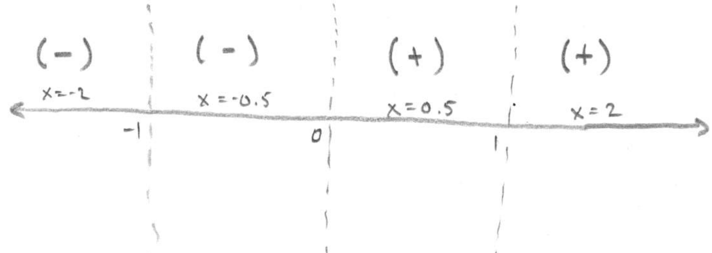
Exam 3 Review Problems | Solutions

1. a) $f(x) = (x^2-1)^{1/3} \Rightarrow f'(x) = \frac{1}{3}(x^2-1)^{-2/3} \cdot 2x = \frac{2x}{3\sqrt[3]{(x^2-1)^2}}$

Critical points:

$f'(x) = 0 \Rightarrow x = 0$

$f'(x)$ undefined $\Rightarrow x = \pm 1$



Classify w/ First Deriv. Test:

$f'(-2) = \frac{-4}{3 \cdot 3^{2/3}} < 0$ $\qquad f'(0.5) = \frac{1}{3(3/4)^{2/3}} > 0$

$f'(-0.5) = \frac{-1}{3 \cdot (\frac{3}{4})^{2/3}} < 0$ $\qquad f'(2) = \frac{4}{3 \cdot 3^{2/3}} > 0$

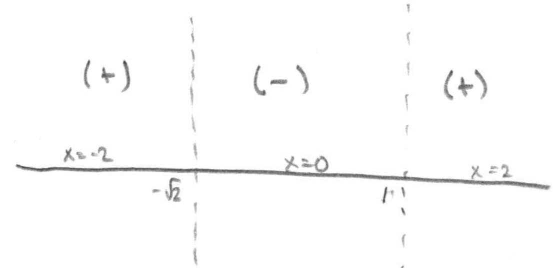
$\Rightarrow$ The critical points at $x = \pm 1$ are neither local max nor min, and the critical point at $x=0$ is a local min.

b) $g(x) = \frac{1}{6}x^3 + |x-1| = \begin{cases} \frac{1}{6}x^3 + x - 1 & x \geq 1 \\ \frac{1}{6}x^3 - (x-1) & x < 1 \end{cases} \Rightarrow g'(x) = \begin{cases} \frac{1}{2}x^2 + 1 & x > 1 \\ \frac{1}{2}x^2 - 1 & x < 1 \end{cases}$

Critical points:

$g'(x) = 0 \Rightarrow \begin{cases} \frac{1}{2}x^2 + 1 = 0 \\ \frac{1}{2}x^2 - 1 = 0 \end{cases} \Rightarrow$ $\begin{array}{l} \text{no solutions} \\ x = \pm\sqrt{2}, \text{ for } x < 1 \Rightarrow x = -\sqrt{2} \end{array}$

$g'(x)$ undefined $\Rightarrow x = 1$ (g has a corner)



First Deriv. Test:

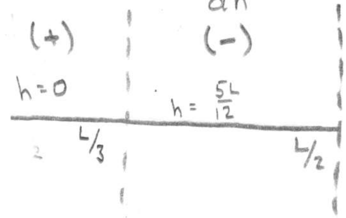
$g'(-2) = 1 > 0$; $g'(0) = -1 < 0$; $g'(2) = 3 > 0$

$\Rightarrow g$ has a local max at $x = -\sqrt{2}$ and a local min at $x = 1$.

c) $A(h) = \frac{h}{2}\sqrt{(L-h)^2 - h^2} = \frac{1}{2}h\sqrt{L^2 - 2Lh} \Rightarrow \frac{dA}{dh} = \frac{1}{2}\sqrt{L^2-2Lh} + \frac{1}{2}h\left(\frac{-2L}{2\sqrt{L^2-2Lh}}\right) = \frac{L^2 - 3Lh}{2\sqrt{L^2-2Lh}}$

Crit pts: $\frac{dA}{dh} = 0 \Rightarrow h = \frac{L}{3}$; $\frac{dA}{dh}$ undefined $\Rightarrow h = \frac{L}{2}$

First Deriv. Test:

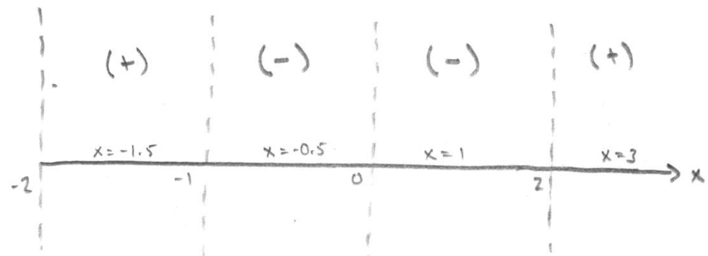


$\left.\frac{dA}{dh}\right|_{h=0} = \frac{1}{2} > 0$; $\left.\frac{dA}{dh}\right|_{h=\frac{5L}{12}} = -\frac{L\sqrt{6}}{8} < 0$

