A NUMERICAL METHOD FOR HIGH ENERGY X-RAY SOURCE SHAPE RECONSTRUCTION

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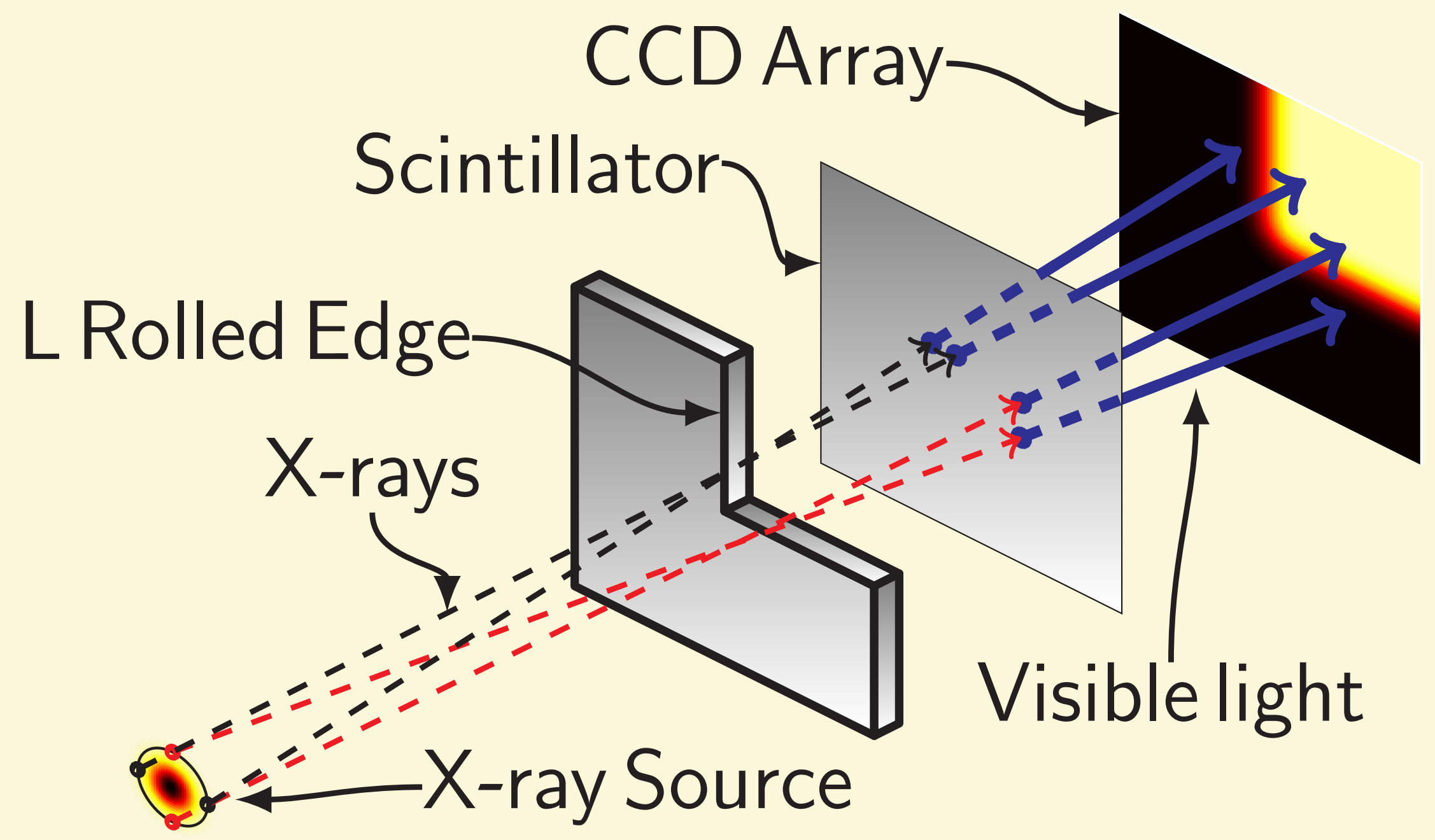
ABSTRACT

Quantifying the source shape of a pulsed power X-ray imaging system is essential in determining a lower bound for the overall resolution of the system. A method of reconstructing the source shape involves solving an inverse problem with an operator matrix of size $n \times n$, where n is the number of pixels in the image. Building and inverting such a large matrix quickly becomes intractable as n increases due to both memory and computational constraints, even in well-posed problems. The ill-posed nature of this problem adds further complications, making direct inversion unstable. Taking advantage of the structure of the operator matrix and utilizing the Projected Newton (PN) and Conjugate Gradient (CG) algorithms, we are able to perform reconstructions of images that are orders of magnitude larger than previously possible.

