Research Statement

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I am interested in algebraic and symplectic geometry. In particular, my research involves the study of moduli spaces, birational geometry, enumerative geometry, and the Gromov-Witten (GW) theory. Two of my projects are described as follows.

1 Smooth compactifications of the spaces of curves in projective spaces

Moduli problems are of central importance in algebraic and symplectic geometry. Among them, the moduli spaces of stable maps play a prominent role. They are the main objects of interest in the theory of GW-invariants, which arise from Gromov's work [Gro] on pseudo-holomorphic curves and Witten's work [Wit] on σ-models in physics and are often related to integer counts of curves in the target space. Moduli spaces, however, are often singular; some of them, including the moduli of stable maps, can possess arbitrary singularities [Vak]. Desingularizing them is arguably one of the hardest problems in algebraic and symplectic geometry.

Suppose the target space is $\mathbb{P}^n$. The space $\mathfrak{M}_g^0$ of the holomorphic maps from *smooth* curves of genus g to $\mathbb{P}^n$ (with a fixed degree d) is of special interest because it corresponds to irreducible curves in $\mathbb{P}^n$. The space $\overline{\mathfrak{M}}_g$ of stable maps from nodal curves of genus g to $\mathbb{P}^n$ (also with the fixed degree d) is a compact space that contains $\mathfrak{M}_g^0$ as a subspace, thus is conventionally referred to as Gromov's *compactification* of $\mathfrak{M}_g^0$ (even though the closure of $\mathfrak{M}_g^0$ in $\overline{\mathfrak{M}}_g$ is usually not $\overline{\mathfrak{M}}_g$). The space $\overline{\mathfrak{M}}_g$ determines a virtual fundamental class that gives rise to the GW-invariants, but $\overline{\mathfrak{M}}_g$ can be arbitrarily singular for general g and d, thus it raises the question:

Question 1 (Q1)**.** For every genus g and every degree d, can we construct a new space (more precisely a Deligne-Mumford stack) $\widetilde{\mathfrak{M}}_g$ and a morphism $\widetilde{\mathfrak{M}}_g \longrightarrow \overline{\mathfrak{M}}_g$ such that all of the following properties are satisfied:

- $\widetilde{\mathfrak{M}}_g$ has smooth irreducible components and at worst normal crossing singularities,
- $\widetilde{\mathfrak{M}}_g \longrightarrow \overline{\mathfrak{M}}_g$ is proper,
- there exists a unique irreducible component of $\widetilde{\mathfrak{M}}_g$ that is birational to $\mathfrak{M}_g^0$, and
- with $(\pi, f)\colon \mathcal{C} \longrightarrow \overline{\mathfrak{M}}_g \times \mathbb{P}^n$ denoting the universal family of $\overline{\mathfrak{M}}_g$, the sheaves $\pi_* f^* \mathscr{O}_{\mathbb{P}^n}(k)$ with $k \geqslant 1$ become locally free on every irreducible component of $\widetilde{\mathfrak{M}}_g$?

An affirmative answer to Q1 would lead to an algebro-geometric definition of the genus g *reduced* GW-invariants, and would allow direct application of Atiyah-Bott localization formula to the computation of GW-invariants of complete intersections. Moreover, it is possible to apply the method of the construction of $\widetilde{\mathfrak{M}}_g$ other singular spaces.

For $g = 1$, Vakil and Zinger in [VZ] first provide an affirmative answer to Q1, followed by Hu and Li in [HL]. This leads to an effective computation of the GW-invariants of complete intersections and ultimately to Zinger's proof [Zin] of the prediction of [BCOV] for genus 1 GW-invariants of the quintic 3-fold. For $g = 2$, we obtain the following:

Theorem 2 ([HLN, HN2])**.** *The answer to Question 1 is affirmative for $g = 2$.*

In [HLN], Hu, Li, and I improve the technique of [HL] and establish a desingularization $\widetilde{\mathfrak{M}}_2$ of $\overline{\mathfrak{M}}_2$. We study the degeneracy loci of certain direct image complexes and construct blowups based on such loci to locally diagonalize the direct image complexes. Compared to its genus 1 counterpart, the blowup procedure in [HLN] is considerably more complicated, in part because of the various topological types of genus 2 curves and of the existence of Weierstrass and conjugate points.

