Genus two reduced quasi-map invariants for CY3 complete intersections (joint with J. Oh, M.-L. Li)

— Sanghyeon Lee, KIAS —

Today's plan

1. Moduli space of stable quasi-maps (= quasi-map space)
   (focused on genus 2, target is $\mathbb{P}^n$)
   + P-fields

2. Local charts, Local equations for genus 2 quasi-map spaces with p-fields and its desingularization

3. Normal cone computation and splitting of virtual cycle
   one of them gives "reduced" invariants

4. Computations to obtain standard versus reduced formula.

5. Further projects

1. Moduli space of stable quasi-maps

genus g degree d stable quasi-map to $X \subseteq \mathbb{P}^n$ c.i
is defined by:

- C : genus g nodal curve with marked points $p_1, \cdots, p_k \in C^{reg}$
- L : deg d line bdle on C,
- $u = (u_0, \cdots, u_n) \in H^0(C, L)^{\oplus n+1}$, base locus of u is finite, does not meets special points (nodes and marked points)

- The image of $C \xrightarrow{u} \mathrm{Tot}(L^{\oplus n+1})$ lies in $L \otimes_{\mathbb{C}} \underbrace{C(X)}_{\text{affine cone of } X \subseteq \mathbb{P}^{n+1}} \subseteq \mathrm{Tot}(L^{\oplus n+1})$

- $\omega_C(\Sigma p_i) \otimes L^{\varepsilon}$ is ample for any $\varepsilon > 0$

  $\Rightarrow$ any genus 0 component of C need at least 3 special points to be stable

  $\Rightarrow$ If $k = 0$ (no marked points), no rational tail is allowed

There exist a moduli space $\overline{Q}_{g,k}(X, d) =: \overline{Q}(X)$ parametrizing those quasi-maps, proper DM stack.

- It carries relative P.O.T $E_\bullet \to L_{\overline{Q}/\mathcal{B}}$ ($-1 \le \tau \le 0$), $L_{\overline{Q}/\mathcal{B}} = L_{\overline{Q}(X)/\mathcal{B}}$, where $\mathcal{B}$: Artin stack of pairs (C, L)

$\mathcal{C} \xrightarrow{\pi} \overline{Q}_{g,k}(X, d)$: canonical curve, $\begin{matrix}\mathcal{L}\\ \downarrow\\ \mathcal{C}\end{matrix}$: univ. line bdle

$u = (u_0, \cdots, u_n)$: univ. section of $\mathcal{L}^{\oplus n+1}$

$\Rightarrow u$ induce a map $\mathcal{C} \xrightarrow{u} \mathcal{L} \otimes_{\mathbb{C}} C(X)$

$\Rightarrow E_\cdot := \left(R^\cdot \pi_* \left(\mathcal{H}om(\mathbb{L}_u , \mathcal{O}_{\mathcal{C}})[1] \right) \right)^\vee$

If $X = \mathbb{P}^n$, $E_\cdot = \left((R^\cdot \pi_* \mathcal{L})^{\oplus n+1} \right)^\vee$

From this P.O.T, we construct virtual cycle

$$[\overline{Q}_{g,k}(X,d)]^{vir} \in A_{vdim}(\overline{Q}_{g,k}(X,d))$$

$$vdim = (\dim X - 3)(1-g) + k - d[line] \cdot \omega_X$$

quasi-map invariant $\langle \underbrace{\psi_1^{a_1}}_{\text{psi-class}} \delta_1, \cdots, \psi_k^{a_k} \underbrace{\delta_k}_{\delta_i \in A^*(X)} \rangle^X_{g,d}$ is defined by

$$\int_{[\overline{Q}_{g,k}(X,d)]^{vir}} \psi_1^{a_1} ev_1^* \delta_1 \cup \cdots \cup \psi_1^{a_k} ev_k^* \delta_k \in \mathbb{Q}$$

— P-fields

Next we consider stable quasi-maps with p-fields

It is given by $(C, L, u, p = (p_1, \cdots, p_m))$

(C, L, u) is a stable quasi-map to $\mathbb{P}^n$

$p_i \in H^0(C, \omega_C \otimes L^{-l_i})$ $(l_1, \cdots, l_m \in \mathbb{Z}_{>0}$, fixed$)$

There is a moduli space $\overline{Q}^p_{g,k}(\mathbb{P}^n, d)$ → DM stack

parametrizing stable maps with p-fields

It carries relative P.O.T $E_\bullet \to L_{\overline{Q}^p/B}$

For $\overline{Q}^p \xrightarrow{\pi} B$, $\begin{matrix} \mathcal{L} \\ \downarrow \\ \mathcal{C} \end{matrix}$, $E_\bullet := (R^\bullet \pi_* \mathcal{L})^{\oplus n+1} \oplus \bigoplus_{i=1}^{m} R^\bullet \pi_* (\omega_C \otimes \mathcal{L}^{-l_i})$

Assume that $X \subseteq \mathbb{P}^n$ c.i defined by $X = \{f_1 = \cdots = f_m = 0\}$
$\deg f_i = l_i$

