THE MODULI STACK OF (ALL) CURVES

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Abstract. The purpose of this note is to prove that there is an algebraic stack $\mathcal{U}$ parameterizing all curves. The curves that appear in the algebraic stack $\mathcal{U}$ are allowed to be arbitrarily singular, non-reduced, disconnected, and reducible. We also prove the boundedness of the open substack of $\mathcal{U}$ parameterizing geometrically connected curves with fixed arithmetic genus g and $\leq e$ irreducible components.

1. Introduction

The purpose of this note is twofold. First, to give an elementary proof that the moduli stack of all curves is algebraic. Second, to prove a boundedness result which does not appear elsewhere in the literature. This is an update to [Smy12, App. B]. Let $\mathcal{U}$ be the functor from schemes to groupoids defined by:

$$\mathcal{U}(T) := \left\{ \begin{array}{c} \text{Flat, proper, finitely-presented morphisms of algebraic} \\ \text{spaces } \mathcal{C} \to T\text{, with one-dimensional geometric fibers} \end{array} \right\}.$$

$\mathcal{U}$ is obviously a stack over the category of schemes in the étale topology. The following theorem is well-known to experts and a proof depending on a variant of Artin's criterion appears in [JHS11, Proposition 2.3]. Here, we give a very elementary argument, using Hilbert schemes to explicitly construct an atlas for $\mathcal{U}$.

Theorem 1.1. *$\mathcal{U}$ is an algebraic stack, locally of finite type over* $\operatorname{Spec}\mathbb{Z}$*, with quasicompact and separated diagonal.*

Before we prove Theorem 1.1, we require the following lemma.

Lemma 1.2. *Let $\pi : C \to S$ be a proper, finitely presented morphism of algebraic spaces. Let $s \in S$ be a closed point such that $\dim_{\kappa(s)} C_s \leq 1$, then there is an étale neighbourhood $(U, \overline{u})$ of $(S, \overline{s})$ such that $C \times_S U \to U$ is projective.*

Proof. By standard limit methods, one quickly reduces to the following situation: $S = \operatorname{Spec} R$, where R is an excellent, strictly henselian local ring, and $s \in S$ is the unique closed point.

First, assume that C is a reduced scheme. Now, let $C_s \to \operatorname{Spec}\kappa(s)$ denote the special fiber of $C \to S$. Since C_s is a proper scheme of dimension 1 over a field, it is projective [Har77, Ex. III.5.8]. Thus, it suffices to show that the map $\operatorname{Pic}(C) \to \operatorname{Pic}(C_s)$ is surjective. Indeed, then C admits a line bundle $\mathscr{L}$ such that the restriction to the central fiber is projective. By [EGA, III.4.7.1], $\mathscr{L}$ is ample.

For this paragraph we utilize the arguments of [SGA4$\frac{1}{2}$, Prop. IV.4.1]. Let $\mathscr{L}_s$ be a line bundle on C_s. Since $C_s \to \operatorname{Spec}\kappa(s)$ is a projective curve, to show that $\operatorname{Pic}(C) \to \operatorname{Pic}(C_s)$ is surjective, it suffices to treat the case where $\mathscr{L}_s = \mathscr{O}_{C_s}(-x_s)$, for some closed point $x_s \in C_s$. In an open neighborhood U_s of $x_s \in C_s$, we have that $x_s = V(f_s) \cap U_s$, for some $f_s \in \mathscr{O}_{C_s}(U_s)$ which is not a 0-divisor. Since $C_s \hookrightarrow C$ is a closed immersion, there is an open subscheme $U \subset C$ such that

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$U \cap C_0 = U_0$. By shrinking U, we may lift the equation $f_s \in \mathscr{O}_{C_s}(U_s)$ to $f \in \mathscr{O}_C(U)$ such that f is not a 0-divisor, and $V(f) \cap U \cap C_s = \{x_s\}$. In particular, the map $V(f) \cap U \to S$ is quasi-finite and separated. Since S is local and strictly henselian, by [EGA, IV.18.12.3], there is a decomposition $V(f) \cap U = V_1 \amalg V_2$, where $V_1 \to S$ is *finite* and contains $\pi^{-1}(s)$. Thus, by further shrinking U, we may assume that the map $V(f) \cap U \to S$ is finite. On C we may now define an effective cartier divisor D as $D|_{C-[V(f)\cap U]} = 0$ and $D|_U = \operatorname{div} f$. The cartier divisor $\mathscr{O}(-D)$ has the property that $\mathscr{O}_{C_s}(-D) = \mathscr{O}_{C_s}(-x_s)$. Since C is reduced and noetherian, by [EGA, IV.21.3.4], $\operatorname{Pic}(C) \to \operatorname{Pic}(C_s)$ is surjective.

