# How Some Constructions in Projective Geometry Vary.

(Anand Patel)

Anand Deopurkar
Eduard Duryev 2019

Eric Riedl
Dennis Tseng 2020

Adam Cartisano
(ongoing)

$$X^n \subset \mathbb{P}^N$$

$n$-dim'l irr. non.deg. projective var. $/ k = \bar{k}$

There are some common TECHNIQUES which help us understand $X$

"Intersect with hyperplane"
"Intersect with $k$-plane"
"finite projection onto a $\mathbb{P}^n$"
"projection to a $\mathbb{P}^m$" more general".

Natural SUBSCHEMES of $X$ are constructed from these techniques: "(Hyper) plane sections of $X$"

"Constructions" → 
- " Ramification divisor of projection $X \longrightarrow \mathbb{P}^n$ "
- " Ramification loci "
- " Multiple-point loci of projection "

Constructed subschemes clearly vary naturally with parameters.

$\boxed{\text{BROAD OBJECTIVE}}$ : Study this variation.

GET:

| Interesting Classification Problems | Difficult Enumerative Problems |
|---|---|
| Which ($X$, construction) have unusually small variation? | If # of parameters = dim (space of constructed schemes) Get challenging enumerative problems. |

- Variation of Hodge Structures / Generic Torelli: ~1980's + 90's

  (R. Donagi)

- Flenner + Manaresi ~1990's

# THIS TALK:

SKETCH 2 RECENT CHAPTERS IN THIS STORY.

# I. Variation of (hyper)plane Slices (of a hypersurface)

with D. Tseng, E. Riedl. (2020).

$X^n \subset \mathbb{P}^{n+1}$ degree $d$ HYPERSURFACE.

GET $\rightsquigarrow$ $\boxed{\Phi : \mathbb{P}^{n+1\,*} \dashrightarrow \mathbb{P} H^0(\mathcal{O}_{\mathbb{P}^n}(d)) /\!\!/ SL_{n+1}}$

$H \longmapsto [H \cap X]$

moduli space of degree $d$ hypersurfaces in $\mathbb{P}^n$.

**Difficult:** (wide open) Classify those $X$'s for which $\Phi$ fails to have maximal rank.

**Easy:** ("Exercise") Show $\Phi$ has max rank if $X$ is assumed GENERAL.

Are there any examples NOT explained by $\mathbb{G}_m / \mathbb{G}_a$ automorphisms.

$\boxed{\text{Medium:}}$ Prove maximum variation for $X$ SMOOTH.

Beauville's Beautiful Warning!

$x^4+y^5+z^7=0$

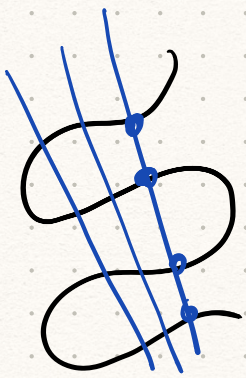
$\boxed{\text{Theorem}}$: In characteristic 0, if $X$ is assumed smooth, and if $\boxed{d \geq n+3}$ then $\Phi$ has maximal rank.

$n=1$ ($X$ plane curve)

$\boxed{\text{Theorem:}}$ If $X \subset \mathbb{P}^2$ is such that $\Phi$ fails to have maximal rank, then $\mathrm{Aut}(\mathbb{P}^2, X)$ is positive dimensional.

(Can write exact list.)

$\mathbb{G}_m$
$\mathbb{G}_a$



(Also generalization to k-plane sections of a hypersurface.)

$$\boxed{\Phi_k : G(k,n+1) \dashrightarrow PH^0(\mathbb{P}^k, O(d))//SL_{n+1}}$$

Techniques: Key Ingredient: To ~~study~~ use stability/semi-stability of $(\mathbb{P}^{n+1}, X)$ lc. pair $\boxed{T_{\mathbb{P}^n}(-\log X)}$ (Guenancia 2019) complex-geometric

**Enumerative Problems**

$$\Phi_{(k)} : \underline{G(k,N)} \dashrightarrow \underline{\mathbb{P}H^0(\mathbb{P}^k, \mathcal{O}(d)) /\!/ SL_{k+1}}$$

$k \neq 1$

In some cases, dimensions of source & target are equal.

Calculate the degree.

(These problems are quite challenging.)

**Theorem:** (Lee, P–, Tseng, 2019)

Let $X \subset \mathbb{P}^4$ be a GENERAL degree 4 hypersurface. Then the map

$$\Phi_2 : G(2,4) \dashrightarrow \mathcal{M}_3$$

$$\Lambda \longmapsto [\Lambda \cap X]$$

moduli space of genus 3 curves.

has degree $\boxed{510{,}720.}$

Indep verified by computer.

$\boxed{\text{Notorious open problem:}}$

(relayed to me by R. Laza)

$X \subset \mathbb{P}^4$ general CUBIC hypersurface.

Calculate the degree of

$$\phi : \mathbb{P}^{4*} \dashrightarrow \underbrace{\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}(3)) /\!\!/ SL_4}.$$

moduli space of cubic surfaces.