- There is a cosection $h^1(E_\bullet^\vee) \xrightarrow{\sigma} \mathcal{O}$ induced from p-fields and eqns $f_1, \cdots, f_m$

- degeneracy locus of σ (where σ vanishes) is $\overline{Q}_{g,k}(X, d) \subset \overline{Q}_{g,k}(\mathbb{P}^n, d) \subset \overline{Q}^p_{g,k}(\mathbb{P}^n, d)$

$$\Rightarrow [\overline{Q}_{g,k}(X,d)]^{vir} = (-1)^{d(\sum_i l_i) + m(1-g)} [\overline{Q}^p_{g,k}(\mathbb{P}^n, d)]^{vir}_\sigma$$

↑ (Kim-Oh, Chang-Li)

localized virtual cycle localized by σ

$\Rightarrow$ we can use $\overline{Q}^p_{g,k}(\mathbb{P}^n, d)$ instead of $\overline{Q}_{g,k}(X,d)$ for computing $\langle \psi_1^{a_1} \delta_1, \cdots, \psi^{a_k} \delta_k \rangle^X_g$

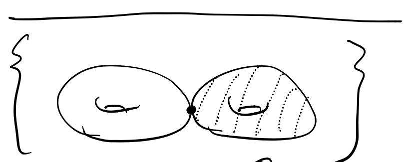
describe

- Some closed locus in $\overline{Q}(\mathbb{P}^n) = \overline{Q}_{2,0}(\mathbb{P}^n, d)$ (or $\overline{Q}^p(\mathbb{P}^n)$)

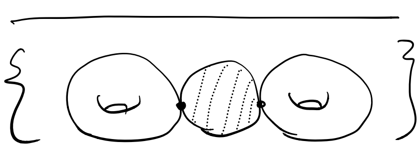
we describe some closed loci in $\overline{Q}(\mathbb{P}^n)$ (or $\overline{Q}^p(\mathbb{P}^n)$) will be used later.

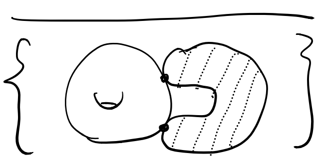
① Let $Q^\circ \subset \overline{Q}(\mathbb{P}^n)$ be an open subset of pairs (C, D) such that C is smooth

Let $\overline{Q}^\circ$ be the closure of Q°, called main component.

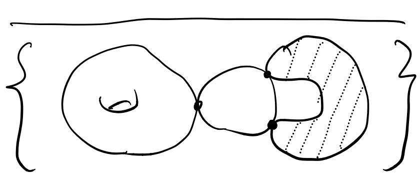
② $Z_1 =$ 

deg L is positive on shaded components

③ $Z_2 =$ 

④ $Z_3 =$ 

Note that $Z_2 \subset Z_1$, $Z_2 \cap Z_3 = \phi$

$Z_1 \cap Z_3 =$ 

Z_1, Z_2, Z_3 are all smooth substack of $\overline{Q}(\mathbb{P}^n)$

2. Local charts and local equations for quasi-map space (with p-fields) and its desingularization.

From now on we consider the case $g=2$, $k=0$, $d \geq 3$

$X \subseteq \mathbb{P}^n$ CY 3 complete intersection

defined by $X = \{ f_1 = \cdots = f_{\underbrace{m}_{=n-3}} = 0 \}$, $\deg f_i = l_i$

Consider an Artin stack $\mathfrak{M}^{div} = \mathfrak{M}^{div}_{2,k,d}$

parametrizing pairs (C, D)

genus g nodal curve with k marked points

deg d eff. div on C.

Consider $x = [(C, L, \overset{(s_0, \cdots, s_n)}{s}, p)] \in \overline{Q}^p(\mathbb{P}^n) = \overline{Q}^p_{2,0}(\mathbb{P}^n, d)$

Assume $x \in \underbrace{Z_1 \cup Z_2 \cup Z_3}$ "boundary"

Since since base locus of s is finite, we may assume $s_0 \neq 0$. Let $D = Z(s_0)$

Construct affine local charts of $\overline{Q}^p(\mathbb{P}^n, d)$ around x.

First, we chose a small affine nbd $V \xrightarrow{sm} \mathfrak{M}^{div}$ around (C, D). Let $\mathcal{D}_V$ be a canonical divisor in the univ. curve $\mathcal{C}_V$ of V.

Now consider a section space

$$\mathcal{F}_e(D_V) := \underbrace{\pi_* \mathcal{O}(D_V)^{\oplus n}}_{s=(s_1,\cdots,s_n)} \oplus \bigoplus_{i=1}^{m} \underbrace{\pi_*\left(\omega_{\mathcal{C}/V} \otimes \mathcal{O}(-l_i D_V)\right)}_{p_i}$$

Let $U \underset{\text{open}}{\subseteq} \mathrm{Tot}(\mathcal{F}_e)$ s.t. for $y \in V$, $\begin{pmatrix} s_1,\cdots,s_n \\ p_1,\cdots,p_m \end{pmatrix} \in \mathcal{F}_e|_{y\in V}$,

$$\left(C_y,\ \mathcal{O}(D_y),\ s=(\overset{s_{D_y}}{1}, s_1,\cdots,s_n),\ p=(p_1,\cdots,p_m)\right)$$

is a stable quasi-map

It induces a smooth morphism $U \longrightarrow \overline{Q}^{p}(\mathbb{P}^{n})$, which is a sm. nbd around x.

