^\varepsilon$ and $\bar{\iota}_{(1,\kappa(\rho))}^\varepsilon$ are as in (4.15) and (4.16), respectively, and G_ρ is the symmetry group of the dual graph of a general element of \widehat{M}_ρ . By (4.14) and (4.20), the normal cone of ι_ρ can be written explicitly as

$$\mathcal{N}_{\iota_\rho} \simeq \nu_\rho^* \gamma_{\kappa(\rho), \{j\} \sqcup S} \oplus \bigoplus_{i \in I} \widehat{L}_{\rho; i} \oplus \nu_\rho^* \left(\gamma_{\kappa(\rho), \{j\} \sqcup S}^\vee \otimes \left(\bigoplus_{i \in [k] \setminus (\{j\} \sqcup S)} L_{\kappa(\rho); i} \right) \right),$$

where $\widehat{L}_{\rho; i}$, $i \in I$, are line bundles corresponding to the smoothing of the nodes labeled by I ; they are analogous to the line bundles in (4.11). We set

$$\nu_\rho^* \widehat{L}_{\rho; j} \equiv \gamma_{\kappa(\rho), \{j\} \sqcup S}, \quad \widehat{L}_{\rho; i} \equiv \nu_\rho^* \left(\gamma_{\kappa(\rho), \{j\} \sqcup S}^\vee \otimes L_{\kappa(\rho); i} \right) \quad \forall i \in k \setminus (\{j\} \sqcup S).$$

Then $\mathcal{N}_{\iota_\rho} = \bigoplus_{i \in \sigma(\rho)} \widehat{L}_{\rho; i}$.

For each $(k, k') \in \text{Ad}_1(m)$, we denote by $(k, k')-1$ its immediate predecessor with respect to the lexicographical order on $\text{Ad}_1(m)$. After $r_{2\mathbf{S}(k, k')-1}$, the immersion ι_ρ with $\rho \in \mathfrak{D}_1(k, k')$ induces the immersion

$$\iota_\rho^{r_{2\mathbf{S}(k, k')-1}} : \widehat{X}_\rho^{r_{2\mathbf{S}(k, k')-1}} \longrightarrow X_\rho^{r_{2\mathbf{S}(k, k')-1}} \subset \mathcal{E}_{(1, k)}^{r_{2\mathbf{S}(k, k')-1}} \subset \widetilde{\mathfrak{M}}^{r_{2\mathbf{S}(k, k')-1}},$$

where $\widehat{X}_\rho^{r_{2\mathbf{S}(k, k')-1}}$ is obtained by blowing up \widehat{X}_ρ successively along some smooth substacks analogously to the construction of $\widehat{\mathfrak{M}}_{(1, k)}^{r_1 \mathbf{P}1\mathbf{S}_j}$ in (4.13). The key fact is that the normal cone of $\iota_\rho^{r_{2\mathbf{S}(k, k')-1}}$ is still in the form

$$(4.25) \quad \mathcal{N}_{\iota_\rho^{r_{2\mathbf{S}(k, k')-1}}} \equiv \bigoplus_{i \in \sigma(\rho)} L_{\rho; i}.$$

Each line bundle $L_{\rho; i}$ above is the pullback of $\widehat{L}_{\rho; i}$ twisted by some exceptional divisors of previous steps. Furthermore, the immersion $\iota_\rho^{r_{2\mathbf{S}(k, k')-1}}$ determines

$$\bar{\iota}_\rho^{r_{2\mathbf{S}(k, k')-1}} : \widehat{X}_\rho^{r_{2\mathbf{S}(k, k')-1}} / G_\rho \longrightarrow X_\rho^{r_{2\mathbf{S}(k, k')-1}} \subset \widetilde{\mathfrak{M}}^{r_{2\mathbf{S}(k, k')-1}},$$

which is isomorphic to its image. Thus, the immersion

$$\iota_\rho^\varepsilon : \mathbb{P}\mathcal{N}_{\iota_\rho^{r_{2\mathbf{S}(k, k')-1}}} \longrightarrow \mathcal{E}_{k, k'}$$

determines an isomorphism

$$\bar{\iota}_\rho^\varepsilon : \mathbb{P}\mathcal{N}_{\bar{\iota}_\rho^{r_{2\mathbf{S}(k, k')-1}}} \longrightarrow \mathcal{E}_{k, k'}.$$

For each $\rho \in \mathfrak{D}_1(k, k')$ and each $\dot{\rho} \in \mathfrak{N}(\rho)$, we set

$$(4.26) \quad \begin{aligned} \widehat{W}_{\rho, \dot{\rho}} &\equiv \mathbb{P} \left(\bigoplus_{i \in \{j\} \sqcup S} L_{\rho; i} \right) \Big|_{\widehat{X}_{w, \rho, \dot{\rho}}^{r_{2\mathbf{S}(k, k')-1}}} \subset \mathbb{P}\mathcal{N}_{\iota_\rho^{r_{2\mathbf{S}(k, k')-1}}}, \\ W_{\rho, \dot{\rho}} &\equiv \iota_\rho^\varepsilon(\widehat{W}_{\rho, \dot{\rho}}) = \bar{\iota}_\rho^\varepsilon(\widehat{W}_{\rho, \dot{\rho}} / G_\rho) \subset \mathcal{E}_{k, k'}. \end{aligned}$$

We are ready to describe a sequential blowup partially derived from r_2 .
Let

$$\text{Ad}_w(m) \equiv \{ (k, k', k'') \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : 1 \leq k \leq k' \leq k'' \leq m \}$$

be endowed with the lexicographical order. For each $(k, k', k'') \in \text{Ad}_w(m)$, set

$$\mathfrak{D}_w(k, k', k'') = \{ (\rho, \dot{\rho}) : \rho \in \mathcal{D}_1(k, k'), \dot{\rho} \in \mathfrak{N}(\rho), |\sigma(\dot{\rho})| = k'' \},$$

$$W_{k, k', k''} = \bigcup_{(\rho, \dot{\rho}) \in \mathfrak{D}_w(k, k', k'')} W_{\rho, \dot{\rho}}.$$

Following the notation in §4.3, we denote by $W_{k, k', k''}^{r_2}$ the proper transform of $W_{k, k', k''}$ after r_2 .

In r_{3p_2} , we blow up $\widetilde{\mathfrak{M}}^{r_{3p_1}}$ successively along the proper transforms of $W_{k, k', k''}^{r_{3p_1}}$, $1 \leq k \leq k' \leq k''$, with respect to the lexicographical order. For each connected component of $\mathfrak{M}_2^{\text{wt}}$ with the total weight m , the relevant blowup loci of r_{3p_2} are indexed by the finite set $\text{Ad}_w(m)$, hence r_{3p_2} will terminate after finitely many steps on each connected component of $\widetilde{\mathfrak{M}}^{r_{3p_1}}$. The proper transform of a substack \mathfrak{N} (of either the original stack $\mathfrak{M}_2^{\text{wt}}$ or its blowup after any intermediate step) after the step $r_{3p_2s_{k, k', k''}}$ (resp. after r_{3p_2}) is denoted by $\mathfrak{N}^{r_{3p_2s_{k, k', k''}}}$ (resp. $\mathfrak{N}^{r_{3p_2}}$). We write

$$\widetilde{\mathfrak{M}}^{r_{3p_2s_{k, k', k''}}} \equiv (\widetilde{\mathfrak{M}}^{r_{3p_2}})^{r_{3p_2s_{k, k', k''}}}, \quad \widetilde{\mathfrak{M}}^{r_{3p_2}} \equiv (\widetilde{\mathfrak{M}}^{r_{3p_3}})^{r_{3p_2}}.$$

Lemma 4.6.1. r_{3p_2} is locally tree-compatible.

Proof. The proof is parallel to that of Lemma 4.5.2, except the local equation should be deduced from Lemma 2.8.1 instead of Lemma 2.8.2. We omit further details. \square

Lemma 4.6.1 and Corollary 3.2.5 lead to the following conclusion.

Corollary 4.6.2. The blowup locus at every step of r_{3p_2} is closed and smooth.

4.7. The third phase of the third round (r_{3p_3}). The blowup loci of r_{3p_3} lie in the relative Picard stack \mathfrak{P}_2 over $\mathfrak{M}_2^{\text{wt}}$ instead of the stack $\mathfrak{M}_2^{\text{wt}}$ itself. For a general point $(C, L) \in \mathfrak{P}_2$ with $\deg L = 2$, we observe that $h^0(L) = 1$; the locus

$$\mathfrak{H} = \{ (C, L) \in \mathfrak{P}_2 : \deg L = 2, h^0(L) = 2 \}$$

is a Cartier divisor of \mathfrak{P}_2 . For $k \geq 0$, let \mathcal{H}_k be the closed substack of \mathfrak{P}_2 whose general points are pairs (C, L) with F denoting the core of C such that $(F, L|_F) \in \mathfrak{H}$ and C has k tails attached to F . Note that if $(C, D) \in \mathfrak{M}_2^{\text{div}}$ with k tails satisfies

$$D \cap F = \{ \delta_1, \delta_2 \}, \quad \delta_1 \text{ is conjugate to } \delta_2,$$

then $(C, \mathcal{O}_C(D)) \in \mathcal{H}_k$.

When the modular blowups of r_1 terminates, let

$$\mathfrak{P}^{r_1} \equiv \mathfrak{P}_2 \times_{\mathfrak{M}_2^{\text{wt}}} \widetilde{\mathfrak{M}}^{r_1}, \quad \mathcal{H}_k^{r_1} \equiv \mathcal{H}_k \times_{\mathfrak{M}_2^{\text{wt}}} \widetilde{\mathfrak{M}}^{r_1} \quad \forall k \geq 0.$$

We denote by $\pi^\dagger : \mathfrak{P}^\dagger \rightarrow \tilde{\mathfrak{P}}^{r_1}$ the sequential blowup successively along the proper transforms of $\mathcal{H}_k^{r_1}$, $k \geq 0$. Since each connected component of $\mathfrak{M}_2^{\text{wt}}$ is of finite type, so is that of $\tilde{\mathfrak{P}}^{r_1}$. The sequential blowup π^\dagger on any connected component will thus terminate after finitely many steps.

Lemma 4.7.1. π^\dagger is locally tree-compatible.

Proof. Given $(C, L) \in \mathfrak{P}_2$, fix $(C, D) \in \mathfrak{M}_2^{\text{div}}$ so that $L = \mathcal{O}_C(D)$. If (C, L) is not contained in any \mathcal{H}_k , obviously π^\dagger is \emptyset -compatible on the pullback of a chart \mathcal{V} containing (C, L) after r_1 . Hereafter we assume (C, L) is contained in at least on \mathcal{H}_k . In particular, this implies $\deg D \cap F \leq 2$.

If $\deg D \cap F = 2$ and $x \equiv (C, c_1(L)) \in \mathfrak{M}_2^{\text{wt}}$ is not contained in $\overline{\mathfrak{M}}_{(i)}$ for any $i \in \llbracket 5 \rrbracket$, then the entire r_1 does not affect a neighborhood of x . With γ denoting the terminally weighted tree of x as in (4.2), we observe that π^\dagger is γ_+ -compatible on a chart \mathcal{V} containing (C, L) , where the extra edge corresponds to the normal direction of \mathfrak{H} in \mathfrak{P}_2 .

For all other possibilities of (C, L) , the proof is similar to that of Lemma 4.3.5 and/or Lemma 4.5.2, hence is omitted. \square

After r_{3p_2} , let

$$\tilde{\mathfrak{P}}^{r_{3p_2}} \equiv \mathfrak{P}_2 \times_{\mathfrak{M}_2^{\text{wt}}} \tilde{\mathfrak{M}}^{r_{3p_2}}, \quad \mathcal{H}_k^{r_{3p_2}} \equiv \mathcal{H}_k \times_{\mathfrak{M}_2^{\text{wt}}} \tilde{\mathfrak{M}}^{r_{3p_2}} \quad \forall k \geq 0.$$

In r_{3p_3} , we blow up the stack $\tilde{\mathfrak{P}}^{r_{3p_2}}$ successively along the proper transforms of $\mathcal{H}_k^{r_{3p_2}}$. As explained before Lemma 4.7.1, the sequential blowup r_{3p_3} will terminate after finitely many steps on each connected component of $\tilde{\mathfrak{P}}^{r_{3p_2}}$. The proper transform of a substack \mathfrak{N} (of either the original stack \mathfrak{P}_2 or its blowup after any intermediate step) after the step $r_{3p_3s_k}$ (resp. after the phase r_{3p_3}) is denoted by $\mathfrak{N}^{r_{3p_3s_k}}$ (resp. $\mathfrak{N}^{r_{3p_3}}$).

It is a direct check that (the proper transform of) π^\dagger in Lemma 4.7.1 is compatible with r_2 - r_{3p_2} . Applying Corollary 3.5.3 repeatedly, we thus obtain the following statement.

Lemma 4.7.2. r_{3p_3} is locally tree-compatible.

Lemma 4.7.2 and Corollary 3.2.5 lead to the following conclusion.

Corollary 4.7.3. The blowup locus at every step of r_{3p_3} is closed and smooth.

4.8. The fourth phase of the third round (r_{3p_4}). The process of r_{3p_4} is analogous to the round r_2 , but the blowup loci lie in the proper transforms of \mathfrak{P}_2 instead of $\mathfrak{M}_2^{\text{wt}}$.

Notice that the main difference between $\overline{\mathfrak{M}}_{(1)}$ and $\overline{\mathfrak{M}}_{(5)}^{r_{1p_4}}$ in (4.8) is the weight of the core curve. Thus, the exceptional divisors obtained in the phases r_{1p_1} and r_{1p_5} share similar properties. In particular, we define

$$X'_\rho \subset \mathcal{E}_{(5,k)}, \quad \rho \in \mathfrak{D}_1(m),$$

analogously to X_ρ in (4.20). For each $\rho = ([k], \{j\}, S, I) \in \mathcal{D}_1(m)$ and every $x \in X'_\rho$, we denote by F_x the core of the curve C_x , by q_x the node labeled by j , and by $p_x = \langle q_x \rangle$ the pivotal node corresponding to q_x . Let

$$\mathcal{Q}_\rho \subset \tilde{\mathfrak{P}}^{r_1 \mathfrak{P}_5 \mathfrak{S}_\kappa(\rho)}, \quad \rho \in \mathcal{D}_1(m),$$

be the closed substack of $\tilde{\mathfrak{P}}^{r_1 \mathfrak{P}_5 \mathfrak{S}_\kappa(\rho)}$ whose *general* points are pairs (x, L) satisfying

$$x \in X'_\rho, \quad (F_x, L|_{F_x} \otimes \mathcal{O}_{F_x}(q_x)) \in \mathfrak{H}.$$

We denote by $\pi^\ddagger: \tilde{\mathfrak{P}}^\ddagger \rightarrow \tilde{\mathfrak{P}}^{r_1}$ the sequential blowup successively along the proper transforms of

$$\mathcal{Q}_{k,k'} = \bigcup_{\rho \in \mathcal{D}_1(k,k')} \mathcal{Q}_\rho \quad (\subset \tilde{\mathfrak{P}}^{r_1 \mathfrak{P}_5 \mathfrak{S}_k}), \quad 1 \leq k \leq k',$$

with respect to the lexicographical order. Similarly, π^\ddagger on each connected component of $\tilde{\mathfrak{P}}^{r_1}$ will terminate after finitely many steps.

The following lemma follows from a combination of the proofs of Lemmas 4.4.1 and 4.5.2.

Lemma 4.8.1. π^\ddagger is supplementary to a sequential blowup derived from $r_1 \mathfrak{P}_5$.

In $r_3 \mathfrak{P}_4$, we blow up $\tilde{\mathfrak{P}}^{r_3 \mathfrak{P}_3}$ successively along the proper transforms of $\mathcal{Q}_{k,k'}^{r_3 \mathfrak{P}_3}$, $1 \leq k \leq k'$. As explained before Lemma 4.8.1, the sequential blowup $r_3 \mathfrak{P}_4$ will terminate after finitely many steps on each connected component of $\tilde{\mathfrak{P}}^{r_3 \mathfrak{P}_3}$. It is a direct check that (the proper transforms of) π^\ddagger is compatible with r_2 - $r_3 \mathfrak{P}_3$. Applying Corollary 3.5.3 repeatedly, we thus obtain the following statement.

Lemma 4.8.2. $r_3 \mathfrak{P}_4$ is locally tree-compatible.

Lemma 4.8.2 and Corollary 3.2.5 lead to the following conclusion.

Corollary 4.8.3. The blowup locus at every step of $r_3 \mathfrak{P}_4$ is closed and smooth.