# II. Variation of Ramification Divisors

(with: A. Deopurkar + E. Duryev. (2019))

$X^n \subset \mathbb{P}^N$ , $L \subset \mathbb{P}^N$ GENERAL linear space of dim $N-n-1$

$\rightsquigarrow$ $\boxed{\pi_L : X^n \longrightarrow \mathbb{P}^n}$ finite covering.

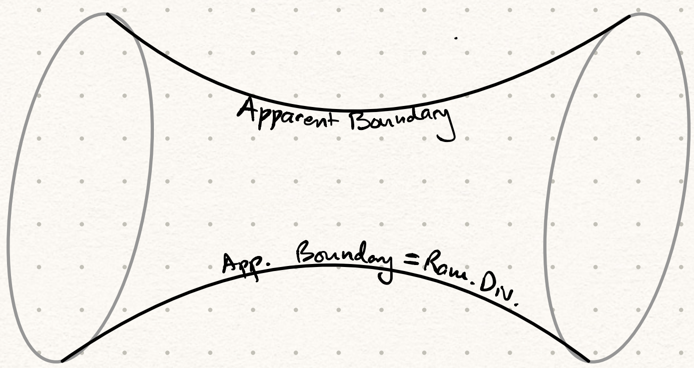
$R_L \subset X$ ramification divisor of $\pi_L$.

$\boxed{\text{"Apparent boundary"}}$

Get projection-ramification map:

$$\boxed{\begin{array}{rcl} \rho : G(N-n-1,N) & \dashrightarrow & |R| \cong \mathbb{P}^m \\ L & \longmapsto & R_L \end{array}}$$

linear system where $R_L$'s live.



$$\boxed{\rho : G(N-n-1, N) \dashrightarrow |R|}$$

Usual **Questions**: Does $\rho$ always have max. rank? Are there interesting enumerative problems?

When $X$ is smooth (e.g.)

First asked (in searchable print) by Flenner & Manaresi (Early 1990's).

**Theorem:** Suppose $X^n \subset \mathbb{P}^N$ is normal, non-degenerate and that the dual variety $X^* \subset \mathbb{P}^{N*}$ is codimension 1. Then ramification divisors vary maximally on $X$, i.e.

set of tangent hyperplane sections ss singular hyperplane section.

$$\boxed{\rho : \mathbb{G}(N-n-1, N) \dashrightarrow |R|}$$ has maximal rank.

**Proof:** We studied carefully what happens when $L \rightsquigarrow$ incident to $X^*$.

**Comments:** • Vastly generalizes a result of Flenner + Manaresi.

• Not perfect. There are simple examples where $X^*$ is small yet still $\rho$ has max rank.

**Thm-Example - Warning**: There exist some $\geq 4$ dimensional rational normal scrolls $X \subset \mathbb{P}^N$ for which $\rho$ fails to have maximal rank.

• (All known examples "Explained by automorphisms") finite generic stab.

$$\rho : \mathbb{G}(N-n-1, N) \overset{\text{equiv.}}{\dashrightarrow} |R|$$

$\mathrm{Aut}(X)$

+ dim'l stabilizer

• Not some known condition regarding, e.g. $\mathrm{II}_X$.

**Theorem:** In "$\rho : \mathbb{G}(N-n-1, N) \dashrightarrow |R|$",

(char 0). $\dim \mathbb{G} \overset{\text{always}}{\leq} \dim |R|$

with equality $\iff$ $X$ is a VARIETY OF MINIMAL DEGREE.

$\|$

$\left\{\begin{array}{l}\text{Quadric hypersurfaces,} \\ \text{rational normal scrolls,} \\ \text{Veronese surface } \subset \mathbb{P}^5.\end{array}\right.$

(Kodaira Vanishing)

$\boxed{\text{Enumerative Problem}}$ When $X$ is a V.M.D.,

compute $\rho$.

Known CASES:

"B+M. Shapiro" Conjecture

Real A.G.

Wronski

(1) $\dim X = 1$ = rat. normal curve: Degree$(\rho)$ = $\boxed{\text{Catalan \#'s.}}$

$X$ = Quadric : $\deg \rho = 1$ (Polarity).

$X$ = Veronese surface : Degree$(\rho) = 3$. $\leftrightarrow$ "A given cubic curve is Hessian cubic plane curves".

$X$ = $\boxed{\text{Surface Scrolls}}$ : Computer suggests strange connection with some combinatorial objects.

$\mathbb{P}^1$-bundles / $\mathbb{P}^1$

higher dimension

OEIS : (# of Baxter permutation).

1, 2, 6, 22, 92, 4 ...

Q:

1. Sometimes $\deg \rho = 0$. Which scrolls have this?

2. Does max variation of ram. divisors hold for all 3-folds? (Enough to prove for $\mathbb{P}^2$-bundles over curves.)

3. Huge: Generalize story to "Variation of ramification loci of arbitrary projections $X^n \to \mathbb{P}^m$"