So we compute $\pi_* \mathcal{O}(D_V)$

(for p-fields, $\pi_*(\omega_{\mathcal{C}/V} \otimes \mathcal{O}(-l_i D_V))$, we can compute similarly using serre duality)

Since base points are finite, we may assume $s_0 \neq 0$.

Let $D = Z(s_0) = a_1 p_1 + \cdots + a_r p_r$ $(a_i \in \mathbb{Z}_{>0})$, $\sum a_i = d$

Since x is contained in the boundary, we observe

$H^0(C, \mathcal{O}(2p_i)) \geq 2$

we can find a meromorphic ftn g on C, which has double pole at p_i (If g has a simple pole, it induces a deg 1 map $C \xrightarrow[g]{\sim} \mathbb{P}^1$ $\times$)

$\Rightarrow g$ induces a 2:1 morphism $C \xrightarrow[2:1]{g} \mathbb{P}^1$

For unramified point $t \in \mathbb{P}^1$, $g^{-1}(t) = P_i' + P_i''$,

$2P_i \sim P_i' + P_i''$. We may assume that P_i', P_i'' distinct from other $P_j \neq P_i$

So that in finite steps, we can construct an isom

$$\mathcal{O}(\underbrace{a_1p_1 + \cdots + a_rp_r}_{=D}) \sim \mathcal{O}(\underbrace{P_1' + P_2' + \cdots + P_d'}_{\text{distinct}})$$

$\Rightarrow \exists\, s_0' \in H^0(C, \mathcal{O}(P_1' + \cdots + P_d'))$ correspond to

$s_0 \in H^0(C, \mathcal{O}(a_1p_1 + \cdots + a_rp_r))$

so that $Z(s_0') = \mathcal{O}(P_1' + \cdots + P_d')$

If V is small enough, we can extend this to define an isom

$$\begin{tikzcd} u: V \arrow[rr, "\sim"] \arrow[dr, "sm"'] & & W \arrow[dl, "sm"] \\ & \mathfrak{M}_{div} & \end{tikzcd}$$

s.t. for $t = (C_t, D_t) \in V$, $u(t) = (C_{u(t)}, D_{u(t)})$,

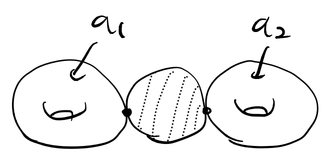
$C_{u(t)} = C_t$, $D_{u(t)} = \underbrace{q_1 + \cdots + q_d}_{\text{distinct}}$

$\Rightarrow$ Can. div $D_W \subset C_W$ is of the form

$D_W = D_1 + \cdots + D_d$, $D_i \cap D_j = \phi \quad \forall i, j$

We have $\mathcal{F}_e(\mathcal{D}_V) \cong u^* \mathcal{F}_e(\mathcal{D}_W)$.
So, for simple notation, we let $W = V$

Use methods in Hu-Li-Niu's paper
(for local chart & equations for $g=2$ stable map & its desing)

We choose additional sections $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B} : V \to \mathcal{C}_V$,
don't intersect to each others.
If $(C, D) \in V$ is of the form 

then img of $\mathcal{A}_1, \mathcal{A}_2$, a_1, a_2 lies in each genus 1 comps, separated

Then we have

$$R^\bullet \pi_* \mathcal{O}(D_1 + \cdots + D_d) = R^\bullet \pi_* \mathcal{O}(D_1 + \cdots + D_d - \mathcal{B}) \oplus \mathcal{O}_V,$$

$$R^\bullet \pi_* \mathcal{O}(D_1 + \cdots + D_d - \mathcal{B}) \overset{\text{qis}}{\sim} \Big[\underbrace{\pi_* \mathcal{O}(D_1 + \cdots + D_d + \mathcal{A}_1 + \mathcal{A}_2 - \mathcal{B})}_{\text{loc. free},\ rk = d} \xrightarrow[ev_{A_1} \oplus ev_{A_2}]{} \mathcal{O}_V^{\oplus 2}$$

from some cohomology computations.

Also we have decomp

$$\pi_* \mathcal{O}(D_1 + \cdots + D_d + \mathcal{A}_1 + \mathcal{A}_2 - \mathcal{B}) \cong \bigoplus_{i=1}^{d} \underbrace{\pi_* \big(\mathcal{O}(D_i + \mathcal{A}_1 + \mathcal{A}_2 - \mathcal{B})\big)}_{\cong\, \mathcal{O}_V}$$

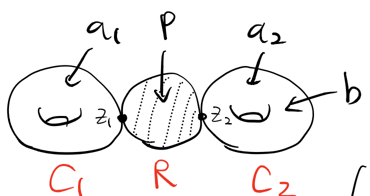
So that we can express $(ev_{A_1} \oplus ev_{A_2})$ as a $2 \times d$ matrix