Corollaries 4.3.6, 4.4.3, 4.5.3, 4.6.2, 4.7.3, and 4.8.3 together imply:

Corollary 4.8.4. The blowup locus at every step of the entire sequence of the modular blowups $\tilde{\mathfrak{P}}_2 \rightarrow \mathfrak{P}_2$ is closed and smooth. In particular, $\tilde{\mathfrak{P}}_2$ is smooth.

5. THE STRUCTURAL HOMOMORPHISM IN THE MODULAR BLOWUPS

5.1. Main statement. Throughout §5, $\tilde{\mathfrak{P}}_2$ denotes the final stack after the three-round blowups and $\pi: \tilde{\mathfrak{P}}_2 \rightarrow \mathfrak{P}_2$ the projection. The following diagram shows the relation between the relevant stacks.

$$\begin{array}{ccc} & \tilde{\mathfrak{P}}_2 & \\ & \pi \downarrow \searrow \varpi & \\ \mathfrak{M}_2^{\text{div}} & \longrightarrow \mathfrak{P}_2 & \longrightarrow \mathfrak{M}_2^{\text{wt}} \end{array}$$

Given $(C, D) \in \mathfrak{M}_2^{\text{div}}$, let $\mathcal{V} \rightarrow \mathfrak{M}_2^{\text{div}}$ be an affine smooth chart containing (C, D) . Then \mathcal{V} can also be considered as affine smooth charts of $(C, \mathcal{O}_C(D)) \in \mathfrak{P}_2$ and of $(C, c_1(D)) \in \mathfrak{M}_2^{\text{wt}}$, respectively. Set

$$\tilde{\mathcal{V}} \equiv \mathcal{V} \times_{\mathfrak{P}_2} \tilde{\mathfrak{P}}_2.$$

For the blowup $\pi^\bullet: \tilde{\mathfrak{P}}^\bullet \rightarrow \mathfrak{P}_2$ after an intermediate step, let $\tilde{\mathcal{V}}^\bullet = \mathcal{V} \times_{\mathfrak{P}_2} \tilde{\mathfrak{P}}^\bullet$.

In this section, we investigate the pullback $\tilde{\varphi}$ of the structural homomorphism φ locally on $\tilde{\mathcal{V}}$. The main statement is as follows.

Theorem 5.1.1. *For all integer $k \geq 1$, the pullback of the derived object $\mathbf{R}\pi_* \mathbf{f}^* \mathcal{O}_{\mathbb{P}^n}(k)$ to the moduli space $\tilde{M}_2(\mathbb{P}^n, d)$ is locally diagonalizable.*

Theorem 5.1.1 follows immediately from Definition 2.1.2 and the statement below.

Proposition 5.1.2. *For every $\tilde{x} \in \tilde{\mathfrak{P}}_2$, there exists an affine smooth chart $\tilde{\mathcal{V}}_{\tilde{x}}$ containing \tilde{x} so that the pullback $\tilde{\varphi}$ of the structural homomorphism φ to $\tilde{\mathcal{V}}_{\tilde{x}}$ is diagonalizable.*

Recall in (4.8), there is a partition of $\mathfrak{M}_2^{\text{wt}}$ into five disjoint substacks:

$$\mathfrak{M}_2^{\text{wt}} = \mathfrak{M}_{(1)}^{\text{mn}} \sqcup \mathfrak{M}_{(2)}^{\text{mn}} \sqcup \mathfrak{M}_{(3)}^{\text{mn}} \sqcup \mathfrak{M}_{(4)}^{\text{mn}} \sqcup \mathfrak{M}^{\text{mn}}.$$

Proposition 5.1.2 is proved in §5.2-5.6, depending on which of the above substacks contains $\varpi(\tilde{x})$.

Remark 5.1.3. Throughout §5, if we write out a matrix explicitly with some columns omitted as “***”, it means each omitted column lies in the span of the shown columns. Since Proposition 5.1.2 is local, if the entries of a row or a column of the pullback of φ have a common factor that is nowhere vanishing on a chart \mathcal{V} , we may omit the factor, because it does not affect the local diagonalizability.

5.2. The structural homomorphism around \mathfrak{M}^{mn} . The main statement of this subsection is as follows.

Proposition 5.2.1. *Proposition 5.1.2 holds if $\varpi(\tilde{x}) \in \mathfrak{M}^{\text{mn}}$.*

Proof. Fix $(C, D) \in \mathfrak{M}_2^{\text{div}}$ so that $\tilde{x} = (C, \mathcal{O}_C(D))$. Let

$$x \equiv (C, c_1(D)) \in \mathfrak{M}^{\text{mn}}$$

and γ be the terminally weighted tree of x as in (4.2). By Basic Assumption (B) in §2.3,

$$D = \delta_1 + \cdots + \delta_m.$$

The definition of \mathfrak{M}^{mn} in (4.8) implies the core F of C satisfies $D \cap F \neq \emptyset$. Moreover, if F' is a connected subcurve of C , then

$$(D \cap F' = D \cap F) \implies F' \supset F.$$

Based on the topology of F and the locations of the points in D , we divide the proof into five cases, with the last being further divided into several subcases. Together they imply Proposition 5.2.1. In each case, it suffices to show that the pullback of φ becomes diagonalizable on a chart containing the image of \tilde{x} after some *intermediate* step of the three-round modular blowups.

Case 1. Assume F is separable. Let F_1 and F_2 be the two genus 1 inseparable components of F . W.l.o.g. we assume

$$\delta_1, a_1 \in F_1, \quad \delta_2, a_2 \in F_2.$$

By Proposition 2.5.1, φ_{11} and φ_{22} do not vanish on \mathcal{V} , whereas φ_{12} and φ_{21} vanish at x . The structural homomorphism φ is thus diagonalizable on \mathcal{V} .

Case 2. Assume F is inseparable and $\deg D \cap F \geq 3$. There then exist $\delta, \delta' \in D \cap F$ that are not conjugate to each other. (Otherwise all the elements of $D \cap F$ would lie on a non-separating bridge of F , which would imply $x \in \mathfrak{M}_{(3)}^{\text{mn}}$.) In addition, F being inseparable implies that $\zeta_{[\delta, a_i]}, \zeta_{[\delta', a_i]} \in \Gamma(\mathcal{O}_{\mathcal{V}}^*)$ for $i = 1, 2$. Hence by Part (1) of Corollary 2.7.3, φ is diagonalizable on \mathcal{V} .

Case 3. Assume F is inseparable, $\deg D \cap F = 2$, and the elements of $D \cap F$ are not conjugate to each other. The argument in Case 2 still holds verbatim.

Case 4. Assume F is inseparable, $\deg D \cap F = 2$, and the elements of $D \cap F$ are conjugate to each other. W.l.o.g. we assume $D \cap F = \{\delta_1, \delta_2\}$. Note that δ_1 cannot be conjugate to any $\delta \in D \setminus F$, for otherwise δ_1 and hence δ_2 would be on a non-separating bridge, which would imply $x \in \mathfrak{M}_{(3)}^{\text{mn}}$. Thus, φ can be written as

$$(5.1) \quad \varphi = \begin{bmatrix} 1 & 0 & \begin{pmatrix} 0 \\ \zeta_{[e]} \end{pmatrix}_{e \in \text{Edg}(\gamma)^t} & * * * \\ 0 & \lambda_{12} & & * * * \end{bmatrix}$$

under suitable trivialization. Since $\deg D \cap F = 2$, the modular blowups up to the end of $r_3 p_2$ do not affect \mathcal{V} . We thus do not distinguish \mathcal{V} and $\tilde{\mathcal{V}}^{r_3 p_2}$.

By Part (2) of Corollary 2.7.3, λ_{12} and ζ_e , $e \in \text{Edg}(\gamma)$, form a γ_+ -labeled subset of local parameters on \mathcal{V} . As in the proof of Lemma 4.7.1, $r_3 p_3$ is γ_+ -compatible on \mathcal{V} . By Proposition 3.3.3, at least one of the pullbacks of the shown entries of the second row of (5.1) divides all the elements of the second row of $\tilde{\varphi}^{r_3 p_3}$ evenly. Thus, $\tilde{\varphi}^{r_3 p_3}$ is diagonalizable on $\tilde{\mathcal{V}}^{r_3 p_3}$.

Case 5. Assume F is inseparable and $\deg D \cap F = 1$. W.l.o.g. we assume $D \cap F = \{\delta_1\}$. Since γ is terminally weighted,

$$D \cap C_{v_e^+} \neq \emptyset \quad \forall e \in \text{Edg}(\gamma)^t,$$

With γ'_y denoting the derived tree of γ at y as in §3.6 and $\zeta_e^{\gamma'_y}$ denoting the local parameters of Lemma 3.6.4, $\tilde{\varphi}^{r_1 p_5}$ can then be rewritten as

$$(5.3) \quad \begin{bmatrix} 1 & 0 & \begin{pmatrix} 0 \\ \zeta_{[e]}^{\gamma'_y} \end{pmatrix}_{e \in \text{Edg}(\gamma'_y)^t} & * & * & * \\ 0 & \tilde{\lambda}_{12} & & * & * & * \end{bmatrix}.$$

Since $\deg D \cap F = 1$, the modular blowups from r_2 until the end of $r_3 p_2$ do not affect a neighborhood $\tilde{\mathcal{V}}_y$ of y . Moreover, the assumptions of Possibility (a) imply $r_3 p_3$ do not affect $\tilde{\mathcal{V}}_y$ either. By Lemma 4.8.2, the blowups in $r_3 p_4$ are supplementary to a sequence of blowups derived from $r_1 p_5$ on $\tilde{\mathcal{V}}_y$. Proposition 3.6.8 then implies $r_3 p_4$ is $\gamma'_{y,+}$ -compatible on $\tilde{\mathcal{V}}_y$, where the additional vertex in $\gamma'_{y,+}$ corresponds to $\tilde{\lambda}_{12}$. By Lemma 3.6.4 and Proposition 3.3.3, at least one of the pullbacks of the shown entries in (5.3) divides the rest evenly when $r_3 p_4$ terminates. Thus, $\tilde{\varphi}$ is diagonalizable on a chart containing the pullback of y .

Possibility (b) is more complicated. We deal with it in Lemma 5.2.2 separately, which completes the proof of Proposition 5.2.1. \square

Lemma 5.2.2. *Let \tilde{x} be as in Proposition 5.1.2, x the image of \tilde{x} in $\mathfrak{M}_2^{\text{wt}}$, and y a lift of x after $r_1 p_5$ satisfying Case 5, Possibility (b) in the proof of Proposition 5.2.1. Then, $\tilde{\varphi}$ is diagonalizable on a neighborhood of the pullback of y .*

Proof. With $\hat{\gamma}$ denoting the reduced rooted tree of x as in (4.1), set

$$\tau_2 \equiv \left(\hat{\gamma}_{[\{e_\bullet\} \leq]} \right)^b.$$

Let $E_* \subset \text{Edg}(\gamma) \cap \text{Edg}(\tau_2)$ be as in §3.5 so that the common branches of γ and τ_2 are indexed by E_* . The definition of τ_2 implies that

$$(5.4) \quad E_* \sqcup \{e_\bullet\} \in \text{ET}(\gamma).$$

W.l.o.g. we assume $D \cap C_\bullet = \{\delta_2\}$. Then by Part (1) of Proposition 2.6.1, (5.2) can be rewritten as

$$\begin{bmatrix} 1 & 0 & \begin{pmatrix} 0 \\ \zeta_{[e_\bullet]}^2 \zeta_{[e]}^{\tau_2} \end{pmatrix}_{e \in \text{Edg}(\tau_2)^t \setminus E_*^\neq} & \begin{pmatrix} 0 \\ \zeta_{[e_\bullet \wedge e]} \zeta_{[e]} \end{pmatrix}_{e \in E_*^t} & * & * & * \\ 0 & \lambda_{12} \zeta_{[e_\bullet]} & & & * & * & * \end{bmatrix}.$$

Here $\zeta_e^{\tau_2} = \zeta_e$ for all $e \in \text{Edg}(\tau_2)$, but the superscript indicates the product $\zeta_{[e]}^{\tau_2}$ is taken within τ_2 .

After $r_1 p_5$, let $\underline{E} = (E_1, \dots, E_t)$ be the ascending sequence of y . By Proposition 3.3.3, near y , $\tilde{\varphi}^{r_1 p_5}$ can be written as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_{[t]} \end{bmatrix} \begin{bmatrix} 1 & 0 & \begin{pmatrix} 0 \\ \tilde{\varepsilon}_{[t]} \tilde{\zeta}_{[e]}^{\tau_2} \end{pmatrix}_{e \in \text{Edg}(\tau_2)^t \setminus E_*^\neq} & 0 & 0 & * & * & * \\ 0 & \tilde{\lambda}_{12} & & \Xi_1 & \Xi_2 & * & * & * \end{bmatrix},$$

$$\text{where } \Xi_1 = \left(\tilde{\varepsilon}_{[s(e_\bullet \wedge e)]} \tilde{\zeta}_{\langle e \rangle_t} \tilde{\zeta}_{\langle e \rangle_t, e} \right)_{e \in E_*^t \setminus \Delta_{y,t}^\neq},$$

$$\Xi_2 = \left(\tilde{\varepsilon}_{[s(e_\bullet \wedge e)]} \tilde{\zeta}_{\langle e \rangle_t, e} \right)_{e \in E_*^t \cap \Delta_{y,t}^\neq}.$$

There are two relevant rooted trees after $r_1\mathfrak{p}_5$: the proper transform $(\tau_2)_y$ of τ_2 as in §3.5 and the rooted tree γ'_y derived from γ at y as in §3.6. The assumptions of Possibility (b) implies $\dot{E}_* = \ddot{E}_* = \emptyset$ and $\hat{E}_* = E_*$ for τ_2 . With $\hat{E}_t \in \text{ET}(\gamma)$ as in (3.22), we thus have

$$\text{Edg}((\tau_2)_y) = (\text{Edg}(\tau_2) \setminus E_*^{\geq}) \sqcup (\hat{E}_t^{\geq} \cap E_*^{\geq}).$$

We require the additional edges respectively in $(\tau_2)_{y,+}$ and in $\gamma'_{y,+}$ to be the same pivotal edge e_+ . Let

$$(5.5) \quad \mathbb{E}_* = \hat{E}_t \cap E_*^{\geq} \subset E_*^{\geq}, \quad \mathbb{E}_{*,+} = \mathbb{E}_* \sqcup \{e_+\}.$$

The former equality in (5.5) implies

$$\mathbb{E}_*^t = E_*^t.$$

We observe that the common branches of $(\tau_2)_{y,+}$ and $\gamma'_{y,+}$ are indexed by $\mathbb{E}_{*,+}$. The construction of the derived tree in §3.6, along with (5.4) and (5.5), implies the collection of the pivotal edges of $\gamma'_{y,+}$ satisfies

$$(5.6) \quad \begin{aligned} E_o(\gamma'_{y,+}) &= \{e_1\} \sqcup (\hat{E}_t \setminus \{\langle e_\bullet \rangle\}^{\geq}) \sqcup \{e_+\} \\ &= \{e_1\} \sqcup \{e \in \hat{E}_t \cap E_*^{\geq} : \langle e \rangle \neq \langle e_\bullet \rangle\} \sqcup \{e_+\} \\ &= \{e_1\} \sqcup \{e \in \mathbb{E}_* : \langle e \rangle \neq \langle e_\bullet \rangle\} \sqcup \{e_+\}. \end{aligned}$$

With $\zeta_e^{(\tau_2)_y}$ as in Proposition 3.5.2, $\zeta_e^{\gamma'_y}$ as in Lemma 3.6.4, and \mathbb{E}_* as in (5.5), $\tilde{\varphi}^{r_1\mathfrak{p}_5}$ can then be rewritten as

$$(5.7) \quad \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_{[t]} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\lambda}_{12} \end{bmatrix} \begin{bmatrix} 0 \\ \zeta_{[e_1]}^{\gamma'_y} \zeta_{[e]}^{(\tau_2)_y} \end{bmatrix}_{e \in \text{Edg}((\tau_2)_y)^t \setminus \mathbb{E}_*^t} \begin{bmatrix} 0 \\ \zeta_{[e_s(e_\bullet \wedge e)]}^{\gamma'_y} \zeta_{[e]}^{(\tau_2)_y} \end{bmatrix}_{e \in \mathbb{E}_*^t} \begin{bmatrix} *** \\ *** \end{bmatrix}.$$

Note that in (5.7),

$$(5.8) \quad \zeta_e^{(\tau_2)_y} = \zeta_e^{\gamma'_y} \quad \forall e \in \mathbb{E}_*^{\geq}.$$

Since $\deg D \cap F = 1$, the modular blowups from r_2 until the end of $r_3\mathfrak{p}_2$ do not affect a neighborhood $\tilde{\mathcal{V}}_y$ of y . By Lemma 4.7.1 and Proposition 3.5.2, $r_3\mathfrak{p}_3$ is $(\tau_2)_{y,+}$ -compatible on $\tilde{\mathcal{V}}_y$. When $r_3\mathfrak{p}_3$ terminates, fix a lift \tilde{y} of y , whose ascending sequence is denoted by $\tilde{E} = (\tilde{E}_1, \dots, \tilde{E}_u)$. We may then define

$$\begin{aligned} \dot{\mathbb{E}}_{*,+} &= \{\langle e \rangle_{\mathbb{E}_{*,+}} : e \in \text{Dom}(\tilde{y}) \cap \mathbb{E}_{*,+}^{\geq}\} \subset \mathbb{E}_{*,+}, \\ \ddot{\mathbb{E}}_{*,+} &= \bigcup_{e \in \dot{\mathbb{E}}_{*,+}} \{e' \in \text{Edg}(\gamma'_{y,+}) : v_{e'}^- = v_e^-\} \subset \text{Edg}(\gamma'_{y,+}), \quad \hat{\mathbb{E}}_* = \mathbb{E}_* \setminus \dot{\mathbb{E}}_* \subset \mathbb{E}_* \end{aligned}$$

for $\gamma'_{y,+}$ as in §3.5.

