Exam 2 Review Problems Solutions

1. a) $h'(x) = \frac{d}{dx}\left(xf(x)\right)(1-g(x)) + xf(x)\cdot(-g'(x))$

$= (f(x)+xf'(x))(1-g(x)) - xf(x)g'(x)$

$h'(10) = (9+10\cdot{-3})(1-1) - 10\cdot 9\cdot 8 = \boxed{-720}$

b) $k'(x) = \dfrac{2f'(x)\sqrt{g(x)} - (2f(x)-1)\cdot\frac{1}{2\sqrt{g(x)}}\cdot g'(x)}{g(x)} = \dfrac{4f'(x)g(x) - (2f(x)-1)g'(x)}{2g(x)^{3/2}}$

$k'(12) = \dfrac{4\cdot{-4}\cdot 4 - (2\cdot 5-1)\cdot 0}{2\cdot 4^{3/2}} = \boxed{-4}$

c) $p'(x) = \dfrac{(f'(x)g(x)+f(x)g'(x))(f(x)+g(x)) - f(x)g(x)(f'(x)+g'(x))}{(f(x)+g(x))^2}$

$= \dfrac{f(x)^2g'(x) + f'(x)g(x)^2}{(f(x)+g(x))^2} \Rightarrow p'(14) = \dfrac{1^2\cdot{-2} + -1\cdot 0^2}{(1+0)^2} = \boxed{-2}$

d) $\frac{d}{dx}\left(f(x^2g(x))\right)\Big|_{x=8} = f'(x^2g(x))\cdot(2xg(x)+x^2g'(x))\Big|_{x=8} = f'(8^2\cdot\tfrac{1}{4})(2\cdot 8\cdot\tfrac{1}{4} + 8^2\cdot 16)$

$= f'(16)(4+1024) = \boxed{-514}$

e) $\frac{d}{dx}\left(f\left(\frac{f(x)}{3g(x)}\right)\right)\Big|_{x=8} = f'\left(\frac{f(x)}{3g(x)}\right)\cdot\dfrac{f'(x)\cdot 3g(x) - 3f(x)g'(x)}{9g(x)^2}\Big|_{x=8}$

$= f'\left(\frac{12}{3\cdot\frac{1}{4}}\right)\cdot\dfrac{-2\cdot\frac{1}{4} - 12\cdot 16}{3(\frac{1}{4})^2} = f'(16)\cdot\frac{-3080}{3} = \boxed{\frac{1540}{3}}$

f) $\frac{d}{dx}\left(f^{-1}(x)\right)\Big|_{x=12} = \dfrac{1}{f'(f^{-1}(12))} = \dfrac{1}{f'(8)} = \boxed{-\frac{1}{2}}$

g) $\frac{d}{dx}\left(f^{-1}(g(x))\right)\Big|_{x=16} = \dfrac{1}{f'(f^{-1}(g(x)))}\cdot g'(x)\Big|_{x=16} = \dfrac{1}{f'(f^{-1}(9))}\cdot 5 = \dfrac{5}{f'(10)} = \boxed{-\frac{5}{3}}$

2. a) $\frac{dw}{dt} = 100\left(\cosh^2(t)+1\right)^{99} \cdot 2\cosh(t)\sinh(t) = \boxed{200\cosh(t)\sinh(t)\left(\cosh^2(t)+1\right)^{99}}$

b) $f'(x) = \frac{1\cdot(2+3x+4x^2) - (1+x)(3+8x)}{(2+3x+4x^2)^2} = \frac{-2-8x-4x^2}{(2+3x+4x^2)^2} = \boxed{\frac{-2(1+4x+2x^2)}{(2+3x+4x^2)}}$

c) $g'(\theta) = -(1+e^{-\theta})^{-2} \cdot -e^{-\theta} = \boxed{\frac{e^{-\theta}}{(1+e^{-\theta})^2}}$

d) $h'(x) = e^{\tan(x)} + xe^{\tan(x)} \cdot \sec^2(x) = \boxed{e^{\tan(x)}\left(1+x\sec^2(x)\right)}$