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1d} \\ c_{21} & c_{22} & \cdots & c_{2d} \end{pmatrix},$$

Note that each column express

$$\begin{pmatrix} c_{1i} \\ c_{2i} \end{pmatrix} : \underset{\underset{\mathcal{O}_V}{\cong}}{\pi_*(\mathcal{O}(D_i + A_1 + A_2 - B))} \xrightarrow[\oplus ev A_2]{ev A_1} \mathcal{O}_V^{\oplus 2}$$

Ex)

$C =$ 

$\Rightarrow H^0(\mathcal{O}_C(p + a_2 - b)) \ni s$ $\to$ $\ker(ev_{a_1})$

$\cong \mathbb{C}$

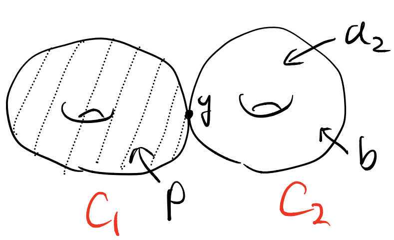
$\because$ $s|_{C_2} \in H^0(C_2, \cancel{\mathcal{O}_{C_2}(a_2 - b)}) \to 0$

$\underset{\text{1-dim'l choice}}{s|_R} \in H^0(R, \mathcal{O}_R(p - y)) \cong \mathcal{O}_R$

$s|_R \in H^0(C_1, \mathcal{O}_{C_1})$ determined by a value at x

$\Rightarrow ev_{a_1} = 0$

If the node z_1 is smoothed

$\Rightarrow$ 

$\Rightarrow H^0(\mathcal{O}_C(p + a_2 - b)) \ni s$

$= 0$

$\because$ $s|_{C_2} \in H^0(C_2, \cancel{\mathcal{O}_{C_2}(a_2 - b)}) \to 0$

$s|_{C_1} \in H^0(C_1, \cancel{\mathcal{O}_{C_1}(p - y)}) \to 0$

$\Rightarrow ev_{a_1} = c \in \mathbb{C}^*$

$\Rightarrow$ Let τ_1 be the node-smoothing parameter on V

($\{\tau_1 = 0\} \subset V$ is the locus where node z_1 is not smoothed)

$\Rightarrow Z_1(ev_{A_1}) = \{\tau_1 = 0\}$, $ev_{A_1} = c \cdot \tau_1^r$ $(c \in \Gamma(\mathcal{O}_V^*), r \in \mathbb{Z}_{>0})$

we can check $r=1$! (see Hu–Li, genus 1 desing for detail)

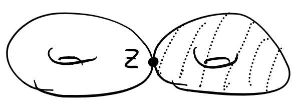
Similarly we can check $ev_{A_2} = c'\cdot\tau_2$ $(c' \in \Gamma(\mathcal{O}_V^*))$

Since $d \geq 3$, we may choose P_1, P_2 are not conjugate

the $2\times d$ matrix becomes

$\begin{pmatrix} \tau_1 & 0 & & 0 & \\ 0 & \tau_2 & & & \end{pmatrix}$ via some basis change of $\mathcal{O}_V^{\oplus d}$ and $\mathcal{O}_V^{\oplus 2}$.

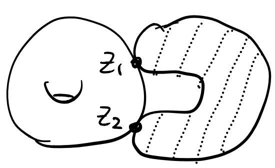
Similarly, if $x = (C, D) \in V$ is of the form



$\Rightarrow$ $2\times d$ matrix is of the form (via suitable basis change)

$\begin{pmatrix} 1 & 0 & & 0 & \\ 0 & \tau_2 & & & \end{pmatrix}$

If $x = (C, D) \in V$ is of the form



We can check $2\times d$ matrix is of the form :

$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_1\tau_1 + b_1\tau_2 & a_2\tau_1 + b_2\tau_2 & \cdots & a_{d-1}\tau_1 + b_{d-1}\tau_2 \end{pmatrix}$

$\{a_1\tau_1 + b_1\tau_2 = 0\}$ is a locus where P_1, P_2 conj

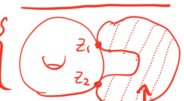
We blow-up this locus $\left(\overset{\text{loc}}{=} \{\tau_1 = \tau_2 = 0\} \subset V \right)$

in the level of $\mathcal{M}^w$

$\nwarrow$ Artin stack of weighted curves

$$\Rightarrow \widetilde{\mathcal{M}}^w := Bl_{\mathcal{M}_3} \mathcal{M}^w, \qquad \widetilde{V} := V \times_{\mathcal{M}^w} \widetilde{\mathcal{M}}^w.$$

$$\widetilde{U} := U \times_{\mathcal{M}^w} \widetilde{\mathcal{M}}^w.$$

$\widetilde{\mathcal{M}_3} = \{$  $\}$

$\uparrow$ positive wt

Let ε be a local parameter of $\widetilde{V}$ correspond to the exc divisor $E \subset \widetilde{V}$