$\Rightarrow A$ has a local max at $h = \frac{L}{3}$ and a min at $h = \frac{L}{2}$ (which is the edge of its domain)

2. a) $f'(x) = 0 \Rightarrow x^2(x-2)^3(x+1) = 0 \Rightarrow x = 0, 2, -1$

$f'(x)$ undefined $\Rightarrow x+2=0 \Rightarrow x=-2$



$f'(-1.5) \approx 68 > 0$

$f'(-0.5) \approx -1.6 < 0$

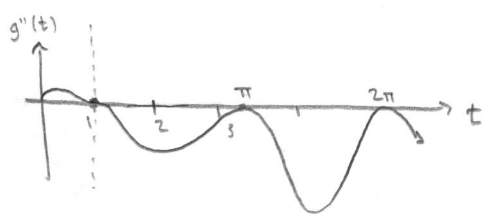
$f'(1) \approx -1.2 < 0$

$f'(3) \approx 16.1 > 0$

f has a local min at $x=-2$, a max at $x=-1$, neither max nor min at $x=0$, and a local min at $x=2$.

b) $g''(t) = -e^{-t}(\ln(t)-1)^2(t^2-1)^3(1-\cos(t))$

Note: e^{-t}, $(\ln(t)-1)^2$, and $(1-\cos(t))$ are all positive, so g'' will only change signs at $t=1$.



$\Rightarrow$ g is concave up for $0<t<1$ and concave down for $t>1$

3. a) f has no critical points on $[-3,2]$, but has a vertical asymptote at $x=0 \Rightarrow f$ is unbounded

$f(-3) = \frac{1}{9}$, $f(2) = \frac{1}{4}$, $\lim_{x\to 0} f(x) = \infty \Rightarrow \boxed{\frac{1}{9} \le f(x) < \infty \text{ on } [-3,2].}$

b) f is continuous on $[-\pi, \pi]$, so it must have a global max + min.

Crit. pts: $f'(x) = 1 - 2\sin(x) = 0 \Rightarrow \sin(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$ and $\frac{5\pi}{6}$ (in $[-\pi,\pi]$)

x	$-\pi$	$\pi/6$	$5\pi/6$	π
$f(x)$	$-\pi-2$	$\pi/6+\sqrt{3}$	$5\pi/6-\sqrt{3}$	$\pi-2$

$\Rightarrow \boxed{-\pi-2 \le f(x) \le \frac{\pi}{6}+\sqrt{3} \text{ on } [-\pi,\pi]}$

c) $\theta'(t) = (-t^2-t+3)^{1/2} + t\cdot\frac{1}{2}(-t^2-t+3)^{-1/2}\cdot(-2t-1) = \dfrac{-4t^2-3t+6}{2\sqrt{-t^2-t+3}} \Rightarrow$ global min at $t = \dfrac{-3-\sqrt{105}}{8}$

$\Rightarrow \boxed{-\frac{3+\sqrt{105}}{32}\sqrt{\frac{51+\sqrt{105}}{2}} \le g(t) < 0 \text{ on } (-2,0)}$

4. a) $f'(x) = 2ax\,e^{-bx} - abx^2 e^{-bx} = axe^{-bx}(2-bx)$

$f'(x)=0 \Rightarrow \boxed{x=0}\ \boxed{x=\frac{2}{b}}$

b) Crit pt at $(5,12) \Rightarrow \begin{cases} f(5)=12 \\ f'(5)=0 \end{cases} \Rightarrow \begin{cases} 25ae^{-5b}=12 \\ 25ae^{-5b}(2-5b)=0 \end{cases}$

$\Rightarrow \boxed{b=\frac{2}{5}} \Rightarrow 25ae^{-5(\frac{2}{5})}=12 \Rightarrow \boxed{a=\frac{12e^2}{25}}$

c) $f''(x) = \frac{24}{625} e^{-\frac{2}{5}x+2}\left(2x^2-20x+25\right)$

$f''(0)>0 \Rightarrow f$ has a local min at $x=0$

$f''(5)<0 \Rightarrow f$ has a local max at $x=5$

5. $S = 2(xy+xz+yz) = 2\left(x(t)y(t)+x(t)z(t)+y(t)z(t)\right)$

$\frac{dS}{dt} = 2\left(x'y+xy'+x'z+xz'+y'z+yz'\right)$

given: $\frac{dS}{dt}=5,\ x=4,\ y=3,\ z=1,\ x'=2,\ y'=-1.5$; want z'