e) $\frac{dy}{dx} = 2e^{2x} \cdot \sin^2(3x) + e^{2x} \cdot 2\sin(3x) \cdot \cos(3x) \cdot 3 = \boxed{2e^{2x}\sin(3x)\left(\sin(3x)+3\cos(3x)\right)}$

f) $r'(t) = \frac{1}{\sin(\frac{t}{k})} \cdot \cos(\frac{t}{k}) \cdot \frac{1}{k} = \boxed{\frac{\cos(t/k)}{k\sin(t/k)}}$ or $\frac{1}{k}\cot(\frac{t}{k})$

g) $\frac{dy}{dx} = \frac{1}{1+(\arcsin(x))^2} \cdot \frac{1}{\sqrt{1-x^2}} = \boxed{\frac{1}{\sqrt{1-x^2}\left(1+\arcsin^2(x)\right)}}$

3. $\frac{d}{dP}\left[\left(P+\frac{n^2a}{V^2}\right)(V-nb)\right] = \frac{d}{dP}(nRT)$

$\frac{d}{dP}\left(PV - nbP + n^2aV^{-1} - n^3abV^{-2}\right) = 0$

$V + P\frac{dV}{dP} - nb - n^2aV^{-2}\frac{dV}{dP} + 2n^3abV^{-3}\frac{dV}{dP} = 0$

$\left(P - n^2aV^{-2} + 2n^3abV^{-3}\right)\frac{dV}{dP} = nb - V$

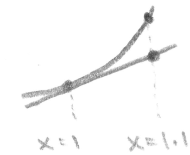
$\frac{dV}{dP} = \frac{nb-V}{P-n^2aV^{-2}+2n^3abV^{-3}} = \boxed{\frac{V^3(nb-V)}{PV^3-n^2aV+2n^3ab}}$

4. $f(x) = xe^x$

a) $m = f'(1) = e^x(1+x)\big|_{x=1} = 2e$; $\quad y_0 = f(1) = e \Rightarrow$ tan line $y = 2e(x-1)+e$
$= 2ex - e$

$\Rightarrow f(1.1) \approx 2e(1.1) - e = \boxed{1.2e}$

$f''(1) = e^x(1+1+x)\big|_{x=1} = 3e > 0 \Rightarrow f$ is concave up at $x=1$

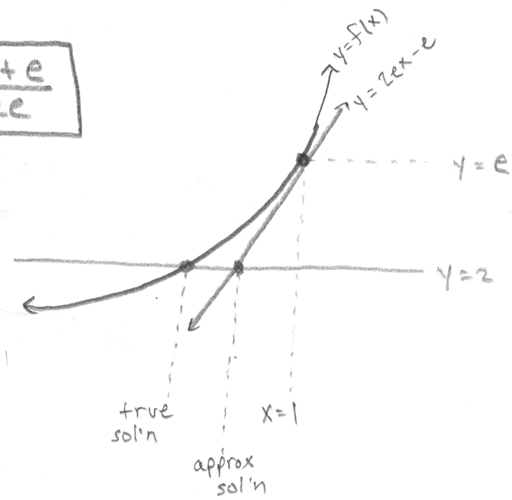


$\Rightarrow$ tangent line approx. is $\boxed{\text{LESS THAN}}$ the true value

b) $f(x) \approx 2ex - e \Rightarrow 2ex - e \approx 2 \Rightarrow \boxed{x \approx \frac{2+e}{2e}}$

$f'(1) > 0$ and $f''(1) > 0 \Rightarrow$

$\Rightarrow$ approx solution is $\boxed{\text{GREATER THAN}}$ true solution.