$\Rightarrow$ pull-back of $ev_{A_1} \oplus ev_{A_2}$, $\widetilde{ev}_{A_1} \oplus \widetilde{ev}_{A_2}$

is of the form:

$$\begin{pmatrix} 1 & 0 & \\ 0 & \varepsilon & 0 \end{pmatrix}$$

Similarly, $R\pi_* \omega_{\mathcal{C}_{\widetilde{V}}/\widetilde{V}} \otimes \mathcal{O}(-l_i D_{\widetilde{V}})$

$$\underset{qis}{\sim} \left[\mathcal{O}_{\widetilde{V}}^{\oplus 2} \longrightarrow \mathcal{O}_{\widetilde{V}}^{\oplus l_i d_i} \right]$$

given by transpose of $2 \times l_i d$ matrices with same form as above

Using these results, we can compute $\mathrm{Tot}(\pi_* \mathcal{F}_e(D_{\widetilde{V}}))$ as subspace of $\mathrm{Tot}\left(\mathcal{O}_{\widetilde{V}}^{\oplus dn} \oplus \mathcal{O}_{\widetilde{V}}^{2m} \right)$
$\cong \widetilde{V} \times \mathbb{C}^{dn+2m}$

and find equations $\left(\mathcal{O}_{\widetilde{V}}^{\oplus dn} \oplus \mathcal{O}_{\widetilde{V}}^{2m} \text{ is an obs. bdle for } \overline{Q}^p/\mathcal{M}^w \right)$

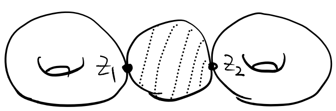
Let $\begin{pmatrix} x_{11}, & x_{12}, & \cdots, & x_{1d} \\ x_{21}, & x_{22}, & \cdots, & x_{2d} \\ & \vdots & & \\ x_{n1}, & x_{n2}, & \cdots, & x_{nd} \end{pmatrix}$ be coordinates of $\mathbb{C}^{dn}$

$\begin{pmatrix} x_{n+1\,1}, & x_{n+1\,2} \\ \vdots & \\ x_{n+m\,1}, & x_{n+m\,2} \end{pmatrix}$ be coordinates of $\mathbb{C}^{2m}$

$\Rightarrow$ rewrite $x_{11} \sim x_{n+m\,1}$ by $x_1 \sim x_{n+m}$

$x_{12} \sim x_{n+m\,2}$ by $y_1 \sim y_{n+m}$

If V is a nbd around $x = (C, D)$ of type



, then $\mathrm{Tot}(\mathcal{F}(\mathcal{D}_{\tilde V})) \subset \tilde V \times \mathbb{C}^{nd+2m}$

is defind by equations:

$x_1 \tau_1 = \cdots = x_{n+m} \tau_1 = 0, \quad y_1 \tau_2 = \cdots = y_{n+m} \tau_2 = 0$

Then the normal cone $C_{\tilde U / \tilde V \times \mathbb{C}^{nd+2m}} =$

$C_{\mathrm{Tot}(\mathcal{F}(\mathcal{D}_{\tilde V})) / \tilde V \times \mathbb{C}^{nd+2m}} \Big|_{\tilde U}$

can be computed by the following

Let $\mathrm{Spec}(R) = \tilde V \times \mathbb{C}^{2d+2m}$ consider some affine chart of the blow-up

$\Rightarrow$ def. ideal : $(x_1\tau_1, \cdots, x_{n+m}\tau_1, y_1\tau_2, \cdots, y_{n+m}\tau_2) \subset R[x_1, \cdots, x_{n+m}, y_1, \cdots, y_{n+m}]$.

Cone parameter: $\alpha_1, \cdots, \alpha_{n+m}, \beta_1, \cdots, \beta_m$

3. Normal cone computation and splitting of virtual cycle

$$\Rightarrow \text{Normal cone}: \underbrace{R[x_1,\cdots,x_{n+m}, y_1,\cdots,y_{n+m}]}_{\bar{R}}[\alpha_1,\cdots,\alpha_{n+m},\beta_1,\cdots\beta_{n+m}]$$

$$\hat{R} \Longrightarrow \Big/ \left(\begin{matrix} x_i\alpha_j - x_j\alpha_i, & y_i\beta_j - y_j\beta_i, \\ y_i\tau_2\alpha_i - \alpha_i\tau_1\beta_i & \end{matrix} \right)$$

$\uparrow$ comes from $\alpha_i \leftrightarrow \tau_1 x_i$, $\beta_i \leftrightarrow \tau_2 y_i$

Normal cone has 3 irreducible components:

① Over $\{x_1 = \cdots = x_{n+m} = y_1 = \cdots = y_{n+m} = 0\} \subset \mathrm{Tot}(\mathcal{F}(\mathcal{D}_V))$

$$\Rightarrow \hat{R} \otimes_{\bar{R}} \bar{R}/(x_1,\cdots,x_{n+m}, y_1,\cdots,y_{n+m})$$

$$\cong \underbrace{\left(\bar{R}/(x_1,\cdots,x_{n+m}, y_1,\cdots,y_{n+m})\right)}_{\cong R}[\alpha_1,\cdots,\alpha_{n+m},\beta_1,\cdots,\beta_{n+m}].$$