$5 = 2\left(2\cdot3+4\cdot-1.5+2\cdot1+4z'+-1.5\cdot1+3z'\right)$

$5 = 2(7z'+0.5) \Rightarrow \boxed{z'=\frac{2}{7}\text{ m/sec}}$

6. a) $\lim_{x\to0}\frac{x}{\arctan(x)} = \frac{0}{0} \Rightarrow$ L'Hopital applies

$\lim_{x\to0}\frac{x}{\arctan(x)} = \lim_{x\to0}\frac{1}{\frac{1}{1+x^2}} = \boxed{1}$

b) $\lim_{x\to\pi}\frac{\sin(x)}{x} = \frac{0}{\pi} = \boxed{0}$

c) $\lim_{x\to0^+}\left(-\ln(x)\right)^x = \infty^0 \Rightarrow$ L'Hopital applies. $y=(-\ln(x))^x \Rightarrow \ln(y) = x\ln(-\ln(x))$

$\ln\left(\lim_{x\to0^+} y\right) = \lim_{x\to0}\frac{\ln(-\ln(x))}{x^{-1}} = \lim_{x\to0}\frac{\frac{1}{-\ln(x)}\cdot\frac{-1}{x}}{-1/x^2} = \lim_{x\to0}\frac{-x}{\ln(x)} = 0$

$\Rightarrow \lim_{x\to0^+} y = e^0 = \boxed{1}$

d) $\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=0$ (what about as $x\to 0$?)

e) $\lim_{x\to\infty} e^{-x}3^x = \lim_{x\to\infty}\frac{3^x}{e^x} = \lim_{x\to\infty}\left(\frac{3}{e}\right)^x = \infty$

f) $\lim_{x\to 0}(3x+4)^{1/x} = \infty$

7. 

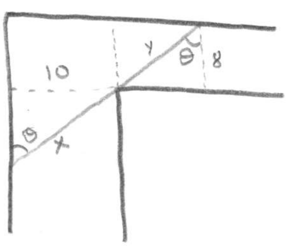
$$x^2+y^2=(10-x)^2 \Rightarrow y^2 = 100-20x \Rightarrow x = \frac{100-y^2}{20}$$

$$\Rightarrow A = xy \Rightarrow A(y) = \left(\frac{100-y^2}{20}\right)y = \frac{100y-y^3}{20} \text{ on } [0,10]$$

$$A'(y) = \frac{100-3y^2}{20} = 0 \Rightarrow y = \sqrt{\frac{100}{3}}$$

y	0	$\sqrt{\frac{100}{3}}$	10
$A(y)$	0	$\frac{100}{3\sqrt{3}}$	0

Since $A(y)$ is continuous on the closed interval $[0,10]$, the max must occur at and endpt or crit pt $\Rightarrow$ the global max of A is $\boxed{\frac{100}{3\sqrt{3}}}$

8. 

$$\sin(\theta) = \frac{10}{x} \Rightarrow x = \frac{10}{\sin(\theta)}$$

$$\cos(\theta) = \frac{8}{y} \Rightarrow y = \frac{8}{\cos(\theta)}$$

$\Rightarrow$ length of pipe is $L(\theta) = \frac{10}{\sin(\theta)} + \frac{8}{\cos(\theta)}$ on $(0, \frac{\pi}{2})$

The minimum of L is the largest pipe that will fit around the corner

i.e. find minimum of $L(\theta) = \frac{10}{\sin(\theta)} + \frac{8}{\cos(\theta)}$ on $(0,\frac{\pi}{2})$

$$L'(\theta) = \frac{-10\cos(\theta)}{\sin^2(\theta)} + \frac{8\sin(\theta)}{\cos^2(\theta)} = 0 \Rightarrow \frac{10\cos(\theta)}{\sin^2(\theta)} = \frac{8\sin(\theta)}{\cos^2(\theta)}$$

$$\Rightarrow 8\sin^3(\theta) = 10\cos^3(\theta) \Rightarrow \tan^3(\theta) = \frac{10}{8} \Rightarrow \tan(\theta) = \sqrt[3]{\frac{5}{4}}$$

$\Rightarrow \theta = \arctan\left(\sqrt[3]{\frac{5}{4}}\right)$. Since $\lim_{\theta\to 0} L = \lim_{\theta\to\pi/2} L = \infty$, the min must occur at $\theta = \arctan\left(\sqrt[3]{\frac{5}{4}}\right) \approx 47.13^\circ$

$$\Rightarrow \boxed{L = \left(4^{1/3}+5^{2/3}\right)^{1/2}\left(10\cdot 5^{-1/3}+8\cdot 4^{-1/3}\right)} \approx 25.4 \text{ Feet.}$$