If $\dot{\mathbb{E}}_{*,+} \cap E_o(\gamma'_{y,+}) \neq \emptyset$, by (5.6) and (5.5), either e_+ or some $e \in \mathbb{E}_*^t$ with $\langle e \rangle \neq \langle e_\bullet \rangle$ (i.e. $\zeta_{[e_s(e_\bullet \wedge e)]}^{\gamma'_y}$ does not vanish at y) becomes dominant after $r_3\mathfrak{p}_3$. Thus by (5.7), $\tilde{\varphi}^{r_3\mathfrak{p}_3}$ is diagonalizable near \tilde{y} .

If $\dot{\mathbb{E}}_{*,+} \cap E_o(\gamma'_{y,+}) = \emptyset$, then

$$\text{Dom}(\tilde{y}) \subset \text{Edg}((\tau_2)_y)^t \setminus \{e \in E_*^t : \langle e \rangle \neq \langle e_\bullet \rangle\} \quad (\subset \text{Edg}(\hat{\gamma})).$$

Hence every $e \in \text{Dom}(\tilde{y})$ satisfies $\langle e \rangle_{\hat{\gamma}} = \langle e_\bullet \rangle_{\hat{\gamma}}$ in $\hat{\gamma}$. Let

$$e_o = \min\{e_\bullet \wedge e : e \in \text{Dom}(\tilde{y})\} \in \text{Edg}(\hat{\gamma}).$$

Obviously, $e_o < e_\bullet$, hence e_o is an edge in γ . Thus,

$$\dot{\mathbb{E}}_* = \mathbb{E}_* \setminus \{e_o\}^{\geq}, \quad \text{Edg}(\gamma'_y) \setminus (\mathbb{E}_* \cup \dot{\mathbb{E}}_*)^{\geq} = \{e_1, \dots, e_{s(e_o)}\},$$

$$\text{Edg}((\gamma'_{y,+})_{\tilde{y}}) = \{e_1, \dots, e_{s(e_o)}\} \sqcup ((\tilde{E}_u \setminus \Delta_{\tilde{y},u})^{\geq} \cap \dot{\mathbb{E}}_*^{\geq}) \sqcup (\Delta_{\tilde{y},u}^{\geq} \cap \dot{\mathbb{E}}_*^{\geq}) \sqcup \{e_+\}.$$

By (5.7), Proposition 3.3.3, and (5.8), near \tilde{y} , $\tilde{\varphi}^{r_3\mathbf{p}_3}$ can be written as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\epsilon}_{[t]} \tilde{\epsilon}_{[u]} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & \tilde{\lambda}_{12} & \zeta_{[e_s(e_o)]}^{\gamma'_y} & \Xi'_1 & \Xi'_2 & * & * & * \end{bmatrix}, \quad \text{where}$$

$$\Xi'_1 = (\zeta_{[e_s(e_o \wedge e)]}^{\gamma'_y} \check{\zeta}_{\langle e \rangle_u} \tilde{\zeta}_{\langle \langle e \rangle_u, e \rangle})_{e \in \dot{\mathbb{E}}_*^t \setminus \Delta_{\tilde{y},u}^{\geq}}, \quad \Xi'_2 = (\zeta_{[e_s(e_o \wedge e)]}^{\gamma'_y} \tilde{\zeta}_{\langle \langle e \rangle_u, e \rangle})_{e \in \dot{\mathbb{E}}_*^t \cap \Delta_{\tilde{y},u}^{\geq}}.$$

Here $\tilde{\epsilon}_i$, $i \in [u]$, are the exceptional parameters associated with the ascending sequence \tilde{E} of \tilde{y} . With $\zeta_e^{(\gamma'_{y,+})_{\tilde{y}}}$ as in Proposition 3.5.2, we rewrite $\tilde{\varphi}^{r_3\mathbf{p}_3}$ as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\epsilon}_{[t]} \tilde{\epsilon}_{[u]} \end{bmatrix} \begin{bmatrix} 1 & \begin{pmatrix} 0 \\ \zeta_e^{(\gamma'_{y,+})_{\tilde{y}}} \end{pmatrix}_{e \in \text{Edg}((\gamma'_{y,+})_{\tilde{y}})^t} \\ * & * & * \\ * & * & * \end{bmatrix}.$$

By Part (2) of Proposition 3.5.2 and Lemma 4.8.1, we conclude $r_3\mathbf{p}_4$ is $(\gamma'_{y,+})_{\tilde{y}}$ -compatible on a neighborhood of \tilde{y} . Thus, the expression of $\tilde{\varphi}^{r_3\mathbf{p}_3}$ in the previous paragraph and Proposition 3.3.3 together imply $\tilde{\varphi} = \tilde{\varphi}^{r_3\mathbf{p}_4}$ is diagonalizable on the pullback of a neighborhood of \tilde{y} . \square

Example 5.2.3. Given $(C, D) \in \mathfrak{M}_2^{\text{div}}$ with the dual graph γ^* of C as in the first diagram in Figure 3. Assume

$$D \cap F = \{\delta_1\}, \quad D \cap C_b = \{\delta_b\}, \quad D \cap C_i = \{\delta_i, \delta_{i'}, \dots\}, \quad i = a, c, d.$$

This implies the core F of C is smooth and $x = (C, c_1(D)) \in \mathfrak{M}^{\text{mn}}$. Under suitable trivialization, φ on a chart \mathcal{V} containing (C, D) can be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & \lambda_{1a}\zeta_a & \lambda_{1b}\zeta_b & \theta_{a,a'}\zeta_a^2 & \theta_{bc}\zeta_b^2\zeta_c & \theta_{bd}\zeta_b^2\zeta_d & * & * & * \end{bmatrix},$$

where $\lambda_{1a}, \lambda_{1b}$ are as in Corollary 2.7.3 and $\theta_{a,a'}, \theta_{bc}, \theta_{bd}$ are as in Proposition 2.6.1. Since $D \cap F \neq \emptyset$, the blowups in $r_1\mathbf{p}_1$ - $r_1\mathbf{p}_4$ do not affect \mathcal{V} . After $r_1\mathbf{p}_5$, given a lift y of x , there are three possible cases: $\text{Dom}(y) = \{e_a, e_b\}$, $\text{Dom}(y) = \{e_a\}$, or $\text{Dom}(y) = \{e_b\}$. In each case, the ascending sequence of y consists of only one traverse section $\{e_a, e_b\}$ of γ^* .

If $\text{Dom}(y) = \{e_a, e_b\}$, the pullbacks of λ_{1a} and λ_{1b} cannot vanish at y simultaneously, because δ_1 cannot be conjugate to δ_a and δ_b simultaneously. Thus, the pullback $\tilde{\varphi}^{r_1\mathbf{p}_5}$ of φ near y can be rewritten as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\epsilon}_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \end{bmatrix},$$

hence $\tilde{\varphi}^{r_1 p_5}$ is diagonalizable near y .

In the second case (i.e. $\text{Dom}(y) = \{e_a\}$), the pullback of φ near y can be written as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & *** \\ 0 & \tilde{\lambda}_{1a} & \tilde{\lambda}_{1b} \check{\zeta}_b & \tilde{\theta}_{a,a'} \tilde{\varepsilon}_1 & \tilde{\theta}_{bc} \tilde{\varepsilon}_1 \check{\zeta}_b^2 \check{\zeta}_c & \tilde{\theta}_{bd} \tilde{\varepsilon}_1 \check{\zeta}_b^2 \check{\zeta}_d & *** \end{bmatrix}.$$

As per our convention in §3.1, we use $\tilde{\sim}$ and $\check{\sim}$ to denote the pullback of a function and the proper transform of a local parameter, respectively. If $\tilde{\lambda}_{1a}(y) \neq 0$, then $\tilde{\varphi}^{r_1 p_5}$ is still diagonalizable near y . If $\tilde{\lambda}_{1a}(y) = 0$, then δ_a and δ_1 are conjugate, hence the node p_{e_a} on C cannot be Weierstrass and consequently $\tilde{\theta}_{a,a'}$ does not vanish near y . By Corollary 2.7.1, $\tilde{\lambda}_{1a}$ vanishes to the first order near y . Furthermore, $\tilde{\lambda}_{1b}$ does not vanish near y because δ_b is not conjugate to δ_1 whenever δ_a is. Thus, the pullback of φ near y can be rewritten as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & *** \\ 0 & \tilde{\lambda}_{1a} & \check{\zeta}_b & \tilde{\varepsilon}_1 & *** \end{bmatrix}.$$

It is then a direct check that r_2 through $r_3 p_3$ do not affect a neighborhood of y , whereas the pullback of φ becomes diagonalizable on the pullback of a chart containing y after $r_3 p_4$.

In the third case (i.e. $\text{Dom}(y) = \{e_b\}$), the pullback of φ near y can be written as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & *** \\ 0 & \tilde{\lambda}_{1a} \check{\zeta}_a & \tilde{\lambda}_{1b} & \tilde{\theta}_{a,a'} \tilde{\varepsilon}_1 \check{\zeta}_a^2 & \tilde{\theta}_{bc} \tilde{\varepsilon}_1 \check{\zeta}_c & \tilde{\theta}_{bd} \tilde{\varepsilon}_1 \check{\zeta}_d & *** \end{bmatrix}.$$

If $\tilde{\lambda}_{1b}(y) \neq 0$, then $\tilde{\varphi}^{r_1 p_5}$ is still diagonalizable near y . If $\tilde{\lambda}_{1b}(y) = 0$, analogous to the second case, the pullback of φ can be written as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & *** \\ 0 & \check{\zeta}_a & \tilde{\lambda}_{1b} & \tilde{\varepsilon}_1 \check{\zeta}_c & \tilde{\varepsilon}_1 \check{\zeta}_d & *** \end{bmatrix}.$$

It is then a direct check that r_2 through $r_3 p_2$ do not affect a neighborhood of y , and after $r_3 p_3$, for every lift \tilde{y} of y , either the pullback of φ becomes diagonalizable near \tilde{y} , or it can be rewritten as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_1 \tilde{\varepsilon}_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & *** \\ 0 & \check{\zeta}_a & \check{\lambda}_{1b} & \tilde{\varepsilon}_1 & *** \end{bmatrix},$$

thus becomes diagonalizable on the pullback of a chart containing \tilde{y} after $r_3 p_4$.

5.3. The structural homomorphism around $\mathfrak{M}_{(4)}^{\text{mn}}$. The main statement of this subsection is as follows.

Proposition 5.3.1. *Proposition 5.1.2 holds if $\varpi(\tilde{x}) \in \mathfrak{M}_{(4)}^{\text{mn}}$.*

Proof. As in the proof of Proposition 5.2.1, we fix $(C, D) \in \mathfrak{M}_2^{\text{div}}$ so that $\tilde{x} = (C, \mathcal{O}_C(D))$ and set

$$x \equiv (C, c_1(D)) \in \mathfrak{M}_{(4)}^{\text{mn}}.$$

We still assume $D = \delta_1 + \cdots + \delta_m$.

By the definition of $\mathfrak{M}_{(4)}^{\text{mn}}$, the core F of x consists of two genus 1 inseparable components F_1 and F_2 as well as a (possibly degenerate) separating bridge B . W.l.o.g. we assume that

$$a_1 \in F_1, \quad a_2 \in F_2, \quad D \cap F \subset F_1 \cup B.$$

Let C_1 be the smallest connected genus 1 subcurve of F containing $D \cap F$. Then C_1 contains a unique irreducible component F'_1 that is the closest to F_2 . The definition of $\mathfrak{M}_{(4)}^{\text{mn}}$ then implies F'_1 does not belong to any non-separating bridge and $D \cap F'_1 \neq \emptyset$.

Let γ be the terminally weighted tree of x and γ_\vee be the rooted tree as in (4.4). Recall that F_1 and F_2 correspond to the terminal vertex o_1 and the root o_2 , respectively, in γ_\vee . Let $E_* \subset \text{Edg}(\gamma) \cap \text{Edg}(\gamma_\vee)$ be as in §3.5 so that the common branches of γ and γ_\vee are indexed by E_* . With $e_1 \in \text{Edg}(\gamma_\vee)$ denoting the edge such that $v_{e_1}^+ = o_1$, we see that

$$E_* \sqcup \{e_1\} \in \text{ET}(\gamma_\vee).$$

Similar to the proof of Proposition 5.2.1, we divide the proof of Proposition 5.3.1 into four cases.

Case 1. Assume $\deg D \cap F \geq 3$. W.l.o.g. we assume

$$\delta_1 \in D \cap F_1, \quad \delta_2 \in D \cap F'_1, \quad \delta_3 \in D \cap C_1.$$

Switching δ_1 and δ_3 if necessary, we may assume that δ_1 and δ_2 are not conjugate, for otherwise these three points would lie on a non-separating bridge. By Proposition 2.5.1, φ can be written as

$$\varphi = \begin{bmatrix} 1 & \begin{pmatrix} 0 \\ \zeta_{[e]}^{\gamma_\vee} \end{pmatrix}_{e \in \text{Edg}(\gamma_\vee)^t} & * * * \\ 0 & & * * * \end{bmatrix}$$

on \mathcal{V} . Here $\zeta_e^{\gamma_\vee} = \zeta_e$ for all $e \in \text{Edg}(\gamma_\vee)$, but as in the proof of Lemma 5.2.2, the superscript indicates the product $\zeta_{[e]}^{\gamma_\vee}$ is taken in γ_\vee .

Since $x \in \mathfrak{M}_{(4)}^{\text{mn}}$, the modular blowups up to the end of $r_1 p_3$ do not affect \mathcal{V} . By the proof of Lemma 4.3.4, the blowups in $r_1 p_4$ are γ_\vee -compatible on \mathcal{V} . By Proposition 3.3.3, at least one of the pullbacks of $\zeta_{[e]}^{\gamma_\vee}$, $e \in \text{Edg}(\gamma_\vee)$, divides all the elements of the second row of $\tilde{\varphi}^{r_1 p_4}$ evenly. Thus $\tilde{\varphi}^{r_1 p_4}$ is diagonalizable on $\tilde{\mathcal{V}}^{r_1 p_4}$.

Case 2. Assume $\deg D \cap F = 2$ and the elements of $D \cap F$ are not conjugate to each other. The argument in Case 1 still holds verbatim.

Case 3. Assume $\deg D \cap F = 2$ and the elements δ_1 and δ_2 of $D \cap F$ are conjugate to each other. Then,

$$\delta_1, \delta_2 \in F'_1 \subset F_1 = C_1.$$

By Proposition 2.5.1, Part (1) of Proposition 2.6.1, and Part (1) of Corollary 2.7.3, φ on \mathcal{V} can be written as

$$(5.9) \quad \begin{bmatrix} 1 & 0 \\ 0 & \lambda_{12} \zeta_{[e_1]}^{\gamma_v} \end{bmatrix} \begin{bmatrix} 0 \\ \zeta_{[e_1]}^{\gamma_v} \zeta_{[e]} \end{bmatrix}_{e \in \text{Edg}(\gamma)^t \setminus E_*^{\geq}} \begin{bmatrix} 0 \\ \zeta_{[e]}^{\gamma_v} \end{bmatrix}_{e \in E_*^t} \begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}.$$

As in Case 1, the modular blowups up to the end of $r_1 p_3$ do not affect \mathcal{V} , and $r_1 p_4$ is γ_v -compatible on \mathcal{V} . Thus, when $r_1 p_4$ terminates, every lift y of x satisfies $\text{Dom}(y) \neq \emptyset$. Let $\underline{E} = (E_1, \dots, E_t)$ be the ascending sequence of y . There are two possibilities:

- (a) $\text{Dom}(y) \neq \{e_1\}$;
- (b) $\text{Dom}(y) = \{e_1\}$.

