THE ÉTALE LOCAL STRUCTURE OF ALGEBRAIC STACKS

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Abstract. We prove that an algebraic stack, locally of finite presentation and quasi-separated over a quasi-separated algebraic space with affine stabilizers, is étale locally a quotient stack around any point with a linearly reductive stabilizer. This result generalizes the main result of [AHR19] to the relative setting and the main result of [AOV11] to the case of non-finite inertia. We also provide various coherent completeness and effectivity results for algebraic stacks as well as structure theorems for linearly reductive groups schemes. Finally, we provide several applications of these results including generalizations of Sumihiro's theorem on torus actions and Luna's étale slice theorem to the relative setting.

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1. Introduction

1.1. **A local structure theorem.** One of the main theorems in this paper provides a local description of many algebraic stacks:

Theorem 1.1 (Local structure). *Suppose that:*

- *S is a quasi-separated algebraic space;*
- *$\mathcal{X}$ is an algebraic stack, locally of finite presentation and quasi-separated over S, with affine stabilizers;*
- *$x \in |\mathcal{X}|$ is a point with residual gerbe $\mathcal{G}_x$ and image $s \in |S|$ such that the residue field extension $\kappa(x)/\kappa(s)$ is finite; and*
- *$h_0 \colon \mathcal{W}_0 \to \mathcal{G}_x$ is a smooth (resp., étale) morphism where $\mathcal{W}_0$ is a gerbe over the spectrum of a field and has linearly reductive stabilizer.*

Then there exists a cartesian diagram of algebraic stacks

$$\begin{tikzcd} \mathcal{G}_w = \mathcal{W}_0 \arrow[r, "h_0"] \arrow[d] & \mathcal{G}_x \arrow[d] \\ {[\operatorname{Spec} A/\mathrm{GL}_n]} = \mathcal{W} \arrow[r, "h"] & \mathcal{X} \end{tikzcd}$$

where $h \colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ is a smooth (resp., étale) pointed morphism and w is closed in its fiber over s. Moreover, if $\mathcal{X}$ has separated (resp., affine) diagonal and h_0 is representable, then h can be arranged to be representable (resp., affine).

Remark 1.2. In the case that $\mathcal{X}$ has finite inertia and h_0 is an isomorphism, this theorem had been established in [AOV11, Thm. 3.2].

In Corollary 17.4, we provide more refined descriptions of the stack $\mathcal{W}$ in terms of properties of the gerbe $\mathcal{W}_0$. For example, (a) if $\mathcal{W}_0$ is affine over a linearly reductive gerbe $\mathcal{G}_0$, then $\mathcal{W}$ is affine over a gerbe $\mathcal{G}$ extending $\mathcal{G}_0$, and (b) if $\mathcal{W}_0 \cong [\operatorname{Spec} B/G_0]$ where $G_0 \to \operatorname{Spec} \kappa(w)$ is a linearly reductive group scheme, then there exists a smooth (resp. étale if $\kappa(x)/\kappa(s)$ is separable) morphism $(S', s') \to (S, s)$ with $\kappa(s') = \kappa(w)$ such that $\mathcal{W} \cong [\operatorname{Spec} C/G]$ where $G \to S'$ is a geometrically reductive group scheme with $G_{s'} \cong G_0$. Moreover, except in bad mixed characteristic situations, the gerbe $\mathcal{G}$ in (a) and the group scheme $G \to S'$ in (b) are linearly reductive, and the adequate moduli space $\mathcal{W} \to \operatorname{Spec} A^{\mathrm{GL}_N}$ is a good moduli space. Over a field, with h_0 an isomorphism, the theorem takes the following form:

Theorem 1.3. *Let $\mathcal{X}$ be a quasi-separated algebraic stack which is locally of finite type over a field k with affine stabilizers. Let $x \in |\mathcal{X}|$ be a point with linearly reductive stabilizer such that its residue field $\kappa(x)$ is finite over k. Then there exists an algebraic stack $\mathcal{W}$ affine over the residual gerbe $\mathcal{G}_x$ of x, a point $w \in |\mathcal{W}|$, and an étale morphism $h \colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ inducing an isomorphism of residual gerbes at w. Moreover, if $\mathcal{X}$ has separated (resp., affine) diagonal, then h can be arranged to be representable (resp., affine).*

Remark 1.4. If $x \in |\mathcal{X}|$ is a k-point, then the residual gerbe $\mathcal{G}_x$ is neutral and the theorem gives an étale morphism $h \colon ([\operatorname{Spec} A/G_x], w) \to \mathcal{X}$ inducing an isomorphism of stabilizer groups at w. This had been established in [AHR19, Thm. 1.1] in the case that k is algebraically closed.

