A smooth compactification of $M_{2,n}(\mathbb{P}^r, d)$ via Goreinstein singularities & log Geometry

joint with L. Battistella

0. Motivations

The spaces

$$\boxed{\overline{M}_{g,n}(\mathbb{P}^r, d) := \left\{ f:(C,p_1,\dots,p_n)\to\mathbb{P}^r \;\middle|\; \begin{array}{l} \cdot\ C \text{ is at worst nodal,} \\ \quad g(C)=g \\ \cdot\ \deg f = d \\ \cdot\ |\mathrm{Aut}(f)|<\infty \end{array} \right\}\Big/\simeq}$$

are fundamental in enumerative geometry.

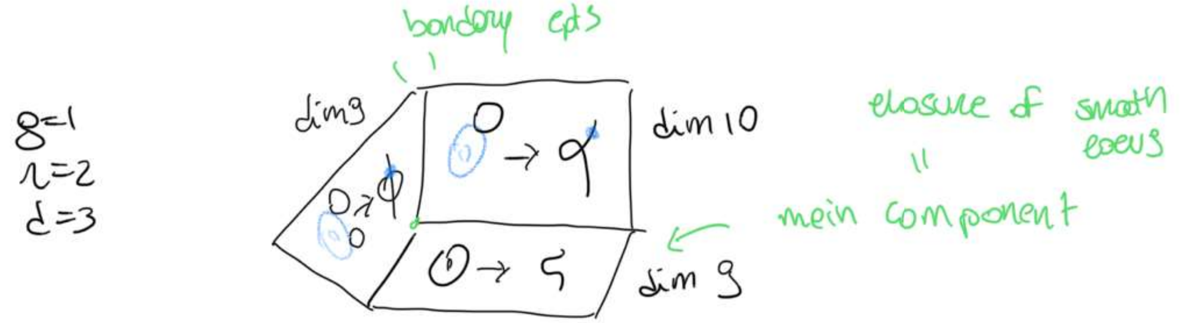
For example for any $Y \subseteq \mathbb{P}^r$ smooth

$$\overline{M}_{g,n}(Y,d) \subseteq \overline{M}_{g,n}(\mathbb{P}^r,d)$$

and we would like to compute $GW_g(Y)$ applying intersection theoretic methods on $\overline{M}_{g,n}(\mathbb{P}^r,d)$

For $\boxed{g=0}$, $\overline{M}_{0,n}(\mathbb{P}^r,d)$ is smooth and indeed we can computing $GW_0(Y)$ integrating suitable chern classes on $[\overline{M}_{0,n}(\mathbb{P}^r,d)]$

For $\boxed{g \geq 1}$ $\overline{M}_{g,n}(\mathbb{P}^r, d)$ has many irreducible components of different dimensions intersecting



$\Rightarrow$ • difficult to compute $GW(\mathbb{P}^2)$ and $GW(Y)$

• the invariants $GW_g(Y)$ also count lower genus curves due to boundary contributions

Motivating question:

Can we find a smooth compactification

$$M_{g,n}(\mathbb{P}^r, d) \subseteq VZ_{g,n}(\mathbb{P}^n, d) \ ?$$

1. Previous work in this direction

For $g=1$ Vakil-Zinger, Hu-Li 2009/2010

$g=2$ Hu-Li-Niu 2018

Describe an explicit sequence of blow-ups (based on the knowled of local equations) which resolve

$$\begin{tikzcd} \overline{M}_1^{main}(\mathbb{P}^n,d) & \text{and} & \overline{M}_2(\mathbb{P}^n,d) \\ \widetilde{M}_1^{main}(\mathbb{P}^n,d) \arrow[u, "bir."] & & \widetilde{M}_2^{main}(\mathbb{P}^n,d) \arrow[u, "bir"] \end{tikzcd}$$

smooth

In $g=1$ 2017 Ranganathan-Santos-Parker-Wise present a different way to resolve

$\overline{M}_1^{main}(\mathbb{P}^2,d)$ RSPW

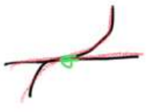
using Smyth genus 1 Gorenstein singularities & Log Geometry