For Possibility (a), at least one edge $e \in E_*^t$ becomes dominant after $r_1 p_4$. By Proposition 3.3.3, the pullback of $\zeta_{[e]}^{\gamma_v}$ then divides all elements of the second row of $\tilde{\varphi}^{r_1 p_4}$ evenly. Hence $\tilde{\varphi}^{r_1 p_4}$ is diagonalizable on a neighborhood of y .

For Possibility (b), by Proposition 3.3.3, near y , the second row of the pullback of (5.9) after $r_1 p_4$ can be written as

$$\begin{aligned} & \tilde{\varepsilon}_{[t]} \left[0, \tilde{\lambda}_{12}, \left(\tilde{\zeta}_{[e]} \right)_{e \in \text{Edg}(\gamma)^t \setminus E_*^{\geq}}, \Xi_1, \Xi_2, * * * \right], \quad \text{where} \\ & \Xi_1 = \left(\tilde{\zeta}_{\langle e \rangle_t} \tilde{\zeta}_{\langle e \rangle_t, e} \right)_{e \in E_*^t \setminus \Delta_{y,t}^{\geq}}, \quad \Xi_2 = \left(\tilde{\zeta}_{\langle e \rangle_t, e} \right)_{e \in E_*^t \cap \Delta_{y,t}^{\geq}}. \end{aligned}$$

Note that after $r_1 p_4$,

$$\dot{E}_* = \ddot{E}_* = \emptyset, \quad \dot{E}_* = E_*$$

for γ . Hence γ_y as in §3.5 satisfies

$$\text{Edg}(\gamma_y) = (\text{Edg}(\gamma) \setminus E_*^{\geq}) \sqcup (\hat{E}_t^{\geq} \cap E_*^{\geq}).$$

With $\zeta_e^{\gamma_y}$ as in Proposition 3.5.2, $\tilde{\varphi}^{r_1 p_4}$ can be rewritten as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_{[t]} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\lambda}_{12} \end{bmatrix} \begin{bmatrix} 0 \\ \zeta_{[e]}^{\gamma_y} \end{bmatrix}_{e \in \text{Edg}(\gamma_y)^t} \begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}.$$

Since $\deg D \cap F = 2$ and $\text{Dom}(y) = \{e_1\}$, the effect of the modular blowups from $r_1 p_5$ on a neighborhood of y is the same as that on a neighborhood of x in Case 4 of the proof of Proposition 5.2.1. Comparing the above expression of $\tilde{\varphi}^{r_1 p_4}$ with (5.1), we conclude that the entire argument of Case 4 of the proof of Proposition 5.2.1 applies to this case.

Case 4. Assume $\deg D \cap F = 1$. W.l.o.g. we assume $D \cap F = \{\delta_1\}$. Then,

$$\delta_1 \in F_1' \subset F_1 = C_1.$$

Similar to (5.2), φ can be written as

$$(5.10) \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \lambda_{1e} \zeta_{[e]} \zeta_{[e_1]}^{\gamma_v} \end{bmatrix}_{e \in \text{Edg}(\gamma)^t \setminus E_*^{\geq}} \begin{bmatrix} 0 \\ \lambda_{1e} \zeta_{[e]}^{\gamma_v} \end{bmatrix}_{e \in E_*^{\geq}} \begin{bmatrix} 0 \\ \chi_{1i} \zeta_{[\delta_i]} \end{bmatrix}_{\delta_i \in D'},$$

where $D' \equiv D \setminus (\{\delta_1\} \sqcup \{\delta_e\}_{e \in \text{Edg}(\gamma)^t})$ and $\chi_{1i} = \det \begin{bmatrix} c_{11} & c_{1i} \\ c_{21} & c_{2i} \end{bmatrix} \in \Gamma(\mathcal{O}_V)$.

As in Case 1, the modular blowups up to the end of $r_1 p_3$ do not affect \mathcal{V} , whereas $r_1 p_4$ is γ_\vee -compatible on \mathcal{V} . So when $r_1 p_4$ terminates, every lift y of x satisfies $\text{Dom}(y) \neq \emptyset$. Let $\underline{E} = (E_1, \dots, E_t)$ be the ascending sequence of y .

If $\text{Dom}(y) \neq \{e_1\}$, since $\tilde{\lambda}_{1e}$ does not vanish at y for all $e \in E_*^t$, the argument in Possibility (a) of Case 3 holds here. If $\text{Dom}(y) = \{e_1\}$, the argument in Possibility (b) of Case 3 implies that near y , $\tilde{\varphi}^{r_1 p_4}$ can be written as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_{[t]} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\lambda}_{1e} \zeta_{[e]}^{\gamma_y} \end{bmatrix}_{e \in \text{Edg}(\gamma_y)^t} \begin{bmatrix} 0 \\ \tilde{\chi}_{1i} \zeta_{[\delta_i]}^{\gamma_y} \end{bmatrix}_{\delta_i \in D'}.$$

Since $\deg D \cap F = 1$ and $\text{Dom}(y) = \{e_1\}$, the effect of the modular blowups from $r_1 p_5$ on a neighborhood of y is the same as that on a neighborhood of x in Case 5 of the proof of Proposition 5.2.1. Comparing the above expression of $\tilde{\varphi}^{r_1 p_4}$ with (5.2), we conclude that the entire argument of Case 5 of the proof of Proposition 5.2.1 applies to the current case. This completes Case 4 of the proof of Proposition 5.3.1. \square

5.4. The structural homomorphism around $\mathfrak{M}_{(3)}^{\text{mn}}$. The main statement of this subsection is as follows.

Proposition 5.4.1. *Proposition 5.1.2 holds if $\varpi(\tilde{x}) \in \mathfrak{M}_{(3)}^{\text{mn}}$.*

Proof. As in the proof of Proposition 5.2.1, we fix $(C, D) \in \mathfrak{M}_2^{\text{div}}$ so that $\tilde{x} = (C, \mathcal{O}_C(D))$ and set

$$x \equiv (C, c_1(D)) \in \mathfrak{M}_{(3)}^{\text{mn}}.$$

We still assume $D = \delta_1 + \dots + \delta_m$.

By the definition of $\mathfrak{M}_{(3)}^{\text{mn}}$, the core F of x contains a *minimal* non-separating bridge $B = B[q_1, q_2]$ so that

$$D \cap B = D \cap F.$$

We denote by C_1^+ and C_2^+ the irreducible components of B containing q_1 and q_2 , respectively. Note that C_1^+ and C_2^+ may be the same.

Let γ be the reduced dual tree of x and γ_\diamond be the rooted tree as in (4.6) so that B corresponds to the fused vertex v_B . Let $E_* \subset \text{Edg}(\gamma) \cap \text{Edg}(\gamma_\diamond)$ be as in §3.5 so that the common branches of γ and γ_\diamond are indexed by E_* . With $\mathbf{e}_1, \mathbf{e}_2 \in \text{Edg}(\gamma_\diamond)^t$ denoting the edges such that

$$v_{\mathbf{e}_1}^+ = v_{\mathbf{e}_2}^+ = v_B,$$

Then, $E_* \sqcup \{\mathbf{e}_1, \mathbf{e}_2\} \in \text{ET}(\gamma_\diamond)$.

If in addition F is separable, w.l.o.g. we assume F_1 and F_2 are the genus 1 inseparable components of F so that $B \subset F_1$. Let γ_\vee be the rooted tree as

in (4.4) so that F_1 and F_2 correspond to the terminal vertex o_1 and the root o_2 , respectively. Let $e_{o_1} \in \text{Edg}(\gamma_\vee)^t$ be such that

$$v_{e_{o_1}}^+ = o_1.$$

Set $\mathbb{E}_* = E_* \cap \text{Edg}(\gamma_\vee)$. It is straightforward that

$$\text{Edg}(\gamma) \cap \text{Edg}(\gamma_\vee) = \text{Edg}(\gamma_\diamond) \cap \text{Edg}(\gamma_\vee) = \mathbb{E}_*^{\geq}.$$

Moreover, the edges in \mathbb{E}_* are not comparable because $\mathbb{E}_* \subset E_*$. Hence the common branches of γ and γ_\vee are indexed by \mathbb{E}_* ; so are the common branches of γ_\diamond and γ_\vee .

Similar to the proofs of Propositions 5.2.1 and 5.3.1, we divide the proof of Proposition 5.4.1 into six cases.

Case 1. Assume $\deg D \cap F \geq 3$ and F is inseparable. W.l.o.g. we assume that

$$\delta_1 \in D \cap F, \quad \delta_2 \in C_1^+, \quad \delta_3 \in C_2^+.$$

By Proposition 2.5.1 and Corollary 2.7.3, φ on \mathcal{V} can thus be written as

$$(5.11) \quad \begin{bmatrix} 1 & \begin{pmatrix} 0 \\ \lambda_{1\delta} \end{pmatrix}_{\delta \in D \cap F \setminus \{\delta_1\}} & \begin{pmatrix} 0 \\ \zeta_{[e]}^{\gamma_\diamond} \end{pmatrix}_{e \in \text{Edg}(\gamma_\diamond)^t \setminus \{\mathbf{e}_1, \mathbf{e}_2\}} & \begin{matrix} * & * & * \\ * & * & * \end{matrix} \end{bmatrix}.$$

Since $x \in \mathfrak{M}_{(3)}^{\text{mn}}$, the modular blowups up to the end of $r_1\mathfrak{p}_2$ do not affect \mathcal{V} . By the proof of Lemma 4.3.3, $r_1\mathfrak{p}_3$ is γ_\diamond -compatible. Let y be a fixed lift of x after $r_1\mathfrak{p}_3$ and $\underline{E} = (E_1, \dots, E_t)$ be its ascending sequence.

If $\text{Dom}(y) \not\subset \{\mathbf{e}_1, \mathbf{e}_2\}$, then by Proposition 3.3.3 and Corollary 2.7.3, at least one of the pullbacks of $\zeta_{[e]}^{\gamma_\diamond}$, $e \in \text{Edg}(\gamma_\diamond)^t \setminus \{\mathbf{e}_1, \mathbf{e}_2\}$, divides all the elements of the second row of $\tilde{\varphi}^{r_1\mathfrak{p}_3}$ evenly. If $\text{Dom}(y) \subset \{\mathbf{e}_1, \mathbf{e}_2\}$, then by Corollary 2.7.3 and the assumption on the location of δ_2 and δ_3 , we conclude that

$$(5.12) \quad \check{\lambda}_{1i} \equiv \tilde{\lambda}_{1i} / \tilde{\varepsilon}_{[t]}$$

does not vanish at y for at least one $i \in \{2, 3\}$. Thus $\tilde{\varphi}^{r_1\mathfrak{p}_3}$ is diagonalizable on $\tilde{\mathcal{V}}^{r_1\mathfrak{p}_3}$.

Case 2. Assume $\deg D \cap F \geq 3$ and F is separable. We choose $\delta_1, \delta_2, \delta_3 \in D \cap F$ as in Case 1. The structural homomorphism φ on \mathcal{V} can be written as

$$\begin{bmatrix} 1 & \begin{pmatrix} 0 \\ \lambda_{1e} \zeta_{[e_{o_1}]}^{\gamma_\vee} \end{pmatrix}_{\delta \in D \cap F \setminus \{\delta_1\}} & \begin{pmatrix} 0 \\ \zeta_{[e]}^{\gamma_\diamond} \zeta_{[e_{o_1}]}^{\gamma_\vee} \end{pmatrix}_{e \in E_*^t \setminus \mathbb{E}_*^t} & \begin{pmatrix} 0 \\ \zeta_{[e]}^{\gamma_\diamond} \zeta_{[e_{o_1} \wedge e]}^{\gamma_\vee} \end{pmatrix}_{e \in \mathbb{E}_*^t} & \begin{matrix} * & * & * \\ * & * & * \end{matrix} \end{bmatrix}.$$

Note that $\zeta_e^{\gamma_\diamond} = \zeta_e^{\gamma_\vee}$ for all $e \in \mathbb{E}_*^{\geq}$.

Similar to Case 1, $r_1\mathfrak{p}_1$ and $r_1\mathfrak{p}_2$ do not affect \mathcal{V} , whereas $r_1\mathfrak{p}_3$ is γ_\diamond -compatible. After $r_1\mathfrak{p}_3$, fix a lift y of x and let $\underline{E} = (E_1, \dots, E_t)$ be its ascending sequence.

If there is $e \in \text{Dom}(y) \cap \mathbb{E}_*^t$ so that $\langle e \rangle_{\gamma_\vee} \neq \langle e_{o_1} \rangle_{\gamma_\vee}$, then $\zeta_{[e_{o_1} \wedge e]}^{\gamma_\vee}$ does not vanish on \mathcal{V} and the pullback of $\zeta_{[e]}^{\gamma_\diamond}$ divides all the elements of $\tilde{\varphi}^{r_1\mathfrak{p}_3}$ evenly. Hence $\tilde{\varphi}^{r_1\mathfrak{p}_3}$ is diagonalizable on $\tilde{\mathcal{V}}^{r_1\mathfrak{p}_3}$.

If $\text{Dom}(y) \cap \{e \in \mathbb{E}_*^t : \langle e \rangle_{\gamma_\vee} \neq \langle e_{o_1} \rangle_{\gamma_\vee}\} = \emptyset$, let

$$e_o = \min \left(\{e_{o_1}\} \sqcup \{e_{o_1} \wedge_{\gamma_\vee} e : e \in \text{Dom}(y) \cap \mathbb{E}_*^t\} \right) \quad (\in \text{Edg}(\gamma_\vee)).$$

Then,

$$\mathring{\mathbb{E}}_* = \mathbb{E}_* \setminus \{e_o\}^{\geq}, \quad \text{Edg}((\gamma_\vee)_y) = \left(\text{Edg}(\gamma_\vee) \setminus (\mathbb{E}_*^{\geq} \cup \{e_o\}^{\geq}) \right) \sqcup (\widehat{E}_t^{\geq} \cap \mathring{\mathbb{E}}_*^{\geq}).$$

By the argument in Case 1 and by Proposition 3.3.3, $\tilde{\varphi}^{r_1 p_3}$ on a neighborhood of y can be rewritten as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_{[t]} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & * & * & * \\ 0 & \zeta_{[e_o]}^{\gamma_\vee} & \Xi_1 & \Xi_2 & * & * & * \end{bmatrix}, \quad \text{where} \\ \Xi_1 = \left(\zeta_{[e_o \wedge e]}^{\gamma_\vee} \check{\zeta}_{\langle e \rangle_t} \tilde{\zeta}_{\langle \langle e \rangle_t, e \rangle} \right)_{e \in \mathring{\mathbb{E}}_*^t \setminus \Delta_{y,t}^{\geq}}, \quad \Xi_2 = \left(\zeta_{[e_o \wedge e]}^{\gamma_\vee} \tilde{\zeta}_{\langle \langle e \rangle_t, e \rangle} \right)_{e \in \mathring{\mathbb{E}}_*^t \cap \Delta_{y,t}^{\geq}}.$$

With $\zeta_e^{(\gamma_\vee)_y}$ as in Proposition 3.5.2, we can rewrite $\tilde{\varphi}^{r_1 p_3}$ as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_{[t]} \end{bmatrix} \begin{bmatrix} 1 & \begin{pmatrix} 0 \\ \zeta_e^{(\gamma_\vee)_y} \end{pmatrix}_{e \in \text{Edg}((\gamma_\vee)_y)^t} & * & * & * \end{bmatrix}.$$

The modular blowups in $r_1 p_4$ is $(\gamma_\vee)_y$ -compatible on neighborhood of y . By Proposition 3.3.3, $\tilde{\varphi}^{r_1 p_4}$ is diagonalizable on the pullback of a neighborhood of y .

Case 3. Assume $\deg D \cap F = 2$ and F is inseparable. W.l.o.g. we assume $D \cap F = \{\delta_1, \delta_2\}$. The structural homomorphism φ on \mathcal{V} can be written as

$$(5.13) \quad \begin{bmatrix} 1 & 0 \\ 0 & \lambda_{12} \end{bmatrix} \begin{pmatrix} 0 \\ \lambda_{1e} \zeta_{[e]} \end{pmatrix}_{e \in \text{Edg}(\gamma)^t \setminus E_*^t} \begin{pmatrix} 0 \\ \zeta_{[e]}^{\gamma_\circ} \end{pmatrix}_{e \in E_*^t} \begin{pmatrix} 0 \\ \chi_{1i} \zeta_{[\delta_i]} \end{pmatrix}_{\delta_i \in D'},$$

where $D' \equiv D \setminus (\{\delta_1, \delta_2\} \sqcup \{\delta_e\}_{e \in \text{Edg}(\gamma)^t})$ and $\chi_{1i} = \det \begin{bmatrix} c_{11} & c_{1i} \\ c_{21} & c_{2i} \end{bmatrix} \in \Gamma(\mathcal{O}_{\mathcal{V}})$.