If C is a now assumed to be a non-reduced scheme, and C_{red} is the reduction, then we have shown that the morphism $C_{\mathrm{red}} \to S$ is projective. If $\mathscr{I}$ denotes the nilradical of C, then there is a k such that $\mathscr{I}^k = (0)$. Thus, it suffices to prove the following: if $i : C' \to C$ is a closed immersion defined by a square zero ideal $\mathscr{J}$ such that C' is projective, then C is projective. To this end, we recall the exponential sequence on C:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{J} \arrow[r, "a \mapsto 1+a"] & \mathscr{O}_C^{\times} \arrow[r] & i_*\mathscr{O}_{C'}^{\times} \arrow[r] & 1 . \end{tikzcd}$$

By taking the long exact sequence of cohomology, we see that the obstruction to lifting a line bundle on C' to a line bundle on C lies in the cohomology group $H^2(C', \mathscr{J})$. Since, C' is a projective S-curve, $H^2(C', \mathscr{J}) = 0$. Consequently, we deduce that $\operatorname{Pic} C \to \operatorname{Pic} C'$ is surjective. Hence, we may lift an ample bundle on C' to a line bundle on C, and any such lift must be ample [EGA, II.4.5.13].

We now treat the case where C is an algebraic space, and it remains to show that it is a scheme. By [LMB, Thm. 16.6], there is a finite and surjective S-map $\widetilde{C} \to C$, where $\widetilde{C}$ is a scheme. Since $\widetilde{C}$ is a proper S-scheme, with special fiber of dimension ≤ 1, we may conclude that $\widetilde{C}$ is a projective S-scheme. In particular, $\widetilde{C}$ has the Chevalley-Kleiman property (i.e. every finite set of points is contained in an open affine). Since S is excellent, we may apply [Kol08, Cor. 48] to conclude that C has the Chevalley-Kleiman property, thus is a scheme. □

Remark 1.3. D. Fulghesu has given an example [Ful09, Example 2.3] of a proper, flat, and finitely presented morphism of algebraic spaces $\pi : \mathcal{C} \to T$ with one-dimensional fibers, which is not representable by schemes. In particular, the morphism π is not Zariski locally projective.

We can now prove Theorem 1.1.

Proof of Theorem 1.1. For any integer $m \geq 1$, let H^m denote the Hilbert scheme of $\mathbb{P}^m_{\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$. Let $H^m_{\mathcal{U}} \subset H^m$ be the subfunctor corresponding to those closed immersions $(i : \mathcal{C} \hookrightarrow \mathbb{P}^m_T)$ satisfying:

(a) the induced morphism $\mathcal{C} \to T$ is an object of $\mathcal{U}(T)$,
(b) for all $t \in T$, $H^1(\mathcal{C}_t, \mathscr{O}_{\mathcal{C}_t}(1)) = 0$.

By Cohomology and Base Change [Har77, Thm. 12.11], the inclusion $H^m_{\mathcal{U}} \subset H^m$ is representable by open immersions. Thus, $H^m_{\mathcal{U}}$ is represented by a scheme, locally of finite type over $\operatorname{Spec}\mathbb{Z}$. Set $U = \amalg_{m\geq 1} H^m_{\mathcal{U}}$, then there is an induced 1-morphism $U \to \mathcal{U}$. By [LMB, Prop. 4.3.2 & Lem. 4.3.3] it remains to prove that:

1. the map $R = U \times_{\mathcal{U}} U$ is a scheme, locally of finite type over $\operatorname{Spec}\mathbb{Z}$;
2. the morphism $R \to U \times U$ is quasicompact and separated;
3. for any 1-morphism $T \to \mathcal{U}$, where T is a scheme, there exists an étale cover $T' \to T$ and a factorization of $T' \to \mathcal{U}$ through $U \to \mathcal{U}$;
4. the two projections $R = U \times_{\mathcal{U}} U \rightrightarrows U$ are smooth.

For (1), we first note that $R = \amalg_{r,m\geq 0} H^r_{\mathcal{U}} \times_{\mathcal{U}} H^m_{\mathcal{U}}$. For any integers r, $m \geq 0$, the 2-fiber product $I_{r,m} := H^r_{\mathcal{U}} \times_{\mathcal{U}} H^m_{\mathcal{U}}$ is the sheaf of isomorphisms between the associated projective families of curves. Thus, $I_{r,m}$—and consequently R—are representable by schemes, locally of finite type over $\operatorname{Spec}\mathbb{Z}$ [Kol96, Thm. 1.10]. Next, we observe that claim (2) follows from [JHS11, Proof of Prop. 3.3].

