Hurwitz – Brill – Noether Theory

joint w/ Kaelin Cook-Powell

|| work H. Larson

Larson – Larson – Vogt

C = curve

Q: Describe all maps $C \longrightarrow \mathbb{P}^r$ of degree d.

Given a line bundle L on C,

let $s_0, s_1, \ldots, s_r \in H^0(C, L)$ basis

$$\begin{array}{ccc} C & \longrightarrow & \mathbb{P}^r \\ \in & & \in \\ p & \longmapsto & [s_0(p), s_1(p), \ldots, s_r(p)] \end{array}$$

Given a map $\varphi : C \to \mathbb{P}^r$

$\varphi^* \mathcal{O}_{\mathbb{P}^r}(1)$ is a line bundle on C.

Brill-Noether varieties:

$$W^r_d(C) := \{ L \in \mathrm{Pic}^d(C) \mid h^0(C, L)$$

$\geq r+1\}$.

Thm: Let C be a <u>general</u> curve of genus g.
Then $W^r_d(C)$ is ...

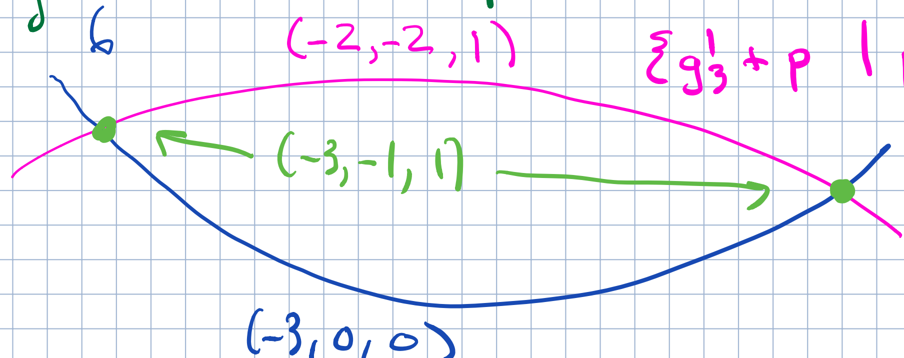
1) equidimensional of dim $g-(r+1)(g-d+r)$
[Griffiths–Harris '80]
2) smooth away from $W^{r+1}_d(C)$
[Gieseker '82]
3) irreducible if dim > 0
[Fulton–Lazarsfeld '81]

What if C is <u>not</u> general?

C is equipped with a map $\pi : C \longrightarrow \mathbb{P}^1$ of degree k.

In other words, (C,π) is an element of the <u>Hurwitz space</u> $\mathcal{H}_{g,k}$.

<u>Ex</u>: $g=5$ $k=3$ $W^1_4(C)$
6



$\{g^1_3+p \mid p\in C\}$

$\{K_C-g^1_3-q \mid q\in C\}$

Let L be a line bundle on C.

$$\pi_* L \simeq \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^1}(\mu_i) = \mathcal{O}(\vec{\mu})$$

$$\vec{\mu} = (\mu_1, \mu_2, \dots, \mu_k).$$

$\vec{\mu}$ is the splitting type of L.

$$W^{\vec{\mu}}(C,\pi) := \{ L \in \mathrm{Pic}(C) \mid \pi_* L \simeq \mathcal{O}(\vec{\mu}) \}$$

<u>Thm</u>: Let $(C,\pi) \in \mathcal{H}_{g,k}$ be <u>general</u>.

Then $W^{\vec{\mu}}(C,\pi)$ is ...

1) equidimensional of dim $g - \sum_{i \neq j} \max\{0, \mu_i - \mu_j - 1\}$

[H. Larson '19, Cook-Powell - J '20]

2) smooth [H. Larson '19]

3) irreducible when dim > 0 [LLV '20]

<u>Degeneration Arguments</u>

$\mathcal{H}_{g,k}$ is irreducible, so any <u>nonempty</u> open subset is dense.