Similar to Case 1, $r_1 p_1$ and $r_1 p_2$ do not affect \mathcal{V} , whereas $r_1 p_3$ is γ_\circ -compatible. After $r_1 p_3$, fix a lift y of x and let $\underline{E} = (E_1, \dots, E_t)$ be its ascending sequence.

If $\text{Dom}(y) \cap E_*^t \neq \emptyset$, choose $e \in \text{Dom}(y) \cap E_*^t$. By Proposition 3.3.3, (5.13), and Corollary 2.7.3, the pullback of $\zeta_{[e]}^{\gamma_\circ}$ divides all the elements of the second row of $\tilde{\varphi}^{r_1 p_3}$, thus $\tilde{\varphi}^{r_1 p_3}$ is diagonalizable near y .

If $\text{Dom}(y) \subset \{\mathbf{e}_1, \mathbf{e}_2\}$, then λ_{12} as in (5.12) has two possibilities: either it does not vanish at y , which along with (5.13) and Proposition 3.3.3 implies $\tilde{\varphi}^{r_1 p_3}$ is diagonalizable on $\tilde{\mathcal{V}}_y$; or $\lambda_{12}(y) = 0$ and hence

$$\check{\lambda}_{1e} \sim 1 \quad \forall e \in \text{Edg}(\gamma)^t \setminus E_*^t, \quad \tilde{\varepsilon}_{[t]} | \check{\chi}_{1i} \quad \forall \delta_i \in D'.$$

By (5.13), $\tilde{\varphi}^{r_1 p_3}$ on a neighborhood of y can be written as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_{[t]} \end{bmatrix} \begin{bmatrix} 1 & 0 & \begin{pmatrix} 0 \\ \check{\zeta}_{[e]} \end{pmatrix}_{e \in \text{Edg}(\gamma)^t \setminus E_*^t} & 0 & 0 & * & * & * \\ 0 & \check{\lambda}_{12} & * & \Xi_1 & \Xi_2 & * & * & * \end{bmatrix}, \quad \text{where} \\ \Xi_1 = \left(\check{\zeta}_{\langle e \rangle_t} \tilde{\zeta}_{\langle \langle e \rangle_t, e \rangle} \right)_{e \in E_*^t \setminus \Delta_{y,t}^{\geq}}, \quad \Xi_2 = \left(\check{\zeta}_{\langle \langle e \rangle_t, e \rangle} \right)_{e \in E_*^t \cap \Delta_{y,t}^{\geq}}.$$

With $\zeta_e^{\gamma_y}$ as in Proposition 3.5.2, the above matrix can be rewritten as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_{[t]} \end{bmatrix} \begin{bmatrix} 1 & 0 & \begin{pmatrix} 0 \\ \zeta_{[e]}^{\gamma_y} \end{pmatrix}_{e \in \text{Edg}(\gamma_y)^t} & * * * \\ 0 & \check{\lambda}_{12} & * * * & * * * \end{bmatrix}.$$

Since F is inseparable, $r_1 p_4$ does not affect a neighborhood of y . Since $\deg D \cap F = 2$ and $\text{Dom}(y) \subset \{\mathbf{e}_1, \mathbf{e}_2\}$, the effect of the modular blowups starting $r_1 p_5$ on a neighborhood of y is the same as that on a neighborhood of x in Case 4 of the proof of Proposition 5.2.1. Comparing the above expression of $\tilde{\varphi}^{r_1 p_3}$ with (5.1), we conclude that $\tilde{\varphi}^{r_3 p_3}$ is diagonalizable on the pullback of a neighborhood of y .

Case 4. Assume $D \cap F = \{\delta, \delta'\}$ and F is separable. Combining the arguments in Case 2 and Case 3, we see that either $\tilde{\varphi}^{r_3 p_3}$ is diagonalizable, or the arguments in Case 1 or Case 3 of the proof of Proposition 5.3.1 apply to the current case.

Case 5. Assume $\deg D \cap F = 1$ and F is inseparable. In this case, B is a smooth rational curve. W.l.o.g. we assume $D \cap F = \{\delta_1\}$. Analogously to (5.13), φ can be written as

$$\begin{bmatrix} 1 & \begin{pmatrix} 0 \\ \lambda_{1e} \zeta_{[e]} \end{pmatrix}_{e \in \text{Edg}(\gamma)^t \setminus E_*^t} \\ 0 & \begin{pmatrix} 0 \\ \zeta_{[e]}^{\gamma_\circ} \end{pmatrix}_{e \in E_*^t} \begin{pmatrix} 0 \\ \chi_{1i} \zeta_{[\delta_i]} \end{pmatrix}_{\delta_i \in D'} \end{bmatrix},$$

where $D' = D \setminus (\{\delta_1\} \sqcup \{\delta_e\}_{e \in \text{Edg}(\gamma)^t})$ and $\chi_{1i} = \det \begin{bmatrix} c_{11} & c_{1i} \\ c_{21} & c_{2i} \end{bmatrix} \in \Gamma(\mathcal{O}_V)$.

Similar to Case 1, $r_1 p_1$ and $r_1 p_2$ do not affect \mathcal{V} , whereas $r_1 p_3$ is γ_\circ -compatible. After $r_1 p_3$, fix a lift y of x and let $\underline{E} = (E_1, \dots, E_t)$ be its ascending sequence. If $\text{Dom}(y) \cap E_*^t \neq \emptyset$, the corresponding argument in Case 3 still holds verbatim.

If $\text{Dom}(y) \subset \{\mathbf{e}_1, \mathbf{e}_2\}$, mimicking the argument in Case 3, we see that $\tilde{\varphi}^{r_1 p_3}$ can be rewritten as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_{[t]} \end{bmatrix} \begin{bmatrix} 1 & \begin{pmatrix} 0 \\ \check{\lambda}_{1e} \zeta_{[e]}^{\gamma_y} \end{pmatrix}_{e \in \text{Edg}(\gamma_y)^t \setminus E_*^t} \\ 0 & \begin{pmatrix} 0 \\ \zeta_{[e]}^{\gamma_y} \end{pmatrix}_{e \in E_*^t} \begin{pmatrix} 0 \\ \check{\chi}_{1i} \zeta_{[\delta_i]}^{\gamma_y} \end{pmatrix}_{\delta_i \in D'} \end{bmatrix}.$$

Since F is inseparable, $r_1 p_4$ does not affect a neighborhood of y . Since $\deg D \cap F = 1$ and $\text{Dom}(y) \subset \{\mathbf{e}_1, \mathbf{e}_2\}$, the effect of the modular blowups starting $r_1 p_5$ on a neighborhood of y is the same as that on a neighborhood of x in Case 5 of the proof of Proposition 5.2.1. Comparing the above expression of $\tilde{\varphi}^{r_1 p_3}$ with (5.2), we conclude that $\tilde{\varphi} = \tilde{\varphi}^{r_3 p_4}$ is diagonalizable on the pullback of a neighborhood of y .

Case 6. Assume $D \cap F = \{\delta\}$ and F is separable. Combining the arguments in Case 2 and Case 5, we see that the argument in Case 5 or the arguments in Case 1, Case 3, or Case 4 of the proof of Proposition 5.3.1 applies to the current case. \square

Example 5.4.2. Given $(C, D) \in \mathfrak{M}_2^{\text{div}}$ with the dual graph γ^* of C as in the third diagram of Figure 4. The edges are denoted by $\mathbf{e}_1, \mathbf{e}_2$, and e_a from left

to right. Assume

$$D \cap B = D \cap F = \{\delta_1, \delta_2, \delta_3\}, \quad D \cap C_a \neq \emptyset.$$

This implies $x = (C, c_1(D)) \in \mathfrak{M}_{(3)}^{\text{mn}}$. Under suitable trivialization, φ on a chart \mathcal{V} containing (C, D) can be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & *** \\ 0 & \lambda_{12} & \lambda_{13} & \zeta_a & *** \end{bmatrix},$$

where $\lambda_{12}, \lambda_{13}$ are as in Corollary 2.7.3.

Since $x \in \mathfrak{M}_{(3)}^{\text{mn}}$, the blowups up to $r_1 p_2$ do not affect φ near x . When $r_1 p_3$ terminates, let y be a fixed lift of x . If $e_a \in \text{Dom}(y)$, then the pullback of φ becomes diagonalizable near y . If $e_a \notin \text{Dom}(y)$, then the pullback of φ can be rewritten as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & *** \\ 0 & \check{\lambda}_{12} & \check{\lambda}_{13} & \check{\zeta}_a & *** \end{bmatrix}.$$

By Part (3) of Corollary 2.7.3, $\check{\lambda}_{12}(y)$ and $\check{\lambda}_{13}(y)$ cannot vanish simultaneously, hence the pullback of φ is diagonalizable near y .

5.5. The structural homomorphism around $\mathfrak{M}_{(2)}^{\text{mn}}$. The main statement of this subsection is as follows.

Proposition 5.5.1. *Proposition 5.1.2 holds if $\varpi(\tilde{x}) \in \mathfrak{M}_{(2)}^{\text{mn}}$.*

Proof. As in the proof of Proposition 5.2.1, we fix $(C, D) \in \mathfrak{M}_2^{\text{div}}$ so that $\tilde{x} = (C, \mathcal{O}_C(D))$ and set

$$x \equiv (C, c_1(D)) \in \mathfrak{M}_{(2)}^{\text{mn}}.$$

We still assume $D = \delta_1 + \cdots + \delta_m$.

By the definition of $\mathfrak{M}_{(2)}^{\text{mn}}$, the core F of x contains two genus 1 inseparable components F_1 and F_2 , as well as a unique *minimal* separating bridge $B = B[q_1, q_2]$ so that

$$D \cap F = D \cap B.$$

Let C_1 and C_2 be the irreducible components of B containing q_1 and q_2 , respectively. Note that C_1 and C_2 may be the same.

Let γ be the reduced dual tree of x , γ_\wedge be the TFMR tree as in (4.5) so that B corresponds to the fused vertex v_B , and γ_\vee be the rooted tree as in (4.4) with the root o_2 . Although F_1 and F_2 respectively correspond to the vertices o_1 and o_2 in both γ_\wedge and γ_\vee , notice that o_1 is a root in γ_\wedge but a terminal vertex in γ_\vee . We set

$$E_{\wedge, i}^t = \{ e \in \text{Edg}(\gamma_\wedge)^t : o_i <_{\gamma_\wedge} v_e^+ \} \quad i = 1, 2.$$

Then, $\text{Edg}(\gamma_\wedge)^t = E_{\wedge, 1}^t \sqcup E_{\wedge, 2}^t$.

Let $E_* \subset \text{Edg}(\gamma) \cap \text{Edg}(\gamma_\wedge)$ be as in §3.5 so that the common branches of γ and γ_\wedge are indexed by E_* . With $\mathbf{e}_1, \mathbf{e}_2 \in \text{Edg}(\gamma_\wedge)^t$ denoting the edges such that $v_{\mathbf{e}_1}^+ = v_{\mathbf{e}_2}^+ = v_B$, we see that

$$E_* \sqcup \{\mathbf{e}_1, \mathbf{e}_2\} \in \text{ET}(\gamma_\wedge).$$

In addition, we set

$$\mathbb{E}_* = E_* \cap \text{Edg}(\gamma_\vee), \quad \mathbb{E}_{*,2} = \mathbb{E}_* \sqcup \{\langle \mathbf{e}_2 \rangle\}.$$

Here $\langle \mathbf{e}_2 \rangle$ is taken in $\text{Edg}(\gamma_\vee)$ (and equivalently in $\text{Edg}(\gamma_\wedge)$). It is straightforward that

$$\mathbb{E}_{*,2}^t = E_{\wedge,2}^t, \quad \text{Edg}(\gamma) \cap \text{Edg}(\gamma_\vee) = \mathbb{E}_*^{\geq}, \quad \text{Edg}(\gamma_\wedge) \cap \text{Edg}(\gamma_\vee) = \mathbb{E}_{*,2}^{\geq}.$$

The edges in \mathbb{E}_* or $\mathbb{E}_{*,2}$ are not comparable.

We divide the proof of Proposition 5.4.1 into four cases.

Case 1. Assume $\deg D \cap F \geq 3$. W.l.o.g. we assume $\delta_1 \in C_1$ and $\delta_2 \in C_2$. If δ_1 is conjugate to δ_2 , then $C_1 = C_2 = B$. By assumption, we may take another point in $D \cap B$, which cannot be conjugate to δ_2 . Thus, w.l.o.g. we further assume δ_1 and δ_2 are not conjugate.

By Proposition 2.5.1, φ is in the form

$$\left[\begin{array}{cc} \left(\begin{array}{c} \zeta_{[e]}^{\gamma_\wedge} \\ 0 \end{array} \right)_{e \in E_{\wedge,1}^t} & \left(\begin{array}{c} 0 \\ \zeta_{[e]}^{\gamma_\wedge} \end{array} \right)_{e \in E_{\wedge,2}^t} & * * * \\ & & * * * \end{array} \right]$$

under suitable trivialization.

Since $x \in \mathfrak{M}_{(2)}^{\text{mn}}$, the modular blowups in $r_1 \mathbf{p}_1$ do not affect \mathcal{V} . By the proof of Lemma 4.3.2, the blowups in $r_1 \mathbf{p}_2$ are γ_\wedge -compatible. After $r_1 \mathbf{p}_2$, fix a lift y of x and let $\underline{E} = (E_1, \dots, E_t)$ be the ascending sequence of y .

Since $\text{Dom}(y) \neq \emptyset$, at least one of

$$(5.14) \quad \text{Dom}(y) \cap E_{\wedge,1}^t \quad \text{and} \quad \text{Dom}(y) \cap E_{\wedge,2}^t,$$

say the former, is not empty. If both are not empty, Proposition 3.4.2 implies $\tilde{\varphi}^{r_1 \mathbf{p}_2}$ is diagonalizable near y .

If $\text{Dom}(y) \subset E_{\wedge,1}^t$, $\tilde{\varphi}^{r_1 \mathbf{p}_2}$ on a neighborhood of y can be rewritten as

$$\left[\begin{array}{cc} 1 & \left(\begin{array}{c} 0 \\ \zeta_{[e]}^{(\gamma_\vee)y} \end{array} \right)_{e \in \text{Edg}((\gamma_\vee)y)^t} & * * * \\ 0 & & * * * \end{array} \right].$$

By (4.9), $r_1 \mathbf{p}_3$ does not affect a neighborhood of y . By the proof of Lemma 4.3.4, the blowups in $r_1 \mathbf{p}_4$ are $(\gamma_\vee)_y$ -compatible. Hence by Proposition 3.3.3, $\tilde{\varphi}^{r_1 \mathbf{p}_4}$ is diagonalizable on the pullback of a neighborhood of y .

Case 2. Assume $\deg D \cap F = 2$ and the elements of $D \cap F$ are not conjugate. The argument in Case 1 still holds verbatim.

Case 3. Assume $\deg D \cap F = 2$ and the elements of $D \cap F$ are conjugate. W.l.o.g. we assume $D \cap F = \{\delta_1, \delta_2\}$. Then $B = C_1 = C_2$. The structural homomorphism φ can then be written as

$$\left[\begin{array}{ccc} \left(\begin{array}{c} \zeta_{[e]}^{\gamma_\wedge} \\ 0 \end{array} \right)_{e \in E_*^t \setminus \mathbb{E}_*^t} & \left(\begin{array}{cc} \zeta_{[\mathbf{e}_1]}^{\gamma_\wedge} & 0 \\ 0 & \lambda_{12} \zeta_{[\mathbf{e}_2]}^{\gamma_\wedge} \end{array} \right) & \left(\begin{array}{c} 0 \\ \zeta_{[e]}^{\gamma_\wedge} \end{array} \right)_{e \in \mathbb{E}_*^t} & \left(\begin{array}{c} 0 \\ \zeta_{[e]} \zeta_{[\mathbf{e}_2]}^{\gamma_\wedge} \end{array} \right)_{e \in \text{Edg}(\gamma)^t \setminus E_*^t} & * * * \\ & & & & * * * \end{array} \right]$$

under suitable trivialization.

As in Case 1, $r_1\mathbf{p}_1$ does not affect \mathcal{V} , whereas $r_1\mathbf{p}_2$ is γ_\wedge -compatible. After $r_1\mathbf{p}_2$, fix a lift y of x and let $\underline{E} = (E_1, \dots, E_t)$ be the ascending sequence of y . One of the two sets in (5.14) is still non-empty.