2. Genus 2 Gorenstein singularities

By a Gorenstein curve C we just mean a curve with a canonical line bdle ω_C

Genus 2 isolated singularities (Battistella)

Type I	Type II

$m=1$		y^2-x^5		
$m=2$		$x(y^2-x^3)$		$y(y-x^3)$
$m=3$		$\subseteq\mathbb{A}^3$ $(z(x-y),$ $z^3-xy)$		$xy(y-x^2)$
$m\geq4$		$\subseteq\mathbb{A}^m$ $\langle x_i(x_j-x_k),$ $x_m(x_i-x_j),$ $x_m^3-x_1x_2\rangle$ $1\leq i\leq j\leq k\leq m-1$		$\subseteq\mathbb{A}^{m-1}$ $\langle x_i(x_j-x_k),$ $x_3(x_1^2-x_2)\rangle$ $1<i<j<k\leq m-1$

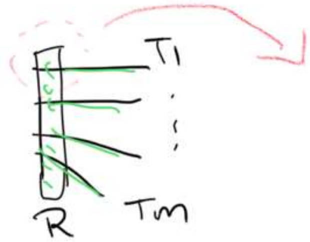
• = special branch

special branches

• = twin branches

Genus 2 non isolated Gorenstein singularities

(just one example)

$\overline{C}=$ 

local equation around this point

$$\boxed{\frac{\mathbb{C}[x,y]}{x^2y}}$$

R is a non reduced curve with $R^{red}\simeq\mathbb{P}^1$

and $\left(\mathcal{I}_{\mathbb{P}^1\hookrightarrow R}\right)^2=0$ in R

we call R a ribbon

and $\overline{C}$ a tailed Ribbon

if $\mathcal{O}_R$ fits in $\boxed{0\to\mathcal{O}_{\mathbb{P}^1}(m-3)\to\mathcal{O}_R\to\mathcal{O}_{\mathbb{P}^1}\to0}$

$\Rightarrow$ $\boxed{g(\overline{C})=2}$

If I have a l. b. of high enough degree on these Gor. genus 2 curves $\Rightarrow$ its $H^1(\bar{C}, \bar{L}) = 0$

Cohomological Lemma

(a) $\bar{C}$ genus 2 Gorenstein isolated singularity without any rational tail nodally attached

$\Rightarrow$ a line bundle $\bar{L}$ on $\bar{C}$ with:

$\boxed{\deg \bar{L}|_{C_i} \geq 0 \ \forall\, C_i \subset \bar{C}}$

$\deg \bar{L}|_{C_s} > 0$ For (at least one) special cpt

tot deg $\bar{L} \geq 3$

has $\boxed{H^1(\bar{C}, \bar{L}) = 0}$

(b) $\bar{C}$ a tailed ribbon and $\bar{L}$ a line ble s.t.

$\deg \bar{L}|_{T_i} > 0$ on at least two tails

or

$\deg \bar{L} > 0$ on R^{red} and at least one tail

$\Rightarrow H^1(\bar{C}, \bar{L}) = 0$

3. The main idea behind our approach

Recall / Notice that obstructions

For $F \in \overline{M}_{2,n}(\mathbb{P}^2, d)$ arise when

$H^1(C, F^* \mathcal{O}(1)) \neq 0$

This happens when:

(1) F contracts a genus 1 or 2 subcurve $Z \subset C$

(2) when $F|_{C_0}$ (C_0 is the minimal genus 2 core) has degree 2

Naive idea

- In case (1), Replace
$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d] & \mathbb{P}^2 \\ \bar{C} \arrow[ur, "\bar{F}"'] & \end{tikzcd}$$
where $\bar{F}_{\bar{C}}$ non trivial on any higher genus curve

$\bar{C}$ is obtained collapsing Z to a point

- In case (2) replace
$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\varphi"'] & \mathbb{P}^2 \\ \bar{C} \arrow[ur, "\bar{F}"'] & \end{tikzcd}$$
where so that tails attached to ribbon have pos. degree

in $\bar{C}$ we replaced C_0 with a Ribbon

$\Rightarrow$ We expect $\bar{F}^* \mathcal{O}(1)$ to satisfy the hp of the cohomological Lemma so