DATA ACQUISITION

High energy X-rays are pulsed from a source, emitting radiation through the scene containing a test object called an L-Rolled Edge (LRE): an L-shaped 90° aperture with beveled inner edges. Photons not attenuated by the LRE are absorbed by a scintillator which emits photons. The visible light is then focused via an optical system onto a charge-coupled device (CCD), which captures the intensity profile as an image.



The Barnea model [2] for the intensity b at the detector from a spot with spacial profile u is

$$b(s,t) := \mathcal{A}(u)(s,t) = \kappa \iint_{\Omega(s,t)} u(x,y)\, dy\, dx,$$

where $\Omega(s,t)$ is the domain of the CCD, and κ is a geometry dependent constant. This can be discretized as the linear inverse problem

$$\mathbf{b} = \mathbf{A}\mathbf{u} + \epsilon,$$

where

- $\mathbf{b} \in \mathbb{R}^{n\times 1}$: column stacked image,
- $\mathbf{u} \in \mathbb{R}^{n\times 1}$: column stacked source profile,
- $\mathbf{A} \in \mathbb{R}^{n\times n}$: discretization of $\mathcal{A}$, highly structured and lower triangular, and
- $\epsilon \in \mathbb{R}^{n\times 1}$: distributed $N(\mathbf{0}, \lambda^{-1}\mathbf{I})$.

BAYESIAN FORMULATION

We impose a prior distribution on $\mathbf{u}$ with a δ-scaled Laplacian precision and hyperpriors λ, δ, and have likelihood for $\mathbf{b}$ given by

$$\pi(\mathbf{u}|\delta) \sim \mathcal{N}(\mathbf{0}, (\delta\mathbf{L})^{-1}), \quad \pi(\lambda),\ \pi(\delta) \sim \Gamma(\alpha,\beta),$$
$$\pi(\mathbf{b}|\mathbf{u},\lambda) \sim \mathcal{N}(\mathbf{A}\mathbf{u}, \lambda^{-1}\mathbf{I})$$

Here, α and β are chosen so that λ, δ have large variance. Using Bayes' Theorem, the resulting posterior distribution is

$$\pi(\mathbf{u},\lambda,\delta|\mathbf{b}) \propto \lambda^{n/2+\alpha-1}\delta^{n/2+\alpha-1}$$
$$\exp\left[-\tfrac{\lambda}{2}\left|\left|\mathbf{b}-\mathbf{A}\mathbf{u}\right|\right|_2^2 - \tfrac{\delta}{2}\left|\left|\mathbf{L}^{\frac{1}{2}}\mathbf{u}\right|\right|_2^2 - \beta\lambda - \beta\delta\right],$$

which is a difficult distribution from which to directly sample. We use the following Gibbs sampler to obtain samples from this posterior.

1) Set λ_0, δ_0, max iterations N, and $k = 0$.
2) Let $\mathbf{R}_k = \lambda_k\mathbf{A}^*\mathbf{A} + \delta_k\mathbf{L}$, and $\mathbf{d}_k = \lambda_k\mathbf{A}^*\mathbf{b}$. Compute a constrained sample via PN as
$$\mathbf{u}_k \sim \mathcal{N}(\mathbf{R}_k^{-1}\mathbf{d}_k, \mathbf{R}_k^{-1})$$
3) Compute samples from
$$\delta_{k+1} \sim \Gamma\left(\tfrac{n_k}{2}+\alpha, \tfrac{1}{2}\mathbf{u}_k^*\mathbf{L}\mathbf{u}_k + \beta\right)$$
$$\lambda_{k+1} \sim \Gamma\left(\tfrac{n}{2}+\alpha, \tfrac{1}{2}\left|\left|\mathbf{A}\mathbf{u}_k - \mathbf{b}\right|\right|^2 + \beta\right)$$
where n_k counts the nonzero entries in $\mathbf{u}_k$.
4) Increment k, and return to 2) if $k < N$.