Let $\operatorname{Spec} \bar{R}/(x_1,\cdots,x_{n+m}, y_1,\cdots,y_{n+m}) \cap \tilde{U} := \tilde{U}^{red}$

$$\Rightarrow \text{Over } \tilde{U}^{red}, \ N_{U/V\times\mathbb{C}^{nd+2m}}|_{\tilde{U}^{red}} \cong \underline{\tilde{U}^{red}} \times \mathbb{C}^{2(n+m)}$$

$\hookrightarrow$ glued to $\tilde{Q}^{p,red}(\mathbb{P}^n) \subset \tilde{Q}^p(\mathbb{P}^n) = \bar{Q}^p(\mathbb{P}^n) \times_{\mathfrak{m}^w} \tilde{\mathfrak{m}}^w$

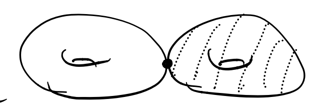
② Over $\{x_1 = \cdots = x_{n+m} = 0, \ \tau_2 = 0\}$
(and $\{y_1 = \cdots = y_{n+m} = 0, \ \tau_1 = 0\}$)

$$\Rightarrow \hat{R} \otimes_{\bar{R}} \bar{R}/(x_1,\cdots,x_{n+m}, \tau_2)$$

$$\cong R/(\tau_2)[y_1,\cdots,y_{n+m}][\alpha_1,\cdots,\alpha_{n+m}][\beta_1,\cdots,\beta_{n+m}] \Big/ (y_i\beta_j - y_j\beta_i)$$

$\Rightarrow$ fiber over $\operatorname{Spec} R/(\tau_2)[\alpha_1,\cdots,\alpha_{n+m}]$ are affine cone of the blow-up $Bl_0\mathbb{A}^{n+m}$ $\Rightarrow$ irreducible

Let $\left(\mathrm{Spec}\,\widetilde{R}/(x_1,\dots,x_{n+m},\tau_2) \cup \mathrm{Spec}\,\widetilde{R}/(y_1,\dots,y_{n+m},\tau_1)\right)$
$\cap\, U = U_1 \Rightarrow C_{U/V\times\mathbb{C}^{nd+2m}}\big|_{U_1}$ glued to an
irred. comp of the cone over $\widetilde{Q}^{p,1}(\mathbb{P}^n)$

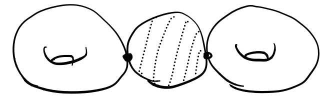
U_1 glued to an irreducible comp $\widetilde{Q}^{p,1}(\mathbb{P}^n) \subset \widetilde{Q}^{p}(\mathbb{P}^n)$
Correspond to $\Big\{$  $\Big\}$ $=: \widetilde{Q}^{p,1}(\mathbb{P}^n)^{red}$

reduced genus 1 quasi-map
defined in Mu-Lin Li's 2019 paper
similarly defined as $\widetilde{Q}^{p,red}(\mathbb{P}^n)$

③ Over $\{\tau_1=\tau_2=0\}$

$$\widehat{R}\otimes_{\bar{R}} \bar{R}/(\tau_1,\tau_2) \cong \left(R/(\tau_1,\tau_2)\frac{[x_1,\dots,x_{n+m}][y_1,\dots,y_{n+m}]}{[\alpha_1,\dots,\alpha_{n+m}][\beta_1,\dots,\beta_{n+m}]}\right) \Big/ \left(x_i\alpha_j - x_j\alpha_i,\ y_i\beta_j - y_j\beta_i\right)$$

$\Rightarrow$ fiber over $\mathrm{Spec}\,R/(\tau_1,\tau_2)$ is a product of
affine cones of the blow-up $Bl_0\mathbb{A}^{n+m}$
$\Rightarrow$ irreducible

Let $U_2 := \mathrm{Spec}\,R/(\tau_1,\tau_2)[x_1,\dots,x_{n+m},y_1,\dots,y_{n+m}] \cap U$
$\Rightarrow C_{U/V\times\mathbb{C}^{nd+2m}}\big|_{U_2}$ glued to an irred comp. over
$\widetilde{Q}^{p,\mu_2}(\mathbb{P}^n,d)$ corresp to $\Big\{$  $\Big\}$

glueing of U_2

Similarly, we can do similar cone computations for other local charts of $\mathfrak{M}^{div}$.

By considering some base changes via $\underset{\{(C,L)\}}{\mathfrak{B}} \to \mathfrak{M}^{w} \leftarrow \mathfrak{M}^{div}$

we obtain local computation of relative intrinsic normal cone $\mathfrak{C}_{\widetilde{Q}^p(\mathbb{P}^n)/\mathfrak{B}}$

$$\Rightarrow \mathfrak{C}_{\widetilde{Q}^p(\mathbb{P}^n)/\mathfrak{B}} = \underbrace{\mathfrak{C}^{red} \cup \mathfrak{C}^1 \cup \mathfrak{C}^2 \cup \mathfrak{C}^3}_{\text{irred comps}}$$