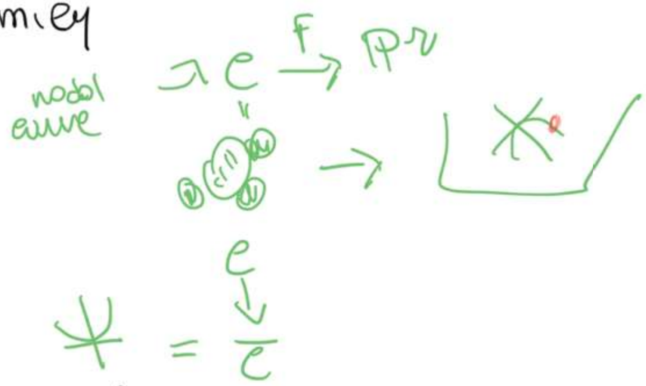
$$\boxed{H^1(\bar{C}, \bar{F}^* \mathcal{O}(1)) = 0}$$

$\rightsquigarrow$ NO OBSTRUCTIONS

Issues with the naive idea

0. the map $C \xrightarrow{\varphi} \overline{C}$ has moduli : what singularity $\overline{C}$ should have is not determined by (C,f)

But [depend from the choice of smoothing family



If I have $\mathcal{C} \overset{\mathcal{L}}{\swarrow}$ smoothing of C $\quad \mathcal{L}$ is π-semi-ample

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"] & \overline{\mathcal{C}} := \mathrm{Proj}\left(\bigoplus \pi_* \mathcal{L}\right) \arrow[d] \\ \Delta & \Delta \end{tikzcd}$$

$\overline{\mathcal{C}}$ is Gorenstein $\Longleftrightarrow$ I can find

$$\boxed{\mathcal{L} = \omega_{\mathcal{C}/\Delta}(D)(\Sigma_i)}$$

supported on contracted locus

away from what is contracted

$f^*\mathcal{O}_{\mathbb{P}^2}(1)$

1. The factorization can only exist for smoothable maps

2. To have a moduli functor we need

a way to construct $\overline{C}$ in families

and ensure
→ Gorenst condition
→ cohomological condition

we need to make sure that this can be done ←

Goal becomes

remembers degree of map ↓

Construct a moduli space ($\widetilde{X} \xrightarrow[e]{\text{bir.}} \mathcal{M}_{2,n}^{wt}$)

containing sufficient information to construct

partial destabiliz.

$$\begin{tikzcd}
 & \widetilde{C} \arrow[dl] \arrow[dr] & \\
e^* C \arrow[rr, dashed] \arrow[dr] & & \overline{C} \arrow[dl, "\pi"] \\
 & \widetilde{X} &
\end{tikzcd}$$

→ $\overline{C}$ Gorenstein

→ wt($\overline{C}$) satisfying the hp of the coh. Lemma

4. Log, Tropical curves and admissible covers

Def: A log scheme (X, M_X) is X scheme

M_X is a sheaf of monoids

+ $M_X \xrightarrow{\alpha} \mathcal{O}_X$

s.t. $\alpha^{-1}(\mathcal{O}_X^*) = \mathcal{O}_X^*$ ghost sheaf

$$\Rightarrow \quad \boxed{0 \to \mathcal{O}_X^* \to M_X \to \overset{\downarrow}{\overline{M}_X} \to 0}$$

Example 0

$X = \operatorname{Spec} k$, + your favourite monoid $\boxed{P = \mathbb{N}^{\oplus r}}$

$$\Rightarrow \quad M_X = k^* \oplus \mathbb{N}^{\oplus r} \longrightarrow k$$

$$(\lambda, v) \to \begin{cases} \lambda & v = 0 \\ 0 & \text{otherwise} \end{cases}$$

Example 1

X smooth proj variety $\boxed{D \subseteq X}$ normal crossing divisor

$$\mathcal{O}_X \xleftarrow{i} M_X = \{ f \in \mathcal{O}_X \mid f|_{X \setminus D} \in \mathcal{O}^*_{X \setminus D} \}$$

functions := inv. outside of D

$$\boxed{\overline{M}_{X,x} = \mathbb{N}^{\oplus k}}$$

$k = \#$ cpts of D meeting at x

Curves

↳ On $\overline{M}_{g,n}$ there is a canonical Log structure induced by the boundary divisor

$$\boxed{(\overline{M}_{g,n}, M_D)}$$

$$\begin{tikzcd}(\mathcal{C}, M_{\mathcal{C}}) \arrow[d, "\pi"] \\ (S, M_S)\end{tikzcd}$$