The computation in 2) involves solving the normal equation

$$(\lambda_k\mathbf{A}^*\mathbf{A} + \delta_k\mathbf{L})\,\mathbf{u}_k = \lambda_k\mathbf{A}^*\mathbf{b},$$

which is a symmetric positive definite system. This is the computationally expensive step, and many iterations of the sampler are required to produce a mean reconstruction. Non-negativity constraints [1] are imposed in step 2).

NON-NEGATIVITY

It is possible to define $\mathbf{A}$, $\mathbf{A}^*$, and $\mathbf{L}$ as functions that act on vectors, as opposed to matrices. Since we have a symmetric positive definite system, we can use both the CG and PN algorithms. The PN algorithm is an iterative method which solves the quadratic optimization problem

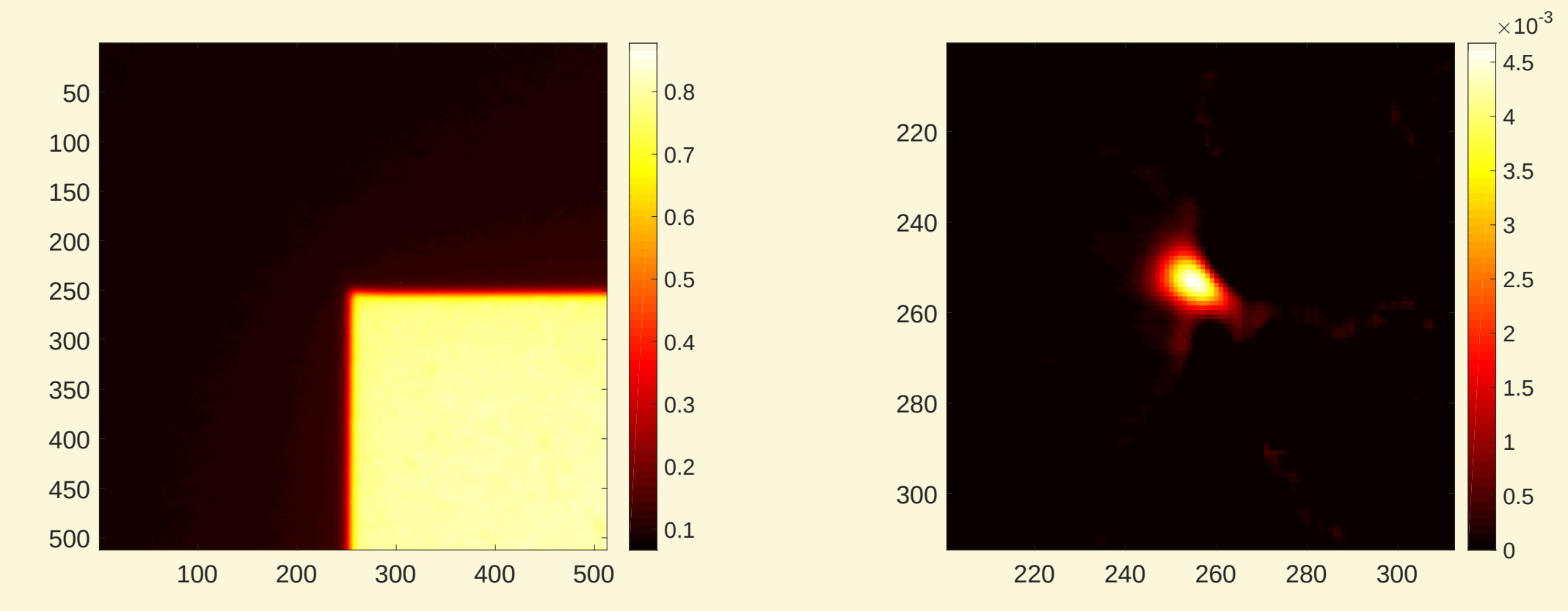
$$\mathbf{u}_k = \underset{\mathbf{u}\geq\mathbf{0}}{\operatorname{argmin}} \left\{\tfrac{1}{2}\mathbf{u}\mathbf{R}_k\mathbf{u} - \mathbf{u}^*(\mathbf{d}_k + \mathbf{w}_k)\right\},$$

where $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k^{-1})$ by a pseudo-Newton iteration of the form