$\mathfrak{C}^1, \mathfrak{C}^2, \mathfrak{C}^3$ lies over boundaries $\widetilde{Z}_1, Z_2, \widetilde{Z}_3 \subseteq \widetilde{Q}^p(\mathbb{P}^n)$

exc div

proper transf of Z_1

does not meet blow-up locus

Take localized Gysin map, we have

$$b_* 0^!_{h^1/h^0(E_\bullet^\vee),6}\left[\mathfrak{C}_{\widetilde{Q}^p(\mathbb{P}^n)/\mathfrak{B}}\right] = \left[\overline{Q}^p(\mathbb{P}^n)\right]^{vir}_6$$

$b: \widetilde{Q}^p(\mathbb{P}^n) \to \overline{Q}^p(\mathbb{P}^n)$

from Costello's pushforward formula

$$= (-1)^{d(\sum_i l_i)+m(1-g)}\left[\overline{Q}^p(\mathbb{P}^n)\right]^{vir}_6$$

$$= \underbrace{\left[\overline{Q}^p(\mathbb{P}^n)^{red}\right]^{vir}_6}_{\text{gives reduced quasi-map inv}} + \left[\overline{Q}^p(\mathbb{P}^n)^1\right]^{vir}_6 + \left[\overline{Q}^p(\mathbb{P}^n)^2\right]^{vir}_6 + \left[\overline{Q}^p(\mathbb{P}^n)^1\right]^{vir}_6$$

boundary cycles

Gives splitting of virtual cycle.

All cycles supported in $\overline{Q}(X)$

$\subset$ CY3

We can check

smooth

$$[\overline{Q}^p(\mathbb{P}^4)^{red}]^{vir}_6 = b_* 0^!_{N,6}\,[\widetilde{Q}(\mathbb{P}^4)^{red}]$$

$$N = \bigoplus_{i=1}^{m} \pi_* \mathcal{L}^{l_i}\Big|_{\widetilde{Q}(\mathbb{P}^4)^{red}} \rightarrow \text{loc. free from } R^{\bullet}\pi_*\mathcal{L} \text{ computation}$$

Cosection $6 : Ob(E^{\bullet\vee}) \longrightarrow \mathcal{O}$

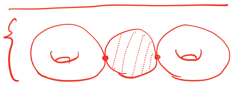
induces cosection $6 : N \longrightarrow \mathcal{O}$

$\Rightarrow \exists$ quantum Lefschetz property for genus 2 reduced quasi-map inv.

4. Computations to obtain standard versus reduced formula.

$\langle\ \rangle^X_{2,d} - \langle\ \rangle^{X,red}_{2,d}$ comes from degree of boundary cycles.

Briefly explain how to compute $[\overline{Q}^p(\mathbb{P}^4)^2]^{vir}_6$

supp. in $\{$  $\}$

Recall the locall computation above,
$\mathcal{C}_2$ locally given by

$$C = \operatorname{Spec}\left(\left(R/(\tau_1,\tau_2)[x_1,\cdots,x_{n+m}][y_1,\cdots,y_{n+m}] \right) [\alpha_1,\cdots,\alpha_{n+m}][\beta_1,\cdots,\beta_{n+m}] \Big/ \left(\begin{matrix} x_i\alpha_j - x_j\alpha_i, \\ y_i\beta_j - y_j\beta_i \end{matrix} \right) \right)$$

over $\underbrace{R/(\tau_1,\tau_2)[x_1,\cdots,x_{n+m},y_1,\cdots,y_{n+m}]}_{=:\overline{R}}$ $\left(\overset{loc}{\cong} \overline{Q}^p(\mathbb{P}^n)^2 \right)$

It can be considered by the following consider

$$\psi : \operatorname{Spec}\left(R/(\tau_1,\tau_2)[x_1,\cdots,x_{n+m},y_1,\cdots,y_{n+m}]\right) \times \mathbb{C}^2 \longrightarrow \operatorname{Spec}(\quad '' \quad) \times \mathbb{C}^{(n+m)} \times \mathbb{C}^{(n+m)}$$

$$(\underbrace{x_1,\cdots,x_{n+m}}_{\psi_1}, \underbrace{y_1,\cdots,y_{n+m}}_{\psi_2})$$

$\Rightarrow \overline{\operatorname{im}\psi} = C$

Let $\operatorname{Spec}(\widehat{R}) := Bl_{\widetilde{B}_2}\left(Bl_{B_1}\operatorname{Spec}(\overline{R})\right)$, $B_1 = \{x_1 = \cdots = x_{n+m} = 0\}$, $B_2 = \{y_1 = \cdots = y_{n+m} = 0\}$ (degen. locus of ψ_1, ψ_2)

let E_1, E_2 be corresponding exc divisors

ψ induces an injection $\operatorname{Spec}\widehat{R} \times \mathbb{C}^2 \overset{\widehat{\psi}}{\hookrightarrow} \operatorname{Spec}\widehat{R} \times \mathbb{C}^{2(n+m)}$
(twisting E_1, E_2)