For higher genus, the blowup construction of $\widetilde{\mathfrak{M}}_g$ may seem formidable. Hu and I thus develop the theory of stacks with twisted fields (STF theory) in [HN1, HN2] that interprets $\widetilde{\mathfrak{M}}_g$ as the Deligne-Mumford stack parameterizing the genus g stable maps with twisted fields for $g=1,2$. The STF theory is more systematic and appears promising for $\widetilde{\mathfrak{M}}_g$ with $g\geqslant 3$. In addition, the theory may possibly be applied to other singular spaces (see below).

Further research.

- To extend the STF theory to the higher genus case, i.e. to construct $\widetilde{\mathfrak{M}}_g$ and confirm Q1 for $g\geqslant 3$;
- to apply the STF theory to other singular spaces, e.g. the moduli space of stable sheaves, which could lead to applications such as the blowup formula for the virtual Euler numbers of [FG] conjectured in [GK];
- to define the genus 2 *reduced* GW-invariants for complete intersections based on the results in [HLN, HN2], find their relations to the usual GW-invariants, compute the genus 2 GW-invariants for quintic 3-folds (which should be consistent with the results of [GJR]);
- for $g=2$, when the target space is just a compact symplectic manifold (instead of $\mathbb{P}^n$), to construct a smaller space $\overline{\mathfrak{M}}_2^0\subset\overline{\mathfrak{M}}_2$ such that $\mathfrak{M}_2^0$ is dense in $\overline{\mathfrak{M}}_2^0$ (for sufficiently regular almost complex structure on the target) in [Niu], which should give rise to the corresponding genus 2 *reduced* GW-invariants.

2 Enumerative counts of positive-genus real curves

The real GW-invariants, first defined in genus 0 cases in [Wel1, Wel2], should count (pseudo-) holomorphic maps from symmetric Riemann surfaces commuting with the involutions on the domain and the target. In [GZ], the real GW-invariants are defined in all genera for many symplectic manifolds, including all odd-dimensional complex projective spaces and the quintic 3-fold.

In [NZ], Zinger and I establish a formula that transforms real GW-invariants of many symplectic 3-folds into signed integer counts of smooth real curves. Suppose (X,ϕ) is a compact real-oriented symplectic 3-fold in the sense of [GZ, Dfn 1.2] and $B\in H_2(X;\mathbb{Z})$ is a Fano class (i.e. $c_1(B)>0$). We show that for every $h\geqslant 0$, there exists an integer invariant $\mathrm{E}_{h,B}^{X,\phi}$ that provides a signed integer count of smooth genus h real curves in the class B, satisfying the following:

Theorem 3 ([NZ])**.** *For every $g\geqslant 0$, the real GW-invariant $\mathrm{GW}_{g,B}^{X,\phi}$ satisfies*

$$\mathrm{GW}_{g,B}^{X,\phi}=\sum_{\substack{0\leqslant h\leqslant g\\ g-h\in 2\mathbb{Z}}}\widetilde{C}_{h,B}\big(\tfrac{g-h}{2}\big)\mathrm{E}_{h,B}^{X,\phi}\,,$$

where the coefficients $\widetilde{C}_{h,B}\big(\frac{g-h}{2}\big)$ satisfy the generating function

$$\sum_{g=0}^{\infty}\widetilde{C}_{h,B}(g)t^{2g}=\left(\frac{\sinh(t/2)}{t/2}\right)^{h-1+c_1(B)/2}.$$

The formula above is the analogue of the Fano case of the Gopakumar-Vafa formula in the real GW-theory, and the integer invariants $\mathrm{E}_{h,B}^{X,\phi}$ are the analogues of the BPS states. Theorem 3 gives rise to many enumerative results. For example, in [NZ], we compute the real GW-invariants for $\mathbb{P}^3$ with conjugate pairs of point constraints up to $g \leqslant 5$ and $d \leqslant 8$ by equivariant localization and transform them into the signed integer counts. These integers provide non-trivial lower bounds for counts of real curves in $\mathbb{P}^3$, e.g. in $\mathbb{P}^3$ there are *at least* 40 genus 5 degree 8 real curves passing through 8 general pairs of conjugate points. The genus 0 numbers coincide with Welschinger's invariants [Wel2].

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