For (3) we fix a 1-morphism $T \to \mathcal{U}$, where T is a scheme. This is equivalent to a morphism of algebraic spaces $\pi : \mathcal{C} \to T$ which is proper, flat, and finitely presented, with one-dimensional geometric fibers. By Lemma 1.2 we may also assume that π is a projective morphism of quasicompact schemes. Let $\mathscr{L}$ be line bundle on $\mathcal{C}$ which is relatively π-ample. We may replace $\mathscr{L}$ by some tensor power for which $\mathscr{L} \cong i^*\mathscr{O}_{\mathbb{P}^m_T}(1)$ for a closed immersion $i : \mathcal{C} \hookrightarrow \mathbb{P}^m_T$ and $H^1(\mathcal{C}_t, \mathscr{L}_t) = 0$ for all $t \in T$. Thus, we see that $T \to \mathcal{U}$ factors through $U \to \mathcal{U}$, as required.

To prove (4), we first fix $m \geq 1$. Then, we fix a square zero closed immersion of local artinian schemes $S \hookrightarrow S'$ such that $\ker(\mathscr{O}_{S'} \to \mathscr{O}_S) \cong \kappa(s)$. Now consider the following lifting problem:

$$\begin{tikzcd}
S \arrow[r] \arrow[d, hook] & H^m_{\mathcal{U}} \arrow[d] \\
S' \arrow[r] \arrow[ur, dashed] & \mathcal{U}.
\end{tikzcd}$$

By [EGA, IV.17.14.2], (4) will be proved if there is always an arrow making the preceding diagram 2-commute. Equivalently, we must show that the following diagram may always be completed:

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, hook, "i"] \arrow[dd, bend right=60, "\pi"'] & \mathcal{C}' \arrow[d, dashed] \arrow[dd, bend left=60, "\pi'"] \\
\mathbb{P}^m_S \arrow[r, hook] \arrow[d] & \mathbb{P}^m_{S'} \arrow[d] \\
S \arrow[r, hook] & S',
\end{tikzcd}$$

where $\pi, \pi' \in \mathcal{U}$, $i : \mathcal{C} \to \mathbb{P}^m_S$ is a closed immersion satisfying $H^1(\mathcal{C}_s, \mathscr{O}_{\mathcal{C}_s}(1)) = 0$, and all squares are cartesian. By [Ill71, III.2.2.4] there is an element of the group:

$$\operatorname{Ext}^1_{\mathscr{O}_{\mathcal{C}}}(i^*\Omega_{\mathbb{P}^m_S/S}, \pi^*\kappa(s)) \cong H^1(\mathcal{C}, \operatorname{Hom}_{\mathscr{O}_{\mathcal{C}}}(i^*\Omega_{\mathbb{P}^m_S/S}, \pi^*\kappa(s))) \cong H^1(\mathcal{C}_s, i_s^*T_{\mathbb{P}^m_{\kappa(s)}/\kappa(s)}),$$

which is zero if and only if the preceding diagram may be completed. The Euler exact sequence [Har77, Thm. II.8.13] shows that this group vanishes, giving us the claim. □

Let $\mathcal{U}_{g,n}$ be the stack of n-pointed, reduced, and connected curves of arithmetic genus g.

Corollary 1.4. *$\mathcal{U}_{g,n}$ is an algebraic stack, locally of finite type over* $\operatorname{Spec}\mathbb{Z}$*, with quasicompact and separated diagonal.*

Proof. Let $\mathcal{U}_n$ denote the stack of n-pointed curves. The 1-morphism $\mathcal{U}_1 \to \mathcal{U}$ is representable by finitely presented algebraic spaces (it is the universal curve). Combining Corollary 1.2 with [EGA, III.7.9.4] and [EGA, IV.12.2.1(viii)] shows that the 1-morphism $\mathcal{U}_{g,0} \to \mathcal{U}$ is representable by open immersions. For $n \geq 1$ we have that $\mathcal{U}_{g,n} = \mathcal{U}_1 \times_{\mathcal{U}} \mathcal{U}_{g,n-1}$, so by Theorem 1.1 the claim follows. □

The following boundedness lemma is needed to show that the substack of curves with a bounded number of irreducible components is of finite type.

Lemma 1.5. *There exists an integer $D_{g,e}$, depending only on g and e, such that any reduced curve of arithmetic genus g with no more than e irreducible components admits a degree d embedding into $\mathbb{P}^{D_{g,e}}$ for some $d \leq D_{g,e}$.*

Proof. It is sufficient to show that there exists an integer $D_{g,e}$ such that any reduced curve of arithmetic genus g with no more than e irreducible components admits a very ample line bundle $\mathscr{L}$ with degree $d \leq D_{g,e}$ and $H^1(C,\mathscr{L}) = 0$.