5. a) $f'(x) = nx^{n-1}e^{-bx} + x^n \cdot -be^{-bx} = x^{n-1}e^{-bx}(n - bx)$

$\Rightarrow$ Critical points at $\boxed{x = \frac{n}{b}}$ and $\boxed{x = 0}$ $\longrightarrow$ b/c $n \geq 1$

$$f''(x) = \frac{d}{dx}\left(e^{-bx}(nx^{n-1} - bx^n)\right) = e^{-bx}(n(n-1)x^{n-2} - nbx^{n-1}) - be^{-bx}(nx^{n-1} - bx^n)$$

$$= x^{n-2}e^{-bx}\left(n(n-1) - 2nbx + b^2x^2\right) \longleftarrow \text{quadratic formula}$$

possible inflection points at $\boxed{x = 0}$ (if $n \geq 2$) and $\boxed{x = \frac{n \pm \sqrt{n}}{b}}$

Second Derivative test: $f''\left(\frac{n}{b}\right) = \left(\frac{n}{b}\right)^{n-2}e^{-n}\left(n^2 - n - 2n^2 + n^2\right) = -n\left(\frac{n}{b}\right)^{n-2}e^{-n} < 0$

$\Rightarrow f$ has a $\boxed{\text{local max}}$ at $x = \frac{n}{b}$

b) $f'(x) = -\frac{2a}{x^3} + 1$; $f''(x) = \frac{6a}{x^4} > 0 \Rightarrow$ always concave up $\boxed{\text{no inflection pts}}$

Crit. pts: $-\frac{2a}{x^3} + 1 = 0 \Rightarrow \boxed{x = \sqrt[3]{2a}}$

$f''\left(\sqrt[3]{2a}\right) > 0 \Rightarrow f\left(\sqrt[3]{2a}\right)$ is a $\boxed{\text{local min}}$

c) $f'(x) = \frac{2}{3}x^{-1/3} + \frac{1}{3}x^{-2/3} = \frac{1}{3}x^{-2/3}\left(2x^{1/3} + 1\right) = \frac{2x^{1/3} + 1}{3x^{2/3}}$

$f''(x) = -\frac{2}{9}x^{-4/3} - \frac{2}{9}x^{-5/3} = -\frac{2}{9}x^{-5/3}\left(x^{1/3} + 1\right) = -\frac{2(x^{1/3} + 1)}{9x^{5/3}}$

Critical points: $f' = 0 \Rightarrow \boxed{x = -\frac{1}{8}}$; f' undefined $\Rightarrow \boxed{x = 0}$

Inflection points: $\boxed{x = -1}$ and possibly $\boxed{x = 0}$

S.D.T. for $x = -\frac{1}{8}$: $f''\left(-\frac{1}{8}\right) = -\frac{2\left(-\frac{1}{2} + 1\right)}{9(-32)} > 0 \Rightarrow f\left(-\frac{1}{8}\right)$ is a $\boxed{\text{local min}}$

$f'' < 0$ | $f'' > 0$ | $f'' < 0$

-1 0 x $\Rightarrow$ $x = 0$ is an inflection point

d) $f(x) = \begin{cases} -\tanh(x) & x \le 0 \\ \tanh(x) & x > 0 \end{cases} \Rightarrow f'(x) = \begin{cases} -\text{sech}^2(x) & x < 0 \\ \text{sech}^2(x) & x > 0 \end{cases} \Rightarrow f'$ is undefined at $x = 0$

$\Rightarrow$ only crit. pt. at $\boxed{x = 0}$

$f' < 0$ | $f' > 0$

$x = 0$

$\Rightarrow f(0)$ is a $\boxed{\text{local min}}$

$f''(x) = \begin{cases} 2\tanh(x)\,\text{sech}^2(x) & x < 0 \\ -2\tanh(x)\,\text{sech}^2(x) & x > 0 \end{cases} \Rightarrow f''(x) \ne 0$, but is undefined at $x = 0$

$\Rightarrow \boxed{\text{no inflection pts}}$

6. a) $x \approx -5$, neither; $x = -2$ local max; $x = -1$ local min; $x = 1$, local max; $x \approx 2.5$ local min

$x = 4$, local max; $x \approx 5.5$, local min

b) $x = -4$, local min; $x = 2$ local max; $x = 4$, neither; $x = 6$ local min