If $\text{Dom}(y) \not\subset \{\mathbf{e}_1, \mathbf{e}_2\}$, w.l.o.g. assume $\text{Dom}(y) \cap \mathbb{E}_*^t \neq \emptyset$. Then similar to Case 1, $\tilde{\varphi}^{r_1\mathbf{p}_4}$ is diagonalizable on the pullback of a neighborhood of y .

If $\text{Dom}(y) = \{\mathbf{e}_1, \mathbf{e}_2\}$, then the blowups from $r_1\mathbf{p}_3$ up to the end of $r_3\mathbf{p}_2$ do not affect a neighborhood \mathcal{V}_y of y , and $\tilde{\varphi}^{r_3\mathbf{p}_2}$ on \mathcal{V}_y can be rewritten as

$$\tilde{\varepsilon}_{[t]} \begin{bmatrix} 1 & 0 & \begin{pmatrix} 0 \\ \zeta_{[e]}^{\gamma_y} \end{pmatrix}_{e \in \text{Edg}(\gamma_y)^t} & * * * \\ 0 & \tilde{\lambda}_{12} & & * * * \end{bmatrix}$$

under suitable trivialization. Comparing the above expression of $\tilde{\varphi}^{r_3\mathbf{p}_3}$ and (5.1), we conclude that the argument in Case 4 of the proof of Proposition 5.2.1 applies to the current case.

If $\text{Dom}(y)$ consists of only one element of $\{\mathbf{e}_1, \mathbf{e}_2\}$, say, $\text{Dom}(y) = \{\mathbf{e}_1\}$, then the blowups in $r_1\mathbf{p}_3$ do not affect a neighborhood \mathcal{V}_y of y , and $\tilde{\varphi}^{r_1\mathbf{p}_3}$ on \mathcal{V}_y can be rewritten as

$$\tilde{\varepsilon}_{[t]} \begin{bmatrix} 1 & 0 & \begin{pmatrix} 0 \\ \zeta_{[e_1]}^{(\gamma_\vee)y} \end{pmatrix}_{e \in \text{Edg}(\gamma)^t \setminus E_*^t} & \begin{pmatrix} 0 \\ \zeta_{[e]}^{(\gamma_\vee)y} \end{pmatrix}_{e \in E_*^t} & * * * \\ 0 & \tilde{\lambda}_{12} \zeta_{[e_1]}^{(\gamma_\vee)y} & & & * * * \end{bmatrix}$$

under suitable trivialization. Comparing the above expression of $\tilde{\varphi}^{r_1\mathbf{p}_3}$ and (5.9), we conclude that the argument in Case 3 of the proof of Proposition 5.3.1 applies to the current case.

Case 4. Assume $\deg D \cap F = \{\delta_1\}$. Then $B = C_1 = C_2$. The structural homomorphism φ can then be written as

$$\left[\begin{array}{c} \begin{pmatrix} \zeta_{[e]}^{\gamma_\wedge} \\ 0 \end{pmatrix}_{e \in E_*^t \setminus \mathbb{E}_*^t} \\ \begin{pmatrix} \zeta_{[\mathbf{e}_1]}^{\gamma_\wedge} \\ \zeta_{[\mathbf{e}_2]}^{\gamma_\wedge} \end{pmatrix} \\ \begin{pmatrix} 0 \\ \zeta_{[e]}^{\gamma_\wedge} \end{pmatrix}_{e \in \mathbb{E}_*^t} \\ \begin{pmatrix} 0 \\ \lambda_{1e} \zeta_{[e]}^{\gamma_\wedge} \zeta_{[\mathbf{e}_2]}^{\gamma_\wedge} \end{pmatrix}_{e \in \text{Edg}(\gamma)^t \setminus E_*^t} \\ \begin{pmatrix} 0 \\ \chi_{1i} \zeta_{[\delta_i]} \end{pmatrix}_{\delta_i \in D'} \end{array} \right],$$

where $D' = D \setminus (\{\delta_1\} \sqcup \{\delta_e\}_{e \in \text{Edg}(\gamma)^t})$ and $\chi_{1i} = \det \begin{bmatrix} c_{11} & c_{1i} \\ c_{21} & c_{2i} \end{bmatrix} \in \Gamma(\mathcal{O}_{\mathcal{V}})$.

As in Case 3, when $r_1\mathbf{p}_2$ terminates, every lift y of x satisfies $\text{Dom}(y) \neq \emptyset$. Let $\underline{E} = (E_1, \dots, E_t)$ be the ascending sequence of y .

If $\text{Dom}(y) \not\subset \{\mathbf{e}_1, \mathbf{e}_2\}$, then similar to Case 3, $\tilde{\varphi}^{r_1\mathbf{p}_4}$ is diagonalizable on the pullback of a neighborhood of y .

If $\text{Dom}(y) = \{\mathbf{e}_1, \mathbf{e}_2\}$, then the blowups in $r_1\mathbf{p}_3$ and $r_1\mathbf{p}_4$ do not affect a neighborhood \mathcal{V}_y of y , and $\tilde{\varphi}^{r_1\mathbf{p}_4}$ on \mathcal{V}_y can be rewritten as

$$\tilde{\varepsilon}_{[t]} \begin{bmatrix} 1 & \begin{pmatrix} 0 \\ \tilde{\lambda}_{1e} \zeta_{[e]}^{\gamma_y} \end{pmatrix}_{e \in \text{Edg}(\gamma_y)^t} & \begin{pmatrix} 0 \\ \chi_{1i} \zeta_{[\delta_i]}^{\gamma_y} \end{pmatrix}_{\delta_i \in D'} & * * * \\ 0 & & & * * * \end{bmatrix}.$$

Comparing the above expression of $\tilde{\varphi}^{r_3\mathbf{p}_3}$ and (5.2), we conclude that the argument in Case 5 of the proof of Proposition 5.2.1 applies to the current case.

If $\text{Dom}(y)$ consists of only one element of $\{\mathbf{e}_1, \mathbf{e}_2\}$, say, $\text{Dom}(y) = \{\mathbf{e}_1\}$, then the blowups in $r_1\mathbf{p}_3$ do not affect a neighborhood \mathcal{V}_y of y , and $\tilde{\varphi}^{r_1\mathbf{p}_3}$ on

\mathcal{V}_y can be rewritten as

$$\tilde{\varepsilon}_{[t]} \begin{bmatrix} 1 & \begin{pmatrix} 0 \\ \tilde{\chi}_{1e} \zeta_{[e]} \zeta_{[e_2]}^{(\gamma v)_y} \end{pmatrix}_{e \in \text{Edg}(\gamma)^t \setminus E_*^t} & \begin{pmatrix} 0 \\ \zeta_{[e]}^{(\gamma v)_y} \end{pmatrix}_{e \in \mathbb{E}_*^t} & \begin{pmatrix} 0 \\ \chi_{1i} \zeta_{[\delta_i]}^{\gamma y} \end{pmatrix}_{\delta_i \in D'} & * * * \\ 0 & & & & * * * \end{bmatrix}$$

Comparing the above expression of $\tilde{\varphi}^{r_1 p_3}$ and (5.10), we conclude that the argument in Case 4 of the proof of Proposition 5.3.1 applies to the current case. \square

5.6. The structural homomorphism around $\overline{\mathfrak{M}}_{(1)}$. The main statement of this subsection is as follows.

Proposition 5.6.1. *Proposition 5.1.2 holds if $\varpi(\tilde{x}) \in \mathfrak{M}_{(1)}^{\text{mn}}$.*

Proof. As in the proof of Proposition 5.2.1, we fix $(C, D) \in \mathfrak{M}_2^{\text{div}}$ so that $\tilde{x} = (C, \mathcal{O}_C(D))$ and set

$$x \equiv (C, c_1(D)) \in \mathfrak{M}_{(1)}^{\text{mn}}.$$

Let γ be the reduced dual tree of x . We still assume $D = \delta_1 + \cdots + \delta_m$. By the definition of $\mathfrak{M}_{(1)}^{\text{mn}}$, the core F of C satisfies $D \cap F = \emptyset$.

It is straightforward that the functions ζ_e , $e \in \text{Edg}(\gamma)$, are coordinate functions labeled by γ on \mathcal{V} , and $r_1 p_1$ is γ -compatible on \mathcal{V} . After $r_1 p_1$, fix a lift y of x and let $\underline{E} = (E_1, \dots, E_t)$ be the ascending sequence of y . With $\text{DPN}(y)$ as in (3.15), let F_y be the *smallest connected* subcurve of F containing $\text{DPN}(y)$ and F_y^c be the closure of $F \setminus F_y$ in F . Notice that F_y^c may be disconnected.

We divide the proof of Proposition 5.6.1 into five cases. Case 1 is further divided into several subcases depending on the number and position of the dominant pivotal branches. Regarding the other three cases, we show that the pullback of φ either becomes locally diagonalizable or can be reduced to Case 1. As in the previous subsections, it suffices to show that in each case the pullback of φ becomes locally diagonalizable after some intermediate step of the three-round modular blowups.

Case 1. Assume that $g_a(F_y^c) = 0$. There are five subcases:

- (a) $|\text{DPN}(y)| \geq 3$;
- (b) $\text{DPN}(y) = \{p_{e_1}, p_{e_2}\}$ and p_{e_1} is not conjugate to p_{e_2} ;
- (c) $\text{DPN}(y) = \{p_{e_1}, p_{e_2}\}$ and p_{e_1} is conjugate to p_{e_2} ;
- (d) $\text{DPN}(y) = \{p_{e_0}\}$ and p_{e_0} is not Weierstrass;
- (e) $\text{DPN}(y) = \{p_{e_0}\}$ and p_{e_0} is Weierstrass.

In Subcases (a) and (b), there exist $p_{e_1}, p_{e_2} \in \text{DPN}(y)$ that are not conjugate. By Corollary 2.7.3, $\tilde{\varphi}^{r_1 p_1}$ near y can be written as

$$\tilde{\varepsilon}_{[t]} \begin{bmatrix} 1 & 0 & * * * \\ 0 & 1 & * * * \end{bmatrix},$$

where the first two columns correspond to some $e'_1, e'_2 \in \text{Dom}(y)$ with $\langle e'_i \rangle = e_i$, $i = 1, 2$.

In Subcase (c), let $e_0^1, e_0^2, \varsigma_0, \varsigma$, and $\sigma(e)$ be as in §3.7. Obviously $\varsigma_0 = 0$. If $|\text{Dom}(y)| \geq 3$ or $|\text{Dom}(y)| = 2$ and at least one $e \in \text{Dom}(y)$ satisfies $\mathbf{w}(v_e^+) > 1$, then $\tilde{\varphi}^{r_1 p_1}$ can be written as

$$(5.15) \quad \begin{aligned} & \tilde{\varepsilon}_{[t]} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & \tilde{\lambda}_{12} & \tilde{\varepsilon}_{[\varsigma]} & \Xi_1 & \Xi_2 & * & * & * \end{bmatrix}, \quad \text{where} \\ & \Xi_1 = \left(\tilde{\varepsilon}_{[\sigma(e)]} \tilde{\zeta}_{\langle e \rangle_t} \tilde{\zeta}_{\langle e \rangle_t, e} \right)_{e \in (\text{Edg}(\gamma)^t \setminus \Delta_{y,t}^{\geq}) \setminus \{e_0^1, e_0^2\} \geq}, \\ & \Xi_2 = \left(\tilde{\varepsilon}_{[\sigma(e)]} \tilde{\zeta}_{\langle e \rangle_t, e} \right)_{e \in \Delta_{y,t}^t \setminus \{e_0^1, e_0^2\} \geq}. \end{aligned}$$

The modular blowups from $r_1 p_2$ up to the end of r_2 do not affect a neighborhood \mathcal{V}_y of y , and $r_3 p_1$ is $\gamma_{y,+}^{(2)}$ -compatible on \mathcal{V}_y . With $\gamma_y^{(2)}$ denoting the bi-dominantly derived tree at y as in §3.7, $\tilde{\varphi}^{r_2}$ can be rewritten as

$$\tilde{\varepsilon}_{[t]} \begin{bmatrix} 1 & 0 & \begin{pmatrix} 0 \\ \zeta_{[e]}^{\gamma_y^{(2)}} \end{pmatrix}_{e \in \text{Edg}(\gamma_y^{(2)})^t} & * & * & * \\ 0 & \tilde{\lambda}_{12} & & * & * & * \end{bmatrix}.$$

By Proposition 3.3.3, $\tilde{\varphi}^{r_3 p_1}$ becomes diagonalizable on the pullback of \mathcal{V}_y .

If $\text{Dom}(y) = \{e, e'\}$ with $\mathbf{w}(v_e^+) = \mathbf{w}(v_{e'}^+) = 1$, then $\varsigma = t$. Set

$$\tau_2 \equiv \left(\hat{\gamma}_{[\{e_0^1, e_0^2\} \leq]} \right)^b.$$

Let $E_* \subset \text{Edg}(\gamma) \cap \text{Edg}(\tau_2)$ be as in §3.5 so that the common branches of γ and τ_2 are indexed by E_* . Mimicking the argument in the paragraph above, we see that $\tilde{\varphi}^{r_1 p_1}$ can be written as

$$\tilde{\varepsilon}_{[t]} \begin{bmatrix} 1 & 0 & \begin{pmatrix} 0 \\ \tilde{\varepsilon}_{[t]} \tilde{\zeta}_{[e]}^{\tau_2} \end{pmatrix}_{e \in \text{Edg}(\tau_2)^t \setminus E_*^{\geq}} & 0 & 0 & * & * & * \\ 0 & \tilde{\lambda}_{12} & & \Xi_1 & \Xi_2 & * & * & * \end{bmatrix},$$

where Ξ_1 and Ξ_2 are as in (5.15). Then either $\tilde{\varphi}^{r_3 p_1}$ becomes diagonalizable as in the previous paragraph, or $\tilde{\varphi}^{r_3 p_3}$ becomes diagonalizable as in the proof of Lemma 5.2.2.

In Subcase (d), let e_\bullet and s be as in §3.6. We have the following two possibilities:

- (d1) none of the pivotal nodes is conjugate to $q_{\langle e_\bullet \rangle}$;
- (d2) there exists a pivotal node $q_{e'}$ conjugate to $q_{\langle e_\bullet \rangle}$.

For Possibility (d1), if either $|\text{Dom}(y)| \geq 2$, or $\text{Dom}(y) = \{e_1\}$ and $\mathbf{w}(v_{e_1}^+) > 1$, similar to the argument in Subcase (c), $\tilde{\varphi}^{r_1 p_1}$ can be written as

$$\tilde{\varepsilon}_{[t]} \begin{bmatrix} 1 & \begin{pmatrix} 0 \\ \zeta_{[e]}^{\gamma'_y} \end{pmatrix}_{e \in \text{Edg}(\gamma'_y)^t} & * & * & * \\ 0 & & * & * & * \end{bmatrix},$$

where γ'_y is the derived tree at y as in §3.6 and $\zeta_e^{\gamma'_y}$ is the local parameter of Lemma 3.6.4. We see that $r_1 p_2$ - $r_1 p_5$ do not affect \mathcal{V}_y , whereas r_2 is γ'_y -compatible on \mathcal{V}_y . Thus, $\tilde{\varphi}^{r_2}$ becomes diagonalizable. If $\text{Dom}(y) = \{e_1\}$ with $\mathbf{w}(v_{e_1}^+) = 1$, let \tilde{y} be a fixed lift of y after $r_1 p_5$. Then either $\tilde{\varphi}^{r_1 p_5}$ is already

diagonalizable near \tilde{y} , or the previous argument in this paragraph applies to the current case with y and γ'_y replaced by \tilde{y} and $(\gamma'_y)_{\tilde{y}}$, respectively.

For Possibility (d2), once again we first assume $|\text{Dom}(y)| \geq 2$ or $\text{Dom}(y) = \{e_1\}$ and $\mathbf{w}(v_{e_1}^+) > 1$. Let \tilde{y} be a fixed lift of y after r_2 . If $\text{Dom}(\tilde{y})$ contains at least one edge e'' with $\langle e'' \rangle \neq e'$, then by the preceding paragraph, $\tilde{\varphi}^{r_2}$ is still diagonalizable. If $\langle e'' \rangle = e'$ for all $e'' \in \text{Dom}(\tilde{y})$, then by the argument in Subcase (c), $\tilde{\varphi}^{r_3 p_1}$ becomes diagonalizable. If $\text{Dom}(y) = \{e\}$ with $\mathbf{w}(v_e^+) = 1$, the argument is parallel to the scenario in Possibility (d1).

In Subcase (e), let e_\bullet and s be as in §3.6 and γ'_y be the derived tree at y . We first assume $\sum_{e \in \text{Dom}(y)} \mathbf{w}(v_e^+) \geq 3$. This guarantees that $r_1 p_2$ - $r_1 p_5$, $r_3 p_3$, and $r_3 p_4$ do not affect the corresponding pullbacks of \mathcal{V}_y . As in Subcase (d), r_2 is γ'_y -compatible on \mathcal{V}_y . When r_2 terminates, we fix a lift \tilde{y} of y . We have the following two possibilities.