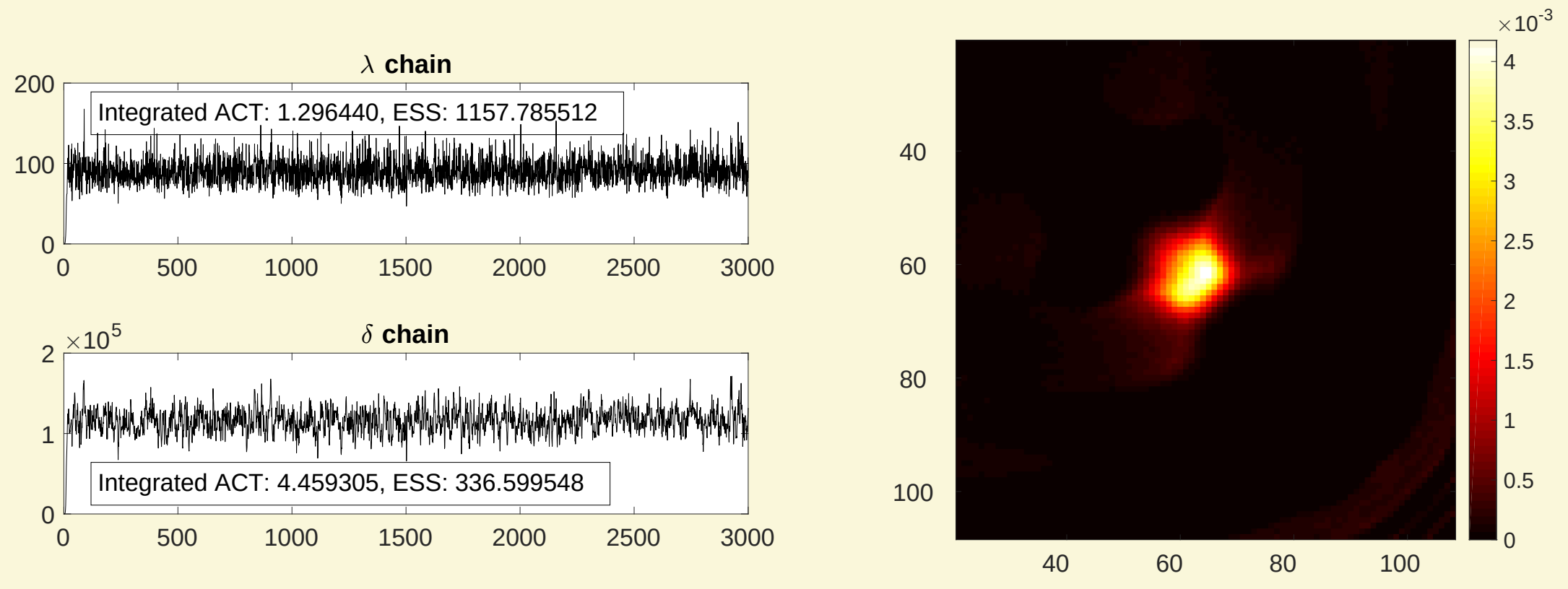
$$\mathbf{u}_{\ell+1} = P\left(\mathbf{u}_\ell - \gamma\mathbf{H}_{\mathbf{R}_k}^{-1}\left(\mathbf{R}_k\mathbf{u}_\ell - \mathbf{d}_k\right)\right),$$

where P is a non-negative projection, $\mathbf{H}_{\mathbf{R}_k}$ is a reduced Hessian dependent on $\mathbf{u}_k$, and γ is a step size. The inversion of the reduced Hessian is computed via CG [3].