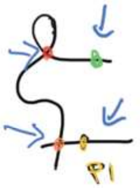
Family of log smooth curves look as follows :

$\overline{M}_{C,x} = \{ (a,b) \in \overline{M}_{S,s}^{\oplus 2} \mid a-b \in \mathbb{Z}S \}$

$S \in \overline{M}_{S,s}$

↳ smoothing parameter

$C =$



$\boxed{\overline{M}_{C,x} = \overline{M}_{S,s}}$

$\overline{M}_{C,P1} = \overline{M}_{S,s} \oplus \mathbb{N}$

$\downarrow$

s

P1 a marked point

More or less

shadow of $xy - t^n$

$\overline{M}_{C,x} = \{ (a',b') \in \overline{M}_{S,s}^{\oplus 2} \mid \underbrace{a'-b' \in \mathbb{Z}S'}\}$

$S' \in \overline{M}_{S,s}$

↑ smoothing param.

$\overset{\text{"}\log x\text{"}}{a'}$ $\overset{\text{"}\log y\text{"}}{b'}$

$\boxed{S' = n}$

Remark : If we consider the canonical structure on the moduli space of curves

$\Rightarrow$ on a point $s \to M_{g,n}$

$$\boxed{\overline{M}_s = \mathbb{N}^{\#\text{ nodes of } C_s}}$$

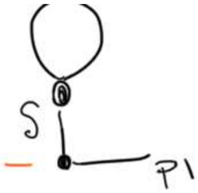
The ghost part of the log structure is recorded in the tropicalization

A tropical curve is Trop(C, M_C)

Γ = dual graph (C)

ghost sheaf of the base

S'



$$+ \quad \ell : E(\Gamma) \longrightarrow \overline{M}_s$$
$$e \longrightarrow s_e$$

we think of s, s' as lengths of these edeges

If $\overline{M}_s = \mathbb{N}$ you can think that s, s' are remembering the speed of a smoothing of C

Notice that in this case s and s' are comparable

$\rightsquigarrow$ Γ is a leveled graph

Admissible covers

A smooth genus 2 curve is hyperelliptic:

$C =$  $\xrightarrow[\tau]{2:1} \mathbb{P}^1$

- τ is branched in 6 pts (called Weierstrass)
- we say p and $\bar{p}$ are conjugated if $\tau(p) = \tau(\bar{p})$
- $\tau^* \mathcal{O}(1) = \omega_C$

Instead of compactifying $M_{2,n}$ to $\overline{M}_{2,n}$ we can consider the

$$\boxed{M_{2,n} \subset A_{2,n} \leftarrow \text{admissible covers}}$$

$$\uparrow$$

$\psi : C \to T$ C nodal, $g=2$ T nodal $g=0$

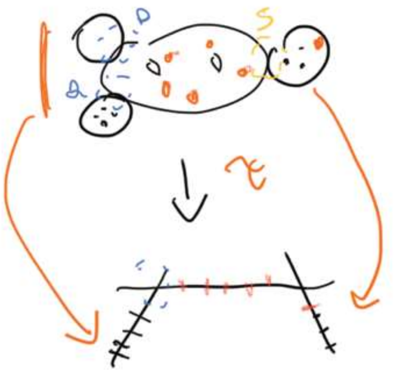
- ψ is 2:1 branched on 6 smooth pt

and

- Around nodes $\mathbb{C}[x,y,e]/(xy-e^i) \leftarrow \mathbb{C}[t,s,e]/(st-e)$ is $t \to x^i$, $s \to y^i$ $i=1,2$