- (e1) $\text{Dom}(\tilde{y})$ contains at least one edge e' such that $\langle e' \rangle_{\gamma'_y} \neq e_1$.
- (e2) All edges $e' \in \text{Dom}(\tilde{y})$ satisfy $\langle e' \rangle_{\gamma'_y} = e_1$.

In Possibility (e1), since $q_{\langle e' \rangle}$ cannot be conjugate to q_{e_0} , $\tilde{\varphi}^{r_2}$ is diagonalizable after r_2 .

In Possibility (e2), the bi-dominantly derived tree $\gamma_{\tilde{y}}^{(2)}$ may or may not exist; c.f. (S1) and (S2) of §3.7. If $\gamma_{\tilde{y}}^{(2)}$ does not exist, then $r_3 p_1$ does not affect $\mathcal{V}_{\tilde{y}}$. By Proposition 2.6.1 and Corollaries 2.7.1-2.7.3, $\tilde{\varphi}^{r_2}$ near \tilde{y} can be rewritten as

$$\tilde{\varepsilon}[t] \tilde{\varepsilon}[u] \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\theta}_{12} \end{bmatrix} \begin{bmatrix} 0 \\ \zeta_{[e]}^{\gamma'_y} \end{bmatrix}_{e \in \text{Edg}((\gamma'_y)'_{e_s, \tilde{y}})} \begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix},$$

where $(\gamma'_y)'_{e_s, \tilde{y}}$ is the TFMR tree partially derived from γ'_y along e_s at \tilde{y} as in §3.6 and $\tilde{\theta}_{12}$ is as in Corollary 2.7.1. The sequential blowup $r_3 p_2$ is $(\gamma'_y)'_{e_s, \tilde{y}, +}$ -compatible on a neighborhood of y' , thus $\tilde{\varphi}^{r_3 p_2}$ is locally diagonalizable on the pullback of $\mathcal{V}_{\tilde{y}}$. If $\gamma_{\tilde{y}}^{(2)}$ exists, similar to the previous argument, we see that $r_3 p_1$ is $\gamma_{\tilde{y}}^{(2)}$ -compatible on $\mathcal{V}_{\tilde{y}}$; when $r_3 p_1$ terminates, near a fixed lift \tilde{y}' of \tilde{y} , either $\tilde{\varphi}^{r_3 p_1}$ is already diagonalizable, or $r_3 p_2$ is $((\gamma'_y)'_{e_s, \tilde{y}, +})_{\tilde{y}'}$ -compatible and $\tilde{\varphi}^{r_3 p_2}$ is diagonalizable near \tilde{y}' .

Regarding the special cases in which $\sum_{e \in \text{Dom}(y)} \mathbf{w}(v_e^+) = 2$ or 1 , $r_3 p_3$ and $r_3 p_4$ may become relevant. The argument is similar to that of Subcase (c) and of Possibility (d1), respectively. We omit further details.

Case 2. Assume that $g_a(F_y^c) = g_a(F_y) = 1$. Let F_1 and F_2 be the core of F_y and F_y^c , respectively. As before, we denote by γ_\vee the rooted tree whose root is o_2 ; c.f. (4.4).

The definition of F_y implies $r_1 p_2$ and $r_1 p_3$ do not affect \mathcal{V}_y , whereas $r_1 p_4$ is $(\gamma_\vee)_y$ -compatible on \mathcal{V}_y . When $r_1 p_4$ terminates, for every lift \tilde{y} of y , either $\tilde{\varphi}^{r_1 p_4}$ is already diagonalizable near \tilde{y} , or the argument in Case 1 applies to the current case with y replaced by \tilde{y} .

Case 3. Assume that $g_a(F_y^c) = 1$, $g_a(F_y) = 0$, and F is inseparable. We denote by F_2 the core of F_y^c . As before, let γ_\diamond be the TFMR tree with a single root o_2 such that F_y corresponds to the unique fused terminal vertex of γ_\diamond ; c.f. (4.6).

Similar to Case 2, r_1p_2 does not affect \mathcal{V}_y , whereas r_1p_3 is $(\gamma_\diamond)_y$ -compatible on \mathcal{V}_y . When r_1p_3 terminates, for every lift \tilde{y} of y , either $\tilde{\varphi}^{r_1p_3}$ is already diagonalizable near \tilde{y} , or the argument in Case 1 applies to the current case with y replaced by \tilde{y} .

Case 4. Assume that $g_a(F_y^c) = 1$, $g_a(F_y) = 0$, and F is separable. Analogous to Case 3, let F_2 be the core of F_y^c . In addition to the TFMR tree γ_\diamond in Case 3, there is another relevant rooted tree γ_\vee whose root o_2 corresponds to F_2 ; c.f. (4.4).

As in Case 3, r_1p_3 is still $(\gamma_\diamond)_y$ -compatible on \mathcal{V}_y . When r_1p_3 terminates, for every lift y' of y , either $\tilde{\varphi}^{r_1p_3}$ is already diagonalizable near y' , or r_1p_4 is $(\gamma_\vee)_{y'}$ -compatible on a neighborhood of y' . When r_1p_4 terminates, for every lift \tilde{y}' of y' , either $\tilde{\varphi}^{r_1p_4}$ is already diagonalizable near \tilde{y}' , or the argument in Case 1 applies to the current case with y replaced by \tilde{y}' .

Case 5. Assume that F_y^c consists of two genus 1 subcurves. We denote by F_1 and F_2 the cores of the connected components of F_y^c . As before, let γ_\wedge be the TFMR tree with the roots o_1 and o_2 respectively corresponding to F_1 and F_2 , and γ_\vee be the rooted tree with the root o_2 corresponding to F_2 ; c.f. (4.5) and (4.4).

The modular blowups in r_1p_2 are $(\gamma_\wedge)_y$ -compatible on \mathcal{V}_y . When r_1p_2 terminates, for every lift y' of y , either $\tilde{\varphi}^{r_1p_2}$ is already diagonalizable near y' , or r_1p_4 is $(\gamma_\vee)_{y'}$ -compatible on a neighborhood of y' . When r_1p_4 terminates, for every lift \tilde{y}' of y' , either $\tilde{\varphi}^{r_1p_4}$ is already diagonalizable near \tilde{y}' , or the argument in Case 1 applies to the current case with y replaced by \tilde{y}' . \square

Example 5.6.2. Let $(C, D) \in \mathfrak{M}_2^{\text{div}}$ be such that C has a *smooth* core F and two tails C_u and C_v , each of which is a *smooth* rational curve, and

$$D \cap F = \emptyset, \quad |D \cap C_i| \geq 2, \quad i = u, v;$$

see the diagram for $\overline{\mathfrak{M}}_{(1,2)}$ in Figure 1. The (pivotal) edges are denoted by e_u and e_v , respectively. Then, $x = (C, c_1(D)) \in \mathfrak{M}_{(1)}^{\text{mn}}$.

In Example 5.6.2, we assume that q_{e_u} and q_{e_v} are not conjugate, and neither of them is a Weierstrass point. Under suitable trivialization, φ on a chart \mathcal{V} containing (C, D) can then be written as

$$(5.16) \quad \begin{bmatrix} c_{1u}\zeta_u & c_{1v}\zeta_v & 0 & 0 & *** \\ c_{2u}\zeta_u & c_{2v}\zeta_v & \zeta_u^2 & \zeta_v^2 & *** \end{bmatrix},$$

where $\zeta_u = \zeta_{e_u}$ and $\zeta_v = \zeta_{e_v}$ are the modular parameters. By Proposition 2.5.1, $c_{ij} \in \Gamma(\mathcal{O}_{\mathcal{V}}^*)$ for $i = 1, 2$ and $j = u, v$. By Corollary 2.7.3,

$$\lambda_{uv} = \det \begin{bmatrix} c_{1u} & c_{1v} \\ c_{2u} & c_{2v} \end{bmatrix} \text{ is nowhere vanishing on } \mathcal{V}.$$

Let y be a lift of x after $r_1\mathfrak{p}_1$ and \mathcal{V}_y be a chart containing y . If $\text{Dom}(y) = \{e_u, e_v\}$, then $\tilde{\varphi}^{r_1\mathfrak{p}_1}$ can be rewritten as

$$\tilde{\varepsilon}_1 \begin{bmatrix} 1 & 0 & *** \\ 0 & 1 & *** \end{bmatrix}.$$

Thus $\tilde{\varphi}^{r_1\mathfrak{p}_1}$ is diagonalizable on \mathcal{V}_y . If $\text{Dom}(y) = \{e_u\}$, $\tilde{\varphi}^{r_1\mathfrak{p}_1}$ can be rewritten as

$$\tilde{\varepsilon}_1 \begin{bmatrix} 1 & 0 & 0 & *** \\ 0 & \check{\zeta}_v & \tilde{\varepsilon}_1 & *** \end{bmatrix}.$$

By the assumption of (C, D) , $r_1\mathfrak{p}_2$ - $r_1\mathfrak{p}_5$ do not affect \mathcal{V}_y . Analyzing the derived tree γ'_y , we see that locally \mathcal{V}_y is blown up along the locus $\{\tilde{\varepsilon}_1 = \check{\zeta}_v = 0\}$ in r_2 , thus $\tilde{\varphi}^{r_2}$ becomes diagonalizable on the pullback on \mathcal{V}_y .

Example 5.6.3. Let $(C, D) \in \mathfrak{M}_2^{\text{div}}$ be the same as in Example 5.6.2, except q_{e_u} and q_{e_v} are *conjugate*. Then on a chart \mathcal{V} containing (C, D) , φ is still in the form of (5.16), but $\lambda_{uv} = \det \begin{bmatrix} c_{1u} & c_{1v} \\ c_{2u} & c_{2v} \end{bmatrix}$ vanishes on the locus where q_{e_u} and q_{e_v} are conjugate by Corollary 2.7.3.

Let y be a lift of $x = (C, c_1(D))$ after $r_1\mathfrak{p}_1$ and \mathcal{V}_y be a chart containing y . If $\text{Dom}(y) = \{e_u, e_v\}$, then $\tilde{\varphi}^{r_1\mathfrak{p}_1}$ can be rewritten as

$$\tilde{\varepsilon}_1 \begin{bmatrix} 1 & 0 & 0 & *** \\ 0 & \tilde{\lambda}_{uv} & \tilde{\varepsilon}_1 & *** \end{bmatrix}.$$

Thus $\tilde{\varphi}^{r_1\mathfrak{p}_1}$ is diagonalizable on \mathcal{V}_y . The topology of C and the assumption on $\text{Dom}(y)$ imply that $r_1\mathfrak{p}_2$ - r_2 do not affect \mathcal{V}_y . Analyzing the rooted tree $\gamma_{y,+}^{(2)}$ supplementary to the bi-dominantly derived tree $\gamma_y^{(2)}$, we see that locally \mathcal{V}_y is blown up along the locus $\{\tilde{\varepsilon}_1 = \tilde{\lambda}_{uv} = 0\}$ in $r_3\mathfrak{p}_1$, thus $\tilde{\varphi}^{r_3\mathfrak{p}_1}$ becomes diagonalizable on the pullback on \mathcal{V}_y .

If $\text{Dom}(y) = \{e_u\}$, $\tilde{\varphi}^{r_1\mathfrak{p}_1}$ can be rewritten as

$$\tilde{\varepsilon}_1 \begin{bmatrix} 1 & 0 & 0 & *** \\ 0 & \tilde{\lambda}_{uv}\check{\zeta}_v & \tilde{\varepsilon}_1 & *** \end{bmatrix}.$$

Similar to the preceding paragraph, for a fixed lift \tilde{y} of y after r_2 , either $\tilde{\varphi}^{r_2}$ is diagonalizable on a chart $\mathcal{V}_{\tilde{y}}$, or $\text{Dom}(\tilde{y}) = \{e_v\}$. In the latter situation, $\tilde{\varphi}^{r_2}$ can be rewritten as

$$\tilde{\varepsilon}_1 \tilde{\varepsilon}_1 \begin{bmatrix} 1 & 0 & 0 & *** \\ 0 & \tilde{\lambda}_{uv} & \tilde{\varepsilon}_1 & *** \end{bmatrix}.$$

As shown in the preceding paragraph, $\tilde{\varphi}^{r_3\mathfrak{p}_1}$ thus becomes diagonalizable on the pullback on $\mathcal{V}_{\tilde{y}}$.

Example 5.6.4. Let $(C, D) \in \mathfrak{M}_2^{\text{div}}$ be such that the dual graph of C is the first graph in Figure 3, the core F is a smooth genus 2 curve, the irreducible components C_a, C_b, C_c , and C_d are all smooth rational curves, and

$$D \cap (F \cup C_b) = \emptyset, \quad |D \cap C_i| \geq 3, \quad i = a, c, d.$$

Assume that q_{e_b} is a Weierstrass point. For a fixed lift y of $x = (C, c_1(D))$ after r_1p_1 , we further assume that $\text{Dom}(y) = \{e_c\}$. By Example 3.6.10, the derived tree γ'_y at y is the last graph in Figure 6.

Let \mathcal{V}_y be a chart containing y . Under suitable trivialization, $\varphi^{r_1p_1}$ on \mathcal{V}_y can then be written as

$$\tilde{\varepsilon}_1\tilde{\varepsilon}_2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & *** \\ 0 & \tilde{\theta}_c\tilde{\varepsilon}_1\tilde{\varepsilon}_2 & \tilde{\varepsilon}_1^2\tilde{\varepsilon}_2^2 & \tilde{\theta}_{cd}\tilde{\varepsilon}_1\check{\zeta}_d & \tilde{\theta}_d\tilde{\varepsilon}_1\tilde{\varepsilon}_2\check{\zeta}_d^2 & \check{\zeta}_a & *** \end{bmatrix},$$

where θ_c, θ_d are as in Proposition 2.6.1 and θ_{cd} is as in Corollary 2.7.1 satisfying

$$\theta_{cd} = f_1\theta_c + g_1\zeta_b = f_2\theta_d + g_2\zeta_b, \quad f_i, g_i \in \Gamma(\mathcal{O}_{\mathcal{V}}^*), \quad i=1, 2, \quad \det \begin{bmatrix} f_1 & g_1 \\ f_2 & g_2 \end{bmatrix} \in \Gamma(\mathcal{O}_{\mathcal{V}}^*).$$

Similar to the previous two examples, r_1p_2 - r_1p_5 do not affect \mathcal{V}_y , and r_2 is γ'_y -compatible on \mathcal{V}_y . When r_2 terminates, fix a lift \tilde{y} of y .

If $e_a \in \text{Dom}(\tilde{y})$, then obviously $\tilde{\varphi}^{r_2}$ becomes diagonalizable. If $e_a \notin \text{Dom}(\tilde{y})$, there are three possibilities for $\text{Dom}(\tilde{y})$: $\{e_2, e_d\}$, $\{e_2\}$, or $\{e_d\}$.

If $\text{Dom}(\tilde{y}) = \{e_2, e_d\}$, then on a chart $\mathcal{V}_{\tilde{y}}$ containing \tilde{y} , $\tilde{\varphi}^{r_2}$ can be rewritten as

$$\tilde{\varepsilon}_1\tilde{\varepsilon}_2 \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_1\tilde{\varepsilon}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & *** \\ 0 & \tilde{\theta}_c & \tilde{\varepsilon}_1 & \check{\zeta}_a & *** \end{bmatrix}.$$

The assumption on $\text{Dom}(\tilde{y})$ implies r_3p_1 does not affect $\mathcal{V}_{\tilde{y}}$. Analyzing the rooted tree $(\gamma'_y)'_{e_2, \tilde{y}, +}$ supplementary to the partially derived tree of γ'_y at \tilde{y} along e_2 , we see that locally $\mathcal{V}_{\tilde{y}}$ is blown up along the locus $\{\tilde{\theta}_c = \tilde{\varepsilon}_1 = \check{\zeta}_a = 0\}$ in r_3p_2 , thus $\tilde{\varphi}^{r_3p_2}$ becomes diagonalizable on the pullback of $\mathcal{V}_{\tilde{y}}$.