RECONSTRUCTION RESULTS



(a) Image data (b) Source reconstruction



(c) δ and λ chains (d) Sampled mean reconstruction

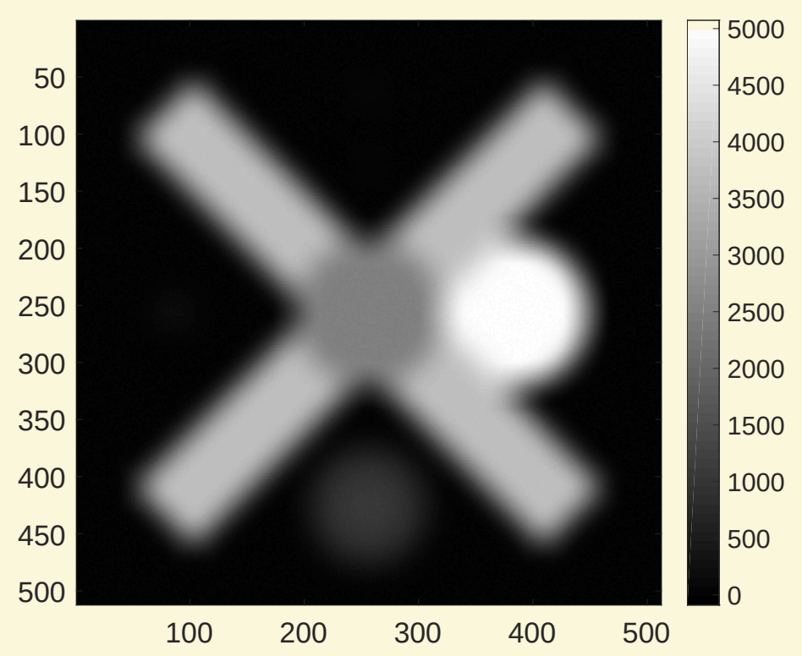
LRE data from Cygnus is shown in (a). A full sized source shape reconstruction is shown in (b). The hyperprior chains (c) and mean source shape reconstruction (d) were produced from the center 128×128 pixels of the image data.

FUTURE WORK

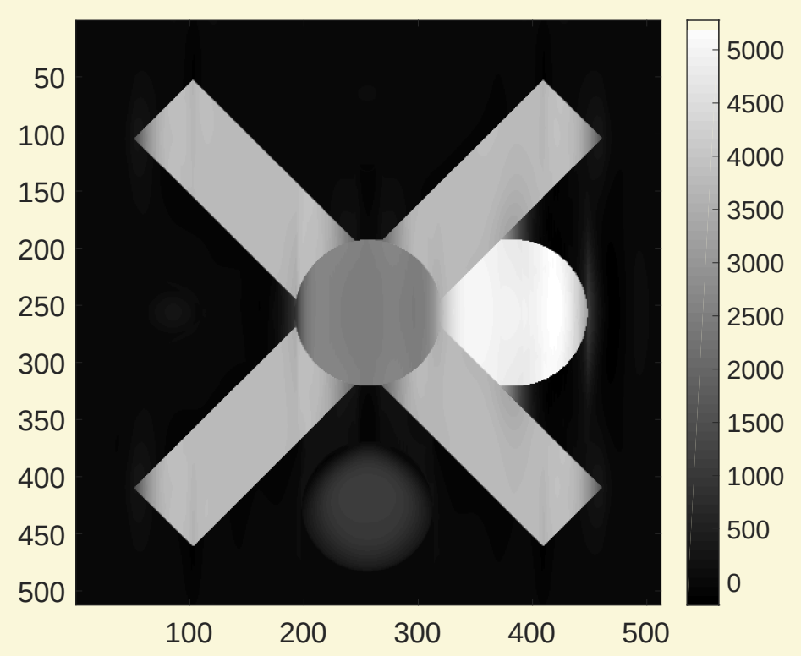
Deconvolution is another ill-posed linear inverse problem that can be formulated in a similar manner. Here, the operator $\mathcal{A}$ corresponds to integral convolution:

$$b(s,t) := \mathcal{A}(u)(s,t) = a(s,t) * u(s,t)$$
$$= \int_\Omega a(s-x, t-y)u(x,y)\, dx\, dy$$

and $\mathbf{A}$ is the discretization.



Blurred Image



Reconstructed Image

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[3] M. Fowler, M. Howard, A. Luttman, S. E. Mitchell, and T. J. Webb, A Stochastic Approach to Quantifying the Blur with Uncertainty for High-energy X-ray Imaging Systems, *Inverse Probl. Sci. Engineering*, 24 (2016).

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