- T marked with $p_1, \dots p_n$ + branch pts $b_1, .. b_6$ is DM-stable

Examples



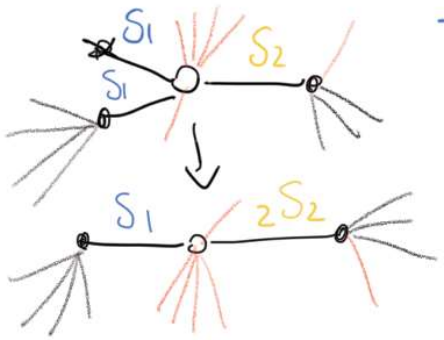
$i=1$ and $p = \bar{a}$

$i=2$ and s is Weierstrass

Also on $A_{2,n}$ there is a natural log-structure

$$\boxed{\mathcal{M}_{\mathcal{A}_{2,n}} = \mathcal{M}_{\mathcal{M}_{2,n}} \oplus \mathcal{M}_{\mathcal{M}_{0,n+6}} \Big/ (\delta, 0) \sim (0, i\delta')}$$

Tropical version



→ edge length of conj. edges are the same = edge lenght of their image in T

→ $\delta^T = 2\delta^C$

As for the curve we can consider variations of $\mathcal{A}_{2,n}$

$\rightsquigarrow \mathcal{A}^{wt}_{2,n}$ ← vertices are decorated with weight and weighted st. condition

Piecewise linear function on the tropicalization

On log schemes we have a distinguished class of line bdles coming from $\Gamma(X, \overline{M}_X^{gp})$

Recall $\boxed{0 \to \mathcal{O}_X^* \to M_X^{gp} \to \overline{M}_X^{gp} \to 0}$

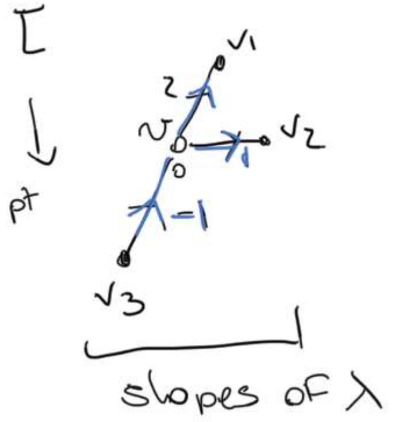
$$\begin{tikzcd} \Rightarrow \Gamma(\overline{M}_X^{gp}) \arrow[r] & H^1(\mathcal{O}_X^*) \end{tikzcd}$$

$$\begin{tikzcd} \alpha \arrow[r] & \mathcal{O}_X(\alpha) \end{tikzcd}$$

If $X \to S$ a log smooth curve

$$\Gamma(\overline{M}_X^{gp}) = \left\{ \text{Piecewise linear Fuctions } \boxed{\lambda : \Gamma \to \overline{M}_S} \;\middle|\; \frac{\lambda(v_1) - \lambda(v_2)}{s_e} \in \mathbb{Z} \right\}$$

$\overset{s_e}{v_1 \text{———} v_2}$



nodes associated to e_1, e_2, e_3

$$\Rightarrow \mathcal{O}_C(\lambda)\big|_{C_v} = \mathcal{O}(2p_1 + p_2 - p_3)$$

5. Main Results & Idea of the proof

Theorem 1 (Battistella, -)

There exists a log étale model (birational)

$$\boxed{\mathcal{A}_{2,n}^{\lambda\text{-}wt} \xrightarrow{e} \mathcal{A}_{2,n}^{wt} \xrightarrow{e'} \mathfrak{M}_{2,n}^{wt}}$$

which is smooth. Moreover

- On $\mathcal{A}^{\lambda\text{-wt}}_{2,n}$ is defined a universal partial destabilization $\boxed{\tilde{C} \to e^* C}$ and a global section $\boxed{\lambda \in \Gamma(\overline{M}_{\tilde{C}})}$

$$\begin{tikzcd} \text{PL } \lambda : \tilde{\Gamma} \arrow[r] \arrow[d] & \overline{M}_S \\ \Gamma & \end{tikzcd}$$

- $\mathcal{A}^{\lambda\text{-wt}}_{2,n}$ parametrizes log admissible covers $\boxed{\psi : C \to T}$ such that

$\boxed{\lambda(v) \text{ and } \lambda(v') \text{ are comparable}}$ either $\lambda(v)-\lambda(v')$ or $\lambda(v')-\lambda(v) \in \overline{M}_S$