Given a curve C satisfying the hypotheses of the Lemma, let $Z \subset C$ be an effective Cartier divisor whose support meets the smooth locus of every irreducible component of C. Since C has no more than e irreducible components, we may assume that $\deg Z \leq e$. Let $\mathscr{L} := \mathscr{O}(Z)$. It suffices to exhibit an integer $m := m(g,e)$, depending only on g and e, such that $\mathscr{L}^m$ is very ample and $H^1(C,\mathscr{L}^m) = 0$. Indeed, we may take $D_{g,e} = me$.

To show that $\mathscr{L}^m$ separates points and tangent vectors, it is sufficient to show that, for any $p \in C$:

$$H^1(C,\mathscr{L}^m \otimes \mathfrak{m}_p) = H^1(C,\mathscr{L}^m \otimes \mathfrak{m}_p^2) = 0.$$

Clearly, the latter vanishing implies the former. The former vanishing also implies that $H^1(C,\mathscr{L}^m) = 0$. Given $p \in C$, let $\pi^{-1}(p) = p_1 + \ldots + p_r$, where $\pi : \tilde{C} \to C$ is the normalization of C. Let $\delta(p)$ denote the δ-invariant of p. We have an exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \pi_*\mathscr{O}_{\tilde{C}}(-2\delta(p)(p_1+\ldots+p_r)) \arrow[r] & \mathfrak{m}_p^2 \arrow[r] & \mathscr{E} \arrow[r] & 0, \end{tikzcd}$$

where $\mathscr{E}$ is a coherent sheaf supported at p. Twisting by $\mathscr{L}^m$ and taking cohomology, we obtain another exact sequence:

$$\begin{tikzcd} H^1(C,\mathscr{L}^m \otimes \pi_*\mathscr{O}_{\tilde{C}}(-2\delta(p)(p_1+\ldots+p_r))) \arrow[r] & H^1(C,\mathscr{L}^m \otimes \mathfrak{m}_p^2) \arrow[r] & 0. \end{tikzcd}$$

By the projection formula:

$$H^1(C,\mathscr{L}^m \otimes \pi_*\mathscr{O}_{\tilde{C}}(-2\delta(p)(p_1+\ldots+p_r))) = H^1(\tilde{C},(\pi^*\mathscr{L})^m(-2\delta(p)(p_1+\ldots+p_r))),$$

which vanishes for $m > 2g-2+2\delta(p)r$. Since $\delta(p) \leq g+e-1$ and $r \leq \delta(p)+1$, we may take $m(g,e) := 2g-2+2(g+e)(g+e-1)$. □

Denote by $\mathcal{U}_{g,n,e}$ the substack of $\mathcal{U}_{g,n}$ having objects those $(\pi,\{\sigma_i\}_{i=1}^n) \in \mathcal{U}_{g,n}$ such that the geometric fibers of π have no more than e irreducible components.

Corollary 1.6. *$\mathcal{U}_{g,n,e}$ is an algebraic stack, of* finite *type over* $\operatorname{Spec}\mathbb{Z}$*, with quasi-compact and separated diagonal.*

Proof. $\mathcal{U}_{g,n,e}$ is an open substack of $\mathcal{U}_{g,n}$ by [EGA, IV.12.2.1(xi)]. By Lemma 1.5, there is an open subscheme of the Hilbert scheme of genus g curves in $\mathbb{P}^{D_{g,e}}$ with degree $\leq D_{g,e}$, mapping surjectively onto $\mathcal{U}_{g,n,e}$. Thus, $\mathcal{U}_{g,n,e}$ is of finite type. □

We conclude with two corollaries that were omitted from [Smy12, App. B].

Corollary 1.7. *If $\mathcal{C} \to \operatorname{Spec}\Bbbk$ is a projective curve, then it has a versal deformation space defined by equations with integral coefficients.*

Note that Corollary 1.7 is a trivial corollary of Theorem 1.1, yet at face value it is entirely non-obvious. For example, if you were to consider a complex curve $\mathcal{C} \to \operatorname{Spec}\mathbb{C}$, with defining equations in some embedding into $\mathbb{P}^N_{\mathbb{C}}$ having lots of transcendental terms, then you would certainly not expect the deformation theory to be governed by equations with integral coefficients. Since the versal deformation of a rigid curve is itself, we immediately obtain the following partial answer to a speculation of R. Vakil in [Vak06]:

Corollary 1.8. *If $\mathcal{C} \to \operatorname{Spec}\Bbbk$ is a rigid, projective curve, then every singularity type of $\mathcal{C}$ is defined over $\mathbb{Z}$.*

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