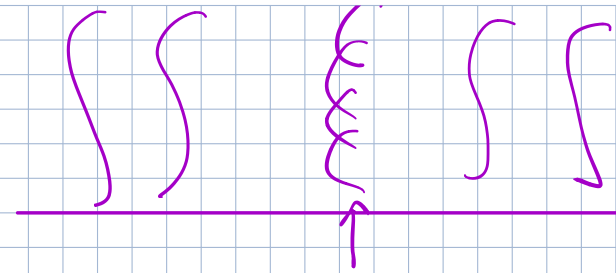
All conditions above are open.

It's enough to find <u>one</u> (C,π) satisfying (1) - (3).

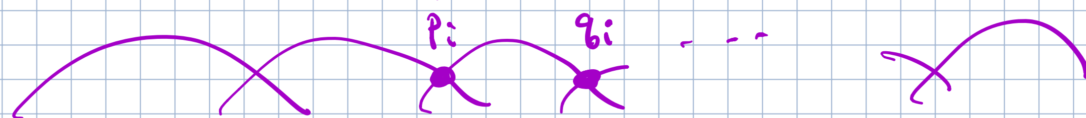
Instead of finding such a pair, you <u>degenerate</u> to a singular curve.

tropical

limit linear series



chain of g elliptic curves



$p_i - q_i$ has torsion order k

<u>Prop</u>: Let (C,π) be this chain of g elliptic curves with torsion profile k. Then

$$W^{\vec{\mu}}(C,\pi) = \bigcup_{\substack{k\text{-uniform} \\ \text{displacement} \\ \text{tableaux } t \\ \text{on } \lambda(\vec{\mu})}} \mathbb{T}(t).$$

$$\lambda(\vec{\mu}) := \{(x,y) \in \mathbb{N}^2 \mid \exists\, m \in \mathbb{Z} \text{ s.t. } x \le h^0(\mathbb{P}^1, \mathcal{O}(\vec{\mu}+m)) \text{ and } y \le h^1(\mathbb{P}^1, \mathcal{O}(\vec{\mu}+m))\}$$

<u>Ex</u>: $g=5$ $k=3$ $\vec{\mu} = (-3,-1,1)$

1	3	4	5
2	5		
3			
5			

<u>Def</u>: A <u>k-uniform displacement tableau</u> t on a partition λ with alphabet $[g]$

partition λ with alphabet $[g]$
is a function $t : \lambda \longrightarrow [g]$ satisfying:

1) t is increasing across rows and down cols.
2) if $t(x,y) = t(x',y')$, then
$$y - x \equiv y' - x' \pmod{k}.$$

Given a k-uddt t on $\lambda(\vec{\mu})$, one can construct a family of limit linear series $\mathbb{T}(t)$.

- if i appears in the tableau, its position determines the line bundle on component i.
- if i does <u>not</u> appear in the tableau, then the l.b. on the i^{th} component can be anything.

$\dim \mathbb{T}(t) = \#$ of symbols in $[g]$ that do <u>not</u> appear in t.

$\lambda(\vec{\mu})$ is an example of a <u>k-core</u>.

$\{k\text{-cores}\} \longleftrightarrow$ infinite Coxeter systems $\tilde{A}_k$.

strong and weak order on the set of k-cores

$\left\{ \begin{array}{l} \text{Standard} \\ \text{Young tableaux} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{saturated chain} \\ \text{in the Young lattice} \end{array} \right\}$

$\left\{ \begin{array}{l} k\text{-"efficient"-uniform} \\ \text{displacement} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{saturated chains in} \\ \text{the weak order lattice} \end{array} \right\}$

tableaux) (...) of k-cores

equidimensionality $\longleftrightarrow$ weak order lattice is a lattice

codimension $\longleftrightarrow$ rank of $\lambda(\vec{u})$ in this lattice

connectedness $\longleftrightarrow$ "braid moves"

containment $\longleftrightarrow$ strong order on k-cores