$\forall\ v, v' \in \Gamma$

- On

$$\begin{tikzcd} & \tilde{C} \arrow[dl] \arrow[dr] & \\ e^* C \arrow[dr] & & \overline{C} \arrow[dl, "\overline{\pi}"] \\ & \mathcal{A}^{\lambda\text{-wt}}_{2,n} & \end{tikzcd}$$

- $\overline{\pi}$ is a flat family of Gor. curves
- the wt on $\overline{C}$ induced by the one of C satisfy the cohomological conditions

Theorem 2 (Battistella, −)

Let $VZ_{2,n}(X,\beta)$ the moduli space parametrizing

- A family $\begin{tikzcd}[row sep=small] C \arrow[r] \arrow[dr] & T \arrow[d] \\ & S \end{tikzcd}$ of λ-aligned log admissible cover

- A map $f : C^{st} \to X$ satisfying the factorization

• A map [illegible]

property, i.e

$$\begin{tikzcd}
 & \tilde{\mathcal{C}} \arrow[dl] \arrow[dr] & \\
\exists\ \mathcal{C}^{st} \arrow[dr, "f"'] & \circlearrowleft & \bar{\mathcal{C}} \arrow[dl, "\bar f"] \\
 & X &
\end{tikzcd}$$

original map

$\bar f$ has high enough degree on each cpt

$\Rightarrow$ (1) $VZ_{2,n}(X,\beta)$ is a proper DM-stack

(2) For $X = \mathbb{P}^r$, this is unobstructed.

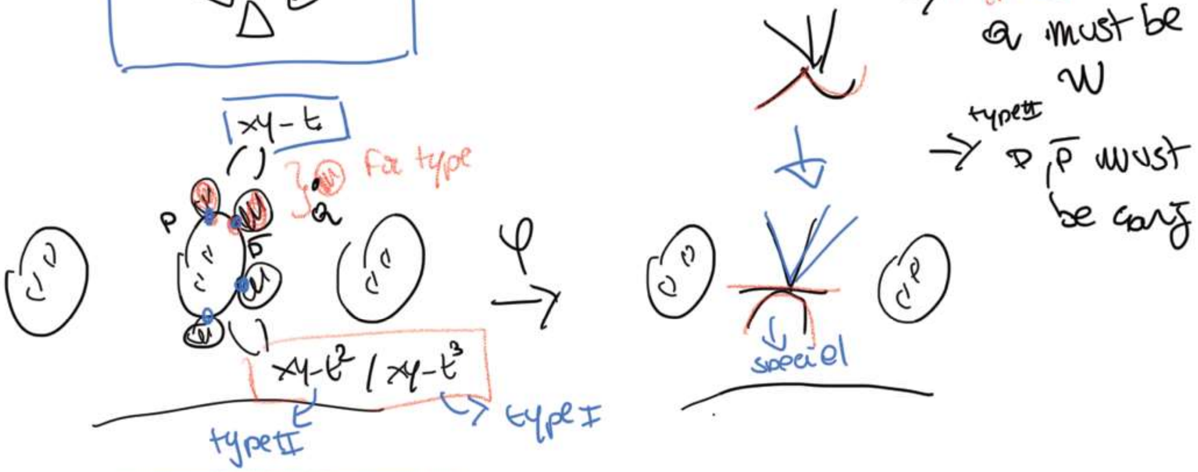
proof of 1

Let $\bar C$ genus 2 Gorenstein, $\begin{matrix}\bar{\mathcal{C}}\\ \downarrow \\ \Delta\end{matrix}$ a smoothing family

and

$$\begin{tikzcd}
\mathcal{C}^{ss} \arrow[rr] \arrow[dr] & & \bar{\mathcal{C}} \arrow[dl] \\
 & \Delta &
\end{tikzcd}$$