If $\text{Dom}(\tilde{y}) = \{e_2\}$, then the partially derived tree $(\gamma'_y)'_{e_2, \tilde{y}}$ of γ'_y along e_2 is isomorphic to γ'_y . On a chart $\mathcal{V}_{\tilde{y}}$ containing \tilde{y} , $\tilde{\varphi}^{r_2}$ can be rewritten as

$$\tilde{\varepsilon}_1\tilde{\varepsilon}_2 \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_1\tilde{\varepsilon}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & *** \\ 0 & \tilde{\theta}_c & \tilde{\varepsilon}_1\tilde{\varepsilon}_2 & \tilde{\varepsilon}_1\check{\zeta}_d & \check{\zeta}_a & *** \end{bmatrix}.$$

Since r_3p_1 does not affect $\mathcal{V}_{\tilde{y}}$ and r_3p_2 is $(\gamma'_y)'_{e_2, \tilde{y}, +}$ -compatible on $\mathcal{V}_{\tilde{y}}$, we see that locally $\mathcal{V}_{\tilde{y}}$ is blown up along the locus $\{\tilde{\theta}_c = \tilde{\varepsilon}_1 = \check{\zeta}_a = 0\}$ and then along the proper transform of $\{\tilde{\theta}_c = \tilde{\varepsilon}_2 = \check{\zeta}_d = \check{\zeta}_a = 0\}$ in r_3p_2 , thus $\tilde{\varphi}^{r_3p_2}$ becomes diagonalizable on the pullback of $\mathcal{V}_{\tilde{y}}$.

If $\text{Dom}(\tilde{y}) = \{e_d\}$, on a chart $\mathcal{V}_{\tilde{y}}$ containing \tilde{y} , $\tilde{\varphi}^{r_2}$ can be rewritten as

$$\tilde{\varepsilon}_1\tilde{\varepsilon}_2 \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon}_1\tilde{\varepsilon}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & *** \\ 0 & \tilde{\theta}_{cd} & \tilde{\varepsilon}_1\tilde{\varepsilon}_2 & \check{\zeta}_a & *** \end{bmatrix}.$$

Similar to the preceding paragraphs, we observe that locally $\mathcal{V}_{\tilde{y}}$ is blown up along the locus $\{\tilde{\theta}_{cd} = \tilde{\varepsilon}_2 = \check{\zeta}_a = 0\}$ in r_3p_1 and then along the proper transform of $\{\tilde{\theta}_{cd} = \tilde{\varepsilon}_1 = \check{\zeta}_a = 0\}$ in r_3p_2 , thus $\tilde{\varphi}^{r_3p_2}$ becomes diagonalizable on the pullback of $\mathcal{V}_{\tilde{y}}$.

Example 5.6.5. Let $(C, D) \in \mathfrak{M}_2^{\text{div}}$ be such that its weighted tree is illustrated in Figure 7. Assume that the irreducible components of C are all smooth,

$$D \cap F = \emptyset, \quad |D \cap C_a| = |D \cap C_b| = 1, \quad |D \cap C_c| = |D \cap C_d| \geq 2,$$

and q_{e_a} and q_{e_b} are *conjugate*. For a fixed lift y of $x = (C, c_1(D))$ after $r_1 p_1$, we further assume that $\text{Dom}(y) = \{e_a\}$.



FIGURE 7. The weighted tree in Example 5.6.5

On a chart \mathcal{V}_y of y , $\varphi^{r_1 p_1}$ can be written as

$$\tilde{\varepsilon}_1 \begin{bmatrix} 1 & 0 & 0 & 0 & *** \\ 0 & \tilde{\varepsilon}_1 \check{\zeta}_c & \lambda_{ab} \check{\zeta}_b & \tilde{\varepsilon}_1 \check{\zeta}_b^2 \check{\zeta}_d & *** \end{bmatrix},$$

where λ_{ab} is as in Corollary 2.7.3. The sequential blowups $r_1 p_2$ - $r_1 p_4$ do not affect \mathcal{V}_y . In $r_1 p_5$, locally \mathcal{V}_y is blown up along the locus $\{\check{\zeta}_c = \check{\zeta}_b = 0\}$. After $r_1 p_5$, fix a lift \tilde{y} of y . There are three possibilities for $\text{Dom}(\tilde{y})$: $\{e_c, e_b\}$, $\{e_c\}$, or $\{e_b\}$.

If $\text{Dom}(\tilde{y}) = \{e_c, e_b\}$, on a chart $\mathcal{V}_{\tilde{y}}$ containing \tilde{y} , $\tilde{\varphi}^{r_1 p_5}$ can be rewritten as

$$\tilde{\varepsilon}_1 \begin{bmatrix} 1 & 0 & 0 & *** \\ 0 & \tilde{\varepsilon}_1 & \tilde{\lambda}_{ab} & *** \end{bmatrix}.$$

The assumption on $\text{Dom}(\tilde{y})$ implies that r_2 does not affect $\mathcal{V}_{\tilde{y}}$. In $r_3 p_1$, locally $\mathcal{V}_{\tilde{y}}$ is blown up along the locus $\{\tilde{\varepsilon}_1 = \tilde{\lambda}_{ab} = 0\}$, thus $\tilde{\varphi}^{r_3 p_1}$ becomes diagonalizable on the pullback of $\mathcal{V}_{\tilde{y}}$.

If $\text{Dom}(\tilde{y}) = \{e_c\}$, on a chart $\mathcal{V}_{\tilde{y}}$ containing \tilde{y} , $\tilde{\varphi}^{r_1 p_5}$ can be rewritten as

$$\tilde{\varepsilon}_1 \begin{bmatrix} 1 & 0 & 0 & *** \\ 0 & \tilde{\varepsilon}_1 & \tilde{\lambda}_{ab} \check{\zeta}_b & *** \end{bmatrix}.$$

Locally $\mathcal{V}_{\tilde{y}}$ is blown up along the locus $\{\tilde{\varepsilon}_1 = \check{\zeta}_b = 0\}$ in r_2 . When r_2 terminates, let \tilde{y}' be a lift of \tilde{y} . If e_1 is dominant, obviously $\tilde{\varphi}^{r_2}$ is diagonalizable; otherwise, we observe that the blowup locus of $r_3 p_1$ near \tilde{y}' is given by $\{\tilde{\varepsilon}_1 = \tilde{\lambda}_{ab} = 0\}$, thus $\tilde{\varphi}^{r_3 p_1}$ becomes diagonalizable on the pullback of $\mathcal{V}_{\tilde{y}}$.

If $\text{Dom}(\tilde{y}) = \{e_b\}$, on a chart $\mathcal{V}_{\tilde{y}}$ containing \tilde{y} , $\tilde{\varphi}^{r_1 p_5}$ can be rewritten as

$$\tilde{\varepsilon}_1 \begin{bmatrix} 1 & 0 & 0 & 0 & *** \\ 0 & \tilde{\varepsilon}_1 \check{\zeta}_c & \tilde{\lambda}_{ab} & \tilde{\varepsilon}_1 \check{\zeta}_b \check{\zeta}_d & *** \end{bmatrix}.$$

It is again a direct check that r_2 does not affect a neighborhood of \tilde{y} , and $\mathcal{V}_{\tilde{y}}$ is blown up along the locus $\{\tilde{\varepsilon}_1 = \tilde{\lambda}_{ab} = 0\}$ in $r_3 p_1$, then *possibly* along the proper transform of $\{\check{\zeta}_c = \tilde{\lambda}_{ab} = \check{\zeta}_d = 0\}$ in $r_3 p_3$, and finally *possibly* along

the proper transform of $\{\check{\zeta}_c = \check{\lambda}_{ab} = \check{\zeta}_b = 0\}$ in $r_3\mathbb{P}^4$. In sum, $\tilde{\varphi}^{r_3\mathbb{P}^4}$ becomes diagonalizable on the pullback of $\mathcal{V}_{\tilde{y}}$.

6. LOCAL DEFINING EQUATIONS OF $\overline{M}_2(\mathbb{P}^n, d)$ AND ITS BLOWUPS

6.1. The local open immersions. As in the first paragraph of §2.3, we cover $\overline{M}_2(\mathbb{P}^n, d)$ by charts $\{U/\mathcal{V}\}$. Let $\mathcal{E}_{\mathcal{V}}$ be the total space of the vector bundle $\rho_*\mathcal{L}(\mathcal{A})^{\oplus n}$ and let

$$p: \mathcal{E}_{\mathcal{V}} \rightarrow \mathcal{V}$$

be the projection. We also set

$$\rho_*(\mathcal{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}}) = \rho_*(\mathcal{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}_1}) \oplus \rho_*(\mathcal{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}_2}).$$

Then the tautological restriction homomorphism

$$(6.1) \quad \text{rest}: \rho_*\mathcal{L}(\mathcal{A})^{\oplus n} \longrightarrow \rho_*(\mathcal{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}})$$

lifts to a section

$$(6.2) \quad \Phi \in \Gamma(\mathcal{E}_{\mathcal{V}}, p^*\rho_*(\mathcal{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}})).$$

By choosing suitable trivialization, the homomorphism “rest” of (6.1) can be identified with the homomorphism

$$(6.3) \quad (0 \oplus \varphi)^{\oplus n}: (\mathcal{O}_{\mathcal{V}} \oplus \rho_*\mathcal{M}(\mathcal{D}))^{\oplus n} \longrightarrow (\rho_*\mathcal{O}_{\mathcal{A}}(\mathcal{A}))^{\oplus n},$$

where φ is the structural homomorphism in (2.4). Then, the section Φ of (6.2) is the lift of $(0 \oplus \varphi)^{\oplus n}$ in (6.3).

Theorem 6.1.1. *Let $\mathcal{U} = \mathcal{V} \times_{\mathfrak{M}_2^{\text{div}}} U$. Then there is a canonical open immersion*

$$(6.4) \quad \mathcal{U} \rightarrow (\Phi = 0) \subset \mathcal{E}_{\mathcal{V}}.$$

Proof. This theorem is the natural extension of [5, Theorem 2.17]; the proofs are also parallel. \square

Corollary 6.1.2. *Let $(C, D) \in \mathfrak{M}_2^{\text{div}}$ be such that its image in $\mathfrak{M}_2^{\text{wt}}$ lies in \mathfrak{M}^{mn} and $D \cap F$ contains a pair of non-conjugate points. Then by shrinking \mathcal{V} if necessary, we have that $\mathcal{U} = \mathcal{V} \times_{\mathfrak{M}_{g,n}^{\text{div}}} U$ is smooth. In particular, the primary component of $\overline{M}_2(\mathbb{P}^n, d)$, $d > 2$, is generically smooth and of expected dimension.*

Proof. We can trivialize $\rho_*\mathcal{M}(\mathcal{D})$ and $\rho_*\mathcal{O}_{\mathcal{A}}(\mathcal{A})$ so that

$$\mathcal{E}_{\mathcal{V}} \cong \mathcal{V} \times (\mathbb{A}^{d+1})^n.$$

Let $w_j^i \in \mathbb{A}^1$ for all $1 \leq i \leq n$ and $0 \leq j \leq d$. Recall that the homomorphism “rest” of (6.1) can be identified with the homomorphism

$$(0 \oplus \varphi)^{\oplus n}: (\mathcal{O}_{\mathcal{V}} \oplus \rho_*\mathcal{M}(\mathcal{D}))^{\oplus n} \longrightarrow (\rho_*\mathcal{O}_{\mathcal{A}}(\mathcal{A}))^{\oplus n}.$$

By Proposition 2.5.1 and Corollary 2.7.3, $0 \oplus \varphi$ can be represented by

$$0 \oplus \varphi = [0, I_2, 0 \cdots, 0].$$

Hence, the equation $\Phi = 0$ is equivalent to the following equations

$$(6.5) \quad w_1^i = w_2^i = 0 \quad \text{for all } 1 \leq i \leq n,$$

which obviously implies that \mathcal{U} is isomorphic to an open subset of $\mathcal{V} \times (\mathbb{A}^{d-1})^n$. Note that $\dim(\mathcal{V} \times (\mathbb{A}^{d-1})^n) = 3+d+(d-1)n = d(n+1)-n+3$ which is the expected dimension of $\overline{M}_2(\mathbb{P}^n, d)$. The assertion then follows. \square

6.2. Induced open immersions of the blowup. We continue with the notation as in previous sections. Let $\tilde{\mathcal{V}} = \mathcal{V} \times_{\mathfrak{P}_2} \tilde{\mathfrak{P}}_2$. Then

$$\mathcal{E}_{\tilde{\mathcal{V}}} = \mathcal{E}_{\mathcal{V}} \times_{\mathcal{V}} \tilde{\mathcal{V}}$$

is the total space of the pull back bundle $\beta^* \rho_* \mathcal{L}(\mathcal{A})^{\oplus n}$, where $\beta : \tilde{\mathcal{V}} \rightarrow \mathcal{V}$ denotes the projection. The tautological restriction homomorphism “rest” of (6.1) pullbacks to give rise to

$$(6.6) \quad \widetilde{\text{rest}} : \beta^* \rho_* \mathcal{L}(\mathcal{A})^{\oplus n} \longrightarrow \beta^* \rho_*(\mathcal{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}}).$$

The section Φ of (6.2) pullbacks to give rise to

$$(6.7) \quad \tilde{\Phi} \in \Gamma(\mathcal{E}_{\tilde{\mathcal{V}}}, \tilde{p}^* \beta^* \rho_*(\mathcal{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}})),$$

where $\tilde{p} : \mathcal{E}_{\tilde{\mathcal{V}}} \rightarrow \tilde{\mathcal{V}}$ denotes the projection. Set $\tilde{\mathcal{U}} = \tilde{\mathcal{V}} \times_{\mathcal{V}} U$. The immersion $\mathcal{U} \rightarrow \mathcal{E}_{\mathcal{V}}$ of Theorem 6.1.1 then naturally induces an open immersion

$$(6.8) \quad \tilde{\mathcal{U}} \longrightarrow (\tilde{\Phi} = 0) \subset \mathcal{E}_{\tilde{\mathcal{V}}}.$$

Lemma 6.2.1. *With notation as above, $\tilde{\mathcal{U}}$ has normal crossing singularities, and its primary component is smooth.*

Proof. First, we trivialize $\beta^* \rho_* \mathcal{M}(\mathcal{D})$ and $\beta^* \rho_* \mathcal{O}_{\mathcal{A}}(\mathcal{A})$ so that

$$\mathcal{E}_{\tilde{\mathcal{V}}} \cong \tilde{\mathcal{V}} \times (\mathbb{A}^{d+1})^n.$$

Let $w_j^i \in \mathbb{A}^1$ for all $1 \leq i \leq n$ and $0 \leq j \leq d$.

By Proposition 5.1.2, $\tilde{\mathcal{V}}$ can be covered by $\{\tilde{\mathcal{V}}'\}$ so that on each $\tilde{\mathcal{V}}' \subset \tilde{\mathcal{V}}$, the homomorphism

$$\beta^* \varphi : \beta^* \rho_* \mathcal{M}(\mathcal{D}) \longrightarrow \beta^* \rho_* \mathcal{O}_{\mathcal{A}}(\mathcal{A})$$

can be diagonalized as

$$\begin{bmatrix} z_1 & 0 & 0 & \cdots & 0 \\ 0 & z_2 & 0 & \cdots & 0 \end{bmatrix} \quad \text{with } z_1 | z_2 \in \Gamma(\mathcal{O}_{\tilde{\mathcal{V}}'}).$$

Then the equation $\tilde{\Phi} = 0$ is equivalent to the following equations

$$(6.9) \quad z_1 w_1^i = 0, \quad z_2 w_2^i = 0 \quad \text{for all } 1 \leq i \leq n.$$

Thus, $\tilde{\mathcal{U}}$ has normal crossing singularities.

Since the primary component of $\tilde{\mathcal{U}} = \tilde{\mathcal{V}} \times_{\mathcal{V}} U$ consists of general points with non-vanishing $z_1 z_2$, we see that the primary component is defined by

$$(6.10) \quad w_1^i = w_2^i = 0 \quad \text{for all } 1 \leq i \leq n.$$

Thus, the primary component of $\tilde{\mathcal{U}}$ is smooth. \square

Theorem 6.2.2. *Assume $d > 2$. Then, $\widetilde{M}_2(\mathbb{P}^n, d)$ has normal crossing singularities and the primary component of $\widetilde{M}_2(\mathbb{P}^n, d)$ is smooth and of expected dimension.*

Proof. The first two statements follows from Lemma 6.2.1. It remains to check the last statement about the dimension. Note that $\dim \mathcal{E}_{\mathcal{Y}}^e = 3 + d + (d + 1)n$. Hence the dimension of the primary component of $\widetilde{\mathcal{U}}$ is

$$3 + d + (d + 1)n - 2n = d(n + 1) - n + 3$$

which is the virtual dimension of $\overline{M}_2(\mathbb{P}^n, d)$. \square

When $d = 2$, the interior of the primary component of $\overline{M}_2(\mathbb{P}^n, 2)$ consists of stable maps that are double covers of smooth rational curves, which is smooth and of dimension $2n + 4$. Since the virtual dimension of $\overline{M}_2(\mathbb{P}^n, 2)$ is $n + 5$, the primary component is of wrong dimension unless $n = 1$. The entire moduli can be treated by hand.

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