The proof of Theorem 1.1 is given in Section 12 and follows the same general strategy as the proof of [AHR19, Thm. 1.1]:

(1) We begin by constructing smooth infinitesimal deformations $h_n \colon \mathcal{W}_n \to \mathcal{X}_n$ where $\mathcal{X}_n$ is the nth infinitesimal neighborhood of $\mathcal{G}_x$ in $\mathcal{X}$. This follows by standard infinitesimal deformation theory.

(2) We show that the system $\mathcal{W}_n$ effectivizes to a *coherently complete* stack $\widehat{\mathcal{W}}$. This is Theorem 1.10.
(3) Tannaka duality [HR19] (see also §1.7.6) then gives us a unique formally smooth morphism $\widehat{h}\colon \widehat{\mathcal{W}} \to \mathcal{X}$.
(4) Finally we apply equivariant Artin algebraization [AHR19, App. A] to approximate $\widehat{h}$ with a smooth morphism $h\colon \mathcal{W} \to \mathcal{X}$.

Steps (3) and (4) are satisfactorily dealt with in [HR19] and [AHR19]. Step (2) is the main technical result of this paper. Theorem 1.10 is far more general than the related results in [AHR19]—even over an algebraically closed field. Steps (1)–(3) are summarized in Theorem 1.11.

The equivariant Artin algebraization results established in [AHR19, App. A] are only valid when $\mathcal{W}_0$ is a gerbe over a point and the morphism $\mathcal{W}_0 \to \mathcal{G}_x$ is smooth. In future work with Halpern-Leistner [AHHR], we will remove these restrictions and also replace $\mathcal{G}_x$ with other substacks. With these results, we can also remove the assumption that $\kappa(x)/\kappa(s)$ is finite in Theorem 1.1.

1.2. **Coherent completeness.** The following definition first appeared in [AHR19, Defn. 2.1].

Definition 1.5. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed immersion of noetherian algebraic stacks. We say that the pair $(\mathcal{X}, \mathcal{Z})$ is *coherently complete* (or $\mathcal{X}$ is *coherently complete along* $\mathcal{Z}$) if the natural functor

$$\mathsf{Coh}(\mathcal{X}) \to \varprojlim_n \mathsf{Coh}(\mathcal{X}_{\mathcal{Z}}^{[n]}),$$

from the abelian category of coherent sheaves on $\mathcal{X}$ to the category of projective systems of coherent sheaves on the nth nilpotent thickenings $\mathcal{X}_{\mathcal{Z}}^{[n]}$ of $\mathcal{Z} \subseteq \mathcal{X}$, is an equivalence of categories.

The following statement was an essential ingredient in all of the main results of [AHR19]: if A is a noetherian k-algebra, where k is a field, and G is a linearly reductive affine group scheme over k acting on $\operatorname{Spec} A$ such that there is a k-point fixed by G and the ring of invariants A^G is a complete local ring, then the quotient stack $[\operatorname{Spec} A/G]$ is coherently complete along the residual gerbe of its unique closed point [AHR19, Thm. 1.3]. For further examples of coherent completeness, see §3.3.

In this article, coherent completeness also features prominently and we need to generalize [AHR19, Thm. 1.3]. To this end, we establish the following theorem where we do not assume a priori that $\mathcal{X}$ has the resolution property, only that the closed substack $\mathcal{Z}$ does.

Theorem 1.6 (Coherent completeness)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal and good moduli space $\pi\colon \mathcal{X} \to X = \operatorname{Spec} A$. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack defined by a coherent ideal $\mathcal{I}$. Let $I = \Gamma(\mathcal{X}, \mathcal{I})$. If $\mathcal{Z}$ has the resolution property, then $\mathcal{X}$ is coherently complete along $\mathcal{Z}$ if and only if A is I-adically complete. If this is the case, then $\mathcal{X}$ has the resolution property.*

An important special case of this theorem is established in Section 5, and the proof is finished in Section 10. The difference between the statement above and formal GAGA for good moduli space morphisms is that the statement above asserts that $\mathcal{X}$ is coherently complete along $\mathcal{Z}$ and not merely along $\pi^{-1}(\pi(\mathcal{Z}))$. Indeed, as a consequence of this theorem, we can easily deduce the following version of formal GAGA (Corollary 1.7), which had been established in [GZB15, Thm. 1.1] with the additional hypotheses that $\mathcal{X}$ has the resolution property and I is maximal, and in [AHR19, Cor. 4.14] in the case that $\mathcal{X}$ is defined over a field and I is maximal.

Corollary 1.7 (Formal GAGA)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal. Suppose there exists a good moduli space $\pi\colon \mathcal{X} \to \operatorname{Spec} R$, where R is noetherian and I-adically complete. Suppose that either*

1. *$I \subseteq R$ is a maximal ideal; or*
2. *$\mathcal{X} \times_{\operatorname{Spec} R} \operatorname{Spec}(R/I)$ has the resolution property.*

Then $\mathcal{X}$ has the resolution property and the natural functor

$$\mathsf{Coh}(\mathcal{X}) \to \varprojlim \mathsf{Coh}\big(\mathcal{X} \times_{\operatorname{Spec} R} \operatorname{Spec}(R/I^{n+1})\big)$$

is an equivalence of categories.

1.3. **Effectivity.** The key method to prove many of the results in this paper is an effectivity result for algebraic stacks. This is similar in spirit to Grothendieck's result on algebraization of formal schemes [EGA, III.5.4.5].

Definition 1.8. A diagram

$$\mathcal{X}_0 \hookrightarrow \mathcal{X}_1 \hookrightarrow \dots$$

is called an *adic sequence* if for each $i \leq j$ there are compatible closed immersions of noetherian algebraic stacks $u_{ij}\colon \mathcal{X}_i \hookrightarrow \mathcal{X}_j$ such that if $\mathcal{I}_{(j)}$ denotes the coherent sheaf of ideals defining u_{0j}, then $\mathcal{I}_{(j)}^{i+1}$ defines u_{ij}.

The sequence of infinitesimal thickenings of a closed substack of a noetherian algebraic stack is adic.

Definition 1.9. Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence of algebraic stacks. An algebraic stack $\widehat{\mathcal{X}}$ is a *completion* of $\{\mathcal{X}_n\}$ if

1. there are compatible closed immersions $\mathcal{X}_n \hookrightarrow \widehat{\mathcal{X}}$ for all n;
2. $\widehat{\mathcal{X}}$ is noetherian with affine diagonal; and
3. $\widehat{\mathcal{X}}$ is coherently complete along $\mathcal{X}_0$.

By Tannaka duality (see §1.7.6), the completion is unique if it exists. Moreover, Tannaka duality implies that if the completion exists, then it is the colimit of $\{\mathcal{X}_n\}_{n\geq 0}$ in the category of noetherian stacks with quasi-affine diagonal (and in the category of algebraic stacks with affine stabilizers if $\mathcal{X}_0$ is excellent).

The following result, which has no precursor for stacks, is our main effectivity theorem. The reader is directed to Definition 2.7 for the definition of *linearly fundamental* stacks.

Theorem 1.10 (Effectivity)**.** *Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence of noetherian algebraic stacks. If $\mathcal{X}_0$ is linearly fundamental, then the completion $\widehat{\mathcal{X}}$ exists and is linearly fundamental.*

We prove Theorem 1.10 in three stages of increasing generality. The case of characteristic zero is reasonably straightforward, being dealt with in Section 6. The case of positive and mixed characteristic requires a short detour on group schemes (Section 7). When $\mathcal{X}_0$ is a gerbe over a field, we establish Theorem 1.10 in Corollary 8.2. This is sufficient for Theorems 1.1 and 1.3. We prove Theorem 1.10 in Section 10, and then use it in Section 11 to establish the existence of formally smooth neighborhoods and completions.

Theorem 1.11 (Formal neighborhoods)**.** *Let $\mathcal{X}$ be noetherian algebraic stack and $\mathcal{X}_0 \subseteq \mathcal{X}$ be a locally closed substack. Let $h_0\colon \mathcal{W}_0 \to \mathcal{X}_0$ be a syntomic (e.g., smooth) morphism. Assume that $\mathcal{W}_0$ is linearly fundamental and that its good moduli space* *is quasi-excellent. Then there is a cartesian diagram*

$$\begin{tikzcd} \mathcal{W}_0 \arrow[r] \arrow[d, "h_0"] & \widehat{\mathcal{W}} \arrow[d, "h"] \\ \mathcal{X}_0 \arrow[r] & \mathcal{X}, \end{tikzcd}$$

where $h\colon \widehat{\mathcal{W}} \to \mathcal{X}$ is flat and $\widehat{\mathcal{W}}$ is noetherian, linearly fundamental and coherently complete along $\mathcal{W}_0$.

Theorem 1.12 (Existence of completions)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine stabilizers. For any point $x \in |\mathcal{X}|$ with linearly reductive stabilizer, the completion of $\mathcal{X}$ at x exists and is linearly fundamental.*

In fact, both theorems above are proven more generally for pro-immersions (Theorem 11.1 and Theorem 11.2).

1.4. **The structure of linearly reductive affine group schemes.** We prove that every linearly reductive group scheme $G \to S$ is étale-locally embeddable (Corollary 13.2) and canonically an extension of a finite flat tame group scheme by a smooth linearly reductive group scheme with connected fibers G^0_{sm} (Theorem 18.9). If S is of equal characteristic, then G is canonically an extension of a finite étale tame group scheme by a linearly reductive group scheme G^0 with connected fibers. In equal positive characteristic, G^0 is of multiplicative type and we say that G is nice.

We also prove that if (S, s) is henselian and $G_s \to \operatorname{Spec} \kappa(s)$ is linearly reductive, then there exists an embeddable linearly reductive group scheme $G \to S$ extending G_s (Proposition 16.8).

1.5. **Applications.** In the course of establishing the results above, we prove several foundational results of independent interest. For instance, we prove that adequate moduli spaces are universal for maps to algebraic spaces (Theorem 3.12) and establish Luna's fundamental lemma for adequate moduli spaces (Theorem 3.14). We also prove that an adequate moduli space $\mathcal{X} \to X$, where the closed points of $\mathcal{X}$ have linearly reductive stabilizers, is necessarily a good moduli space (Theorem 9.3 and Corollary 13.11). We have also resolved the issue (see [AHR19, Question 1.10]) of representability of the local quotient presentation in the presence of a separated diagonal (Proposition 12.5(2)).

In Sections 13, 14, 19 and 20 we establish the following consequences of our results and methods.

(1) We provide the following refinement of Theorem 1.1: if $\mathcal{X}$ admits a good moduli space X, then étale-locally on X, $\mathcal{X}$ is of the form $[\operatorname{Spec} A/\mathrm{GL}_n]$ (Theorem 13.1).
(2) We prove that a good moduli space $\mathcal{X} \to X$ necessarily has affine diagonal as long as $\mathcal{X}$ has separated diagonal and affine stabilizers (Theorem 13.1).
(3) We prove compact generation of the derived category of an algebraic stack admitting a good moduli space (Proposition 14.1).
(4) We prove algebraicity results for stacks parameterizing coherent sheaves (Theorem 14.6), Quot schemes (Corollary 14.7), and Hom stacks (Theorem 14.9).
(5) We provide generalizations of Sumihiro's theorem on torus actions (Theorem 19.1 and Corollary 19.2).
(6) We prove a relative version of Luna's étale slice theorem (Theorem 19.4).
(7) We prove the existence of henselizations of algebraic stacks at points with linearly reductive stabilizer (Theorem 20.3).

(8) We prove that two algebraic stacks are étale locally isomorphic near points with linearly reductive stabilizers if and only if they have isomorphic henselizations or completions (Theorem 20.5).

Finally, Theorem 1.1 and its refinements are fundamental ingredients in the recent preprint of the first author with Halpern-Leistner and Heinloth on establishing necessary and sufficient conditions for an algebraic stack to admit a good moduli space [AHH18].

1.6. **Roadmap.** This paper is naturally divided into five parts:

(I) Sections 1 to 3 consist of the introduction and basic setup. We provide definitions and properties of reductive group schemes, fundamental stacks and local, henselian and coherently complete pairs. Section 3 ends with two applications—universality of adequate moduli spaces (Theorem 3.12) and Luna's fundamental lemma for adequate moduli spaces (Theorem 3.14).

(II) Sections 4 to 12 contain most of the central theorems of this paper: formal functions for good moduli spaces (Corollary 4.2), coherent completeness for good moduli spaces (Theorem 1.6 established in Section 10 with the preliminary version Proposition 5.1), effectivity of adic sequences (Theorem 1.10 established in increasing generality in Sections 6, 8 and 10), the existence of formal neighborhoods (Theorem 1.11 established in Section 11), and the local structure theorem (Theorem 1.1 established in Section 12).

(III) Sections 13 and 14 contain our first applications of the local structure theorem: the resolution property holds étale locally on a good moduli space (Theorem 13.1), compact generation of the derived category of stacks admitting a good moduli space (Proposition 14.1) and algebraicity results (Theorem 14.6, Corollary 14.7, and Theorem 14.9).

(IV) Sections 15 to 17 contain technical results on approximation of linearly fundamental stacks (Theorem 15.3) and good moduli spaces (Corollary 15.5) and various results on deforming objects over henselian pairs in Section 16 which allow us to provide refinements of Theorem 1.1 in Corollary 17.4.

(V) Sections 18 to 20 contain our final applications: structure results of linearly reductive group schemes (Theorem 18.9), our generalizations of Sumihiro's theorem on torus actions (Theorem 19.1 and Corollary 19.2), a relative version of Luna's slice theorem (Theorem 19.4) and the existence of henselizations (Theorem 20.3).

1.7. **Notation and conventions.**

1.7.1. If $\mathcal{X}$ is a locally noetherian algebraic stack, we let $\mathsf{Coh}(\mathcal{X})$ be the abelian category of coherent $\mathcal{O}_{\mathcal{X}}$-modules.

1.7.2. If $\mathcal{X}$ is a locally noetherian algebraic stack and $\mathcal{Z} \subseteq \mathcal{X}$ is a closed substack, we denote by $\mathcal{X}_{\mathcal{Z}}^{[n]}$ the nth order thickening of $\mathcal{Z}$ in $\mathcal{X}$ (i.e. if $\mathcal{Z}$ is defined by a sheaf of ideals $\mathcal{I}$, then $\mathcal{X}_{\mathcal{Z}}^{[n]}$ is defined by $\mathcal{I}^{n+1}$). If $i\colon \mathcal{Z} \to \mathcal{X}$ denotes the closed immersion, then we write $i^{[n]}\colon \mathcal{X}_{\mathcal{Z}}^{[n]} \to \mathcal{X}$ for the nth order thickening of i.

1.7.3. Throughout this paper, we use the concepts of cohomologically affine morphisms and adequately affine morphisms slightly modified from the original definitions of [Alp13, Defn. 3.1] and [Alp14, Defn. 4.1.1]: a quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *cohomologically affine* (resp. *adequately affine*) if (1) f_* is exact on the category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules (resp. if for every surjection $\mathcal{A} \to \mathcal{B}$ of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras, then any section s of $f_*(\mathcal{B})$ over a smooth morphism $\operatorname{Spec} A \to \mathcal{Y}$ has a positive power that lifts to a section of $f_*(\mathcal{A})$), and (2) this property is stable under arbitrary base change. In

[Alp13, Defn. 3.1] and [Alp14, Defn. 4.1.1], condition (2) was not required. If $\mathcal{Y}$ has quasi-affine diagonal (e.g., $\mathcal{Y}$ is a quasi-separated algebraic space), then (2) holds automatically ([Alp13, Prop. 3.10(vii)] and [Alp14, Prop. 4.2.1(6)].

1.7.4. We also use throughout the concepts of good moduli spaces [Alp13, Defn. 4.1] and adequate moduli spaces [Alp14, Defn. 5.1.1]: a quasi-compact and quasi-separated morphism $\pi\colon \mathcal{X} \to X$ of algebraic stacks, where X is an algebraic space, is a *good moduli space* (resp. an *adequate moduli space*) if π is cohomologically affine (resp. adequately affine) and $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

1.7.5. See Definition 2.1 for our terminology regarding group schemes. In particular, we assume that linearly and geometrically reductive group schemes are necessarily affine (even though this was not the convention in [Alp13, Defn. 12.1] and [Alp14, Defn. 9.1.1]). See also Remark 2.6.

1.7.6. We freely use the following form of Tannaka duality, which was established in [HR19]. Let $\mathcal{X}$ be a noetherian algebraic stack with affine stabilizers and let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack such that $\mathcal{X}$ is coherently complete along $\mathcal{Z}$. Let $\mathcal{Y}$ be a noetherian algebraic stack with affine stabilizers. Suppose that either

(1) $\mathcal{X}$ is locally the spectrum of a G-ring (e.g., quasi-excellent), or
(2) $\mathcal{Y}$ has quasi-affine diagonal.

Then the natural functor

$$\mathrm{Hom}(\mathcal{X}, \mathcal{Y}) \to \varprojlim_n \mathrm{Hom}\big(\mathcal{X}_{\mathcal{Z}}^{[n]}, \mathcal{Y}\big)$$

is an equivalence of categories. This statement follows directly from [HR19, Thms. 1.1 and 8.4]; cf. the proof of [AHR19, Cor. 2.8].

1.7.7. An algebraic stack $\mathcal{X}$ is said to have the *resolution property* if every quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module of finite type is a quotient of a locally free sheaf. By the main theorems of [Tot04] and [Gro17], a quasi-compact and quasi-separated algebraic stack is isomorphic to $[U/\mathrm{GL}_N]$, where U is a quasi-affine scheme and N is a positive integer, if and only if the closed points of $\mathcal{X}$ have affine stabilizers and $\mathcal{X}$ has the resolution property. Note that when this is the case, $\mathcal{X}$ has affine diagonal.

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2. Reductive group schemes and fundamental stacks

In this section, we recall various notions of reductivity for group schemes (Definition 2.1) and introduce certain classes of algebraic stacks that we will refer to as fundamental, linearly fundamental, and nicely fundamental (Definition 2.7). The reader may prefer to skip this section and only refer back to it after encountering these notions. In particular, *nice* group schemes and *nicely* fundamental stacks do not make an appearance until Section 7 and Section 8, respectively. We also recall various relations between these notions. Besides some approximation results at the end, this section is largely expository.

2.1. **Reductive group schemes.**

Definition 2.1. Let G be a group algebraic space which is affine, flat and of finite presentation over an algebraic space S. We say that $G \to S$ is

(1) *embeddable* if G is a closed subgroup of $\mathrm{GL}(\mathcal{E})$ for a vector bundle $\mathcal{E}$ on S;
(2) *linearly reductive* if $BG \to S$ is cohomologically affine [Alp13, Defn. 12.1];
(3) *geometrically reductive* if $BG \to S$ is adequately affine [Alp14, Defn. 9.1.1];
(4) *reductive* if $G \to S$ is smooth with reductive and connected geometric fibers [$\mathrm{SGA3}_{\mathrm{III}}$, Exp. XIX, Defn. 2.7]; and
(5) *nice* if there is an open and closed normal subgroup $G^0 \subseteq G$ that is of multiplicative type over S such that $H = G/G^0$ is finite and locally constant over S and $|H|$ is invertible on S.

Linearly reductive group schemes are the focus of this paper, but we need to consider geometrically reductive and nice group schemes for the following two reasons.

- In positive characteristic GL_n is geometrically reductive but not linearly reductive.
- A linearly reductive group scheme G_0 defined over the residue field $\kappa(s)$ of a point s in a scheme S deforms to a linearly reductive group scheme over the henselization at s (Proposition 16.8) but in general only deforms to a geometrically reductive group scheme over an étale neighborhood of s (see Remark 2.4). The reason is that the functor parameterizing linearly reductive group schemes is not limit preserving in mixed characteristic (see Remark 2.16).

Remark 2.2 (Relations between the notions)*.* We have the implications:

$$\text{nice} \implies \text{linearly reductive} \implies \text{geometrically reductive} \Longleftarrow \text{reductive}.$$

The first implication follows since a nice group algebraic space G is an extension of the linearly reductive groups G^0 and H, and is thus linearly reductive [Alp13, Prop. 2.17]. The second implication is immediate from the definitions, and is reversible in characteristic 0 [Alp14, Rem. 9.1.3]. The third implication is Seshadri's generalization [Ses77] of Haboush's theorem, and is reversible if $G \to S$ is smooth with geometrically connected fibers [Alp14, Thm. 9.7.5]. If k is a field of characteristic p, then GL_n is reductive over k but not linearly reductive, and a finite non-reduced group scheme (e.g., α_p) is geometrically reductive but not reductive.

Remark 2.3 (Positive characteristic)*.* The notion of niceness is particularly useful in positive characteristic and was introduced in [HR15, Defn. 1.1] for affine group schemes over a field k. If k is a field of characteristic p, an affine group scheme G of finite type over k is nice if and only if the connected component of the identity G^0 is of multiplicative type and p does not divide the number of geometric components of G. In this case, by Nagata's theorem [Nag62] and its generalization to the non-smooth case (cf. [HR15, Thm. 1.2]), G is nice if and only if it is linearly reductive; moreover, this is also true over a base of equal characteristic p (Theorem 18.9). In mixed characteristic, we prove that every linearly reductive group scheme $G \to S$ is canonically an extension of a finite tame linearly reductive group scheme by a smooth linearly reductive group scheme (Theorem 18.9), and that $G \to S$ is nice étale-locally around any point of characteristic p (Corollary 13.6).

Remark 2.4 (Mixed characteristic)*.* Consider a scheme S, a point $s \in S$ and a linearly reductive group scheme G_0 over $\kappa(s)$. If G_0 is nice (e.g., if s has positive characteristic), then it deforms to a nice group scheme $G' \to S'$ over an étale neighborhood $S' \to S$ of s (Proposition 7.1). If s has characteristic 0 but there is no open neighborhood of $s \in S$ defined in characteristic 0[1] then G_0 need not deform to a linearly reductive group scheme $G \to S'$ over an étale neighborhood $S' \to S$ of s. For example, take $G_0 = \mathrm{GL}_{2,\kappa(s)}$. However, G_0 does deform to a geometrically reductive embeddable group scheme over an étale neighborhood of s (Proposition 16.8 and Lemma 2.12).

Remark 2.5 (Embeddability and geometric reductivity). Any affine group scheme of finite type over a field is embeddable. It is not known to which extent general affine group schemes are embeddable—even over the dual numbers [Con10]. Thomason proved that certain reductive group schemes are embeddable [Tho87, Cor. 3.2]; in particular, if S is a normal, quasi-projective scheme, then every reductive group scheme $G \to S$ is embeddable. There is an example [$\mathrm{SGA3}_{\mathrm{II}}$, Exp. X,§1.6] of a 2-dimensional torus over the nodal cubic curve that is not locally isotrivial and hence not Zariski-locally embeddable. We will eventually show that every linearly reductive group scheme $G \to S$ is embeddable if S is a normal quasi-projective scheme (Corollary 18.10) and always étale-locally embeddable (Corollary 13.2).

If G is a closed subgroup of $\mathrm{GL}(\mathcal{E})$ for a