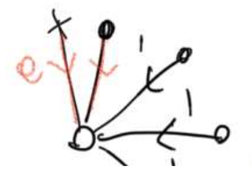
← Gorenstein



$$\Rightarrow \boxed{\varphi^* \omega_{\bar{\mathcal{C}}} = \omega_{\mathcal{C}}(\lambda)} \quad \text{where } \lambda \text{ is}$$

defined on Γ_0 and looks like

this



$e = 2$ in type II

and $p = \bar p$

$e = 3$ in type I
and the tail is
attached to a W
pt

→ the special branch is the
closest to C_0

→ All the other are at the same distance

ration $d(C_{sp}, C_0) \,/\, d(\text{other branche are})$ $\langle$ $1/2$, $1/3$

And

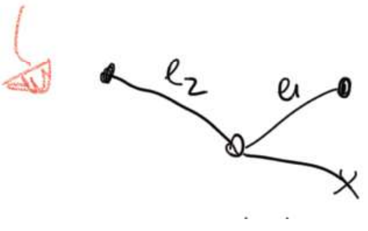
if we started from a Ribbon

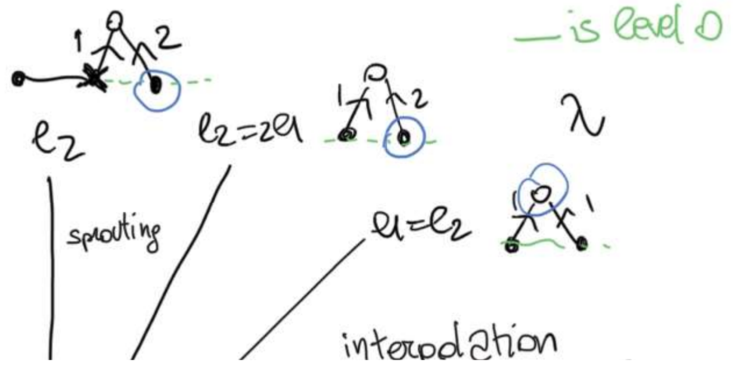
To give $\overline{C}$ starting from $e \to T$

→ choose a special vertex v_D which tell us what the slopes of λ are

→ compare $\lambda(v), \lambda(e)$ for each vertex

↳ information included in $\Lambda_{2,n}^{\lambda\text{-wt}}$





blow-up of $A^{wt}_{2,w}$ givin $A^{x-wt}_{2,n}$

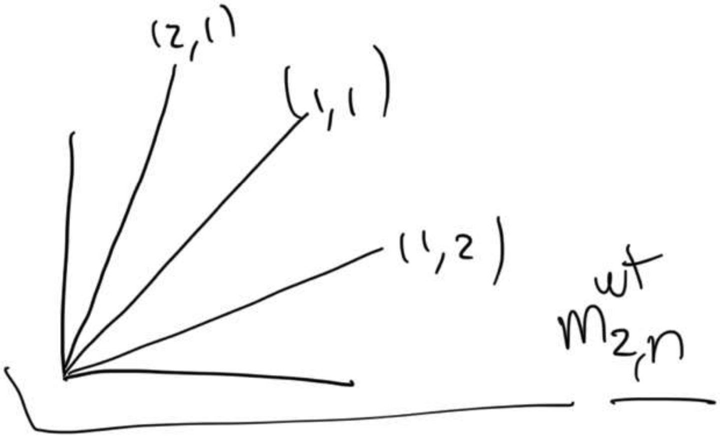
As the lengths vary we need to move the special vertex

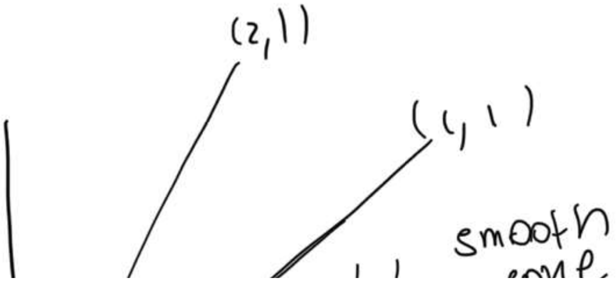
⇝ subdividing $A^{wt}_{2,n}$ we move x aligned

on each cone we can compare the value of the vertices

& know that the special branch is the unique one with slope ≠1

⇝ get $\tilde{e}$ and $\bar{e}$





we have

e quotient
sing.

$$\begin{tikzcd}
 & A_2^{wt} \\
 & \cap \\
C \arrow[r] & T \\
 & \cap \\
 & \text{weighted-stable}
\end{tikzcd}$$

A_2 $C \to T$