For the projection $p : \widehat{R} \longrightarrow \overline{R}$, we have

$$p_*(\operatorname{im}\widehat{\varphi}) = C$$

We observe φ glued to the natural morphism

coarse moduli of int. normal cone

$$\pi^* N_{\mathcal{M}^2/\widetilde{\mathcal{M}}^w} \xrightarrow{\varphi} C_{\widetilde{Q}^p(\mathbb{P}^n)/\widetilde{\mathcal{M}}^w}\Big|_{\widetilde{Q}^p(\mathbb{P}^n)^2} \hookrightarrow \text{obs. sheaf}$$

$$\left(\pi : \widetilde{Q}^p(\mathbb{P}^n) \longrightarrow \widetilde{\mathcal{M}}^w\right) \qquad = H^1\left(E_{\widetilde{Q}^p(\mathbb{P}^n)/\widetilde{\mathcal{M}}^{w\vee}}\right)\Big|_{\widetilde{Q}^p(\mathbb{P}^n)^2}$$

loc. free from $R^\bullet\pi_*\mathcal{O}(D_{\widetilde{\nabla}})$ computation

Because it is locally induced from the ideal inclusion

$$(\tau_1 x_1, \cdots, \tau_1 x_{n+m}, \tau_2 y_1, \cdots, \tau_2 y_{n+m}) \subset (\tau_1, \tau_2).$$

cone $\qquad$ normal bdle

$\Rightarrow C_{\widetilde{Q}^p(\mathbb{P}^n)/\widetilde{\mathcal{M}}^w}\Big|_{\widetilde{Q}^p(\mathbb{P}^n)^2} = \overline{\operatorname{im}\varphi}$ from local comp.

of $\widetilde{Z}_2$

For $p : \widehat{Q}^p(\mathbb{P}^n) = Bl_Z\, \widetilde{Q}^p(\mathbb{P}^n)$ $\left(\begin{array}{l} Z \text{ is the intersection with} \\ \text{the locus } \widetilde{Q}^1(\mathbb{P}^n)^{red} \end{array}\right)$

and exc. divisor E, $\qquad = \overline{\{}$  $\overline{\}}$

reduced genus 1 quasi-map

$\Rightarrow \varphi$ induces an injection

$$\widehat{\varphi} : p^*\pi^* N_{\mathcal{M}^2/\widetilde{\mathcal{M}}^w}(E) \longrightarrow p^*(\text{Obs.bdle})$$

s.t. $p_* \operatorname{Im}\widehat{\varphi} = C_{\widetilde{Q}^p(\mathbb{P}^n)/\widetilde{\mathcal{M}}^w}\Big|_{\widetilde{Q}^p(\mathbb{P}^n)^2}$

Then

$$0^{!}_{\mathrm{Obs}} \, c_{\widetilde{Q}^p(\mathbb{P}^n)/\widetilde{\mathfrak{M}}^w}\Big|_{\widetilde{Q}^p(\mathbb{P}^n)^2}$$

← Not localized

$$= p_*\Big(C_{top}\big(p^*\mathrm{obs}/p^*\pi^* N_{\mathfrak{M}^2/\widetilde{\mathfrak{M}}^w}(E)\big)\Big)$$

$$= p_*\left(\frac{c(p^*\mathrm{obs})}{c(p^*\pi^* N_{\mathfrak{M}^2/\widetilde{\mathfrak{M}}^w})}\right)_{\mathrm{rk\,obs}-2}$$

use some intersection theory argument to erase E twisting term

$$= \left(\frac{c(\mathrm{obs})}{c(\pi^* N_{\mathfrak{M}^2/\widetilde{\mathfrak{M}}^w})}\right)_{\mathrm{rk\ obs}-2}$$

↳ expressed by univ. tan. bundles at nodes

By some further cohomology comp (similar to Zinger's) we obtain closed formula for $\langle\ \rangle^X_{2,d} - \langle\ \rangle^X_{2,d}$

For exact argument, we need "localized" Gysin map, it use some modified method (see -L-Oh 20' for detail)

We can compute other boundary cycles similarly.

As a result, we obtain the closed formula:

$$\langle\ \rangle^X_{2,d} - \langle\ \rangle^{X,red}_{2,d}$$

$$= \frac{-1}{192}\left(\int_{[\overline{Q}_{0,2}(X,d)]^{vir}} \psi_1\cdot\psi_2 + 2\big(1 - ev_2^* c_2(TX)\big)\right)$$

$$- \frac{1}{24}\int_{[\overline{Q}_{1,1}(X,d)^{red}]}\psi_1 .$$

5. Further projects

- Using this formula + Wall crossing btw quasi-map inv and GW-inv

  We are going to give another way for computation of genus 2 GW-invs of quintic 3-folds and another CY3 c.i $\subset \mathbb{P}^n$

- We believe that all these methods directly applied to the case $X =$ CY3 c.i $\subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.
  Also going to compute genus 2 GW-inv in this case

- We assumed $d \geq 3$ to avoid "conjugacy point" issue. If $D = P_1 + P_2$ then dimension of sections $H^0(C, D)$ differs by whether P_1, P_2 is conjugate or not
  $\Rightarrow$ "conjugacy parameter" on M^{div} affects to $(2 \times d)$-matrix we considered above

  But, we want to remove this assumption for full-generality and for genus 2 GW computation in all degree. So we will also try to remove $d \geq 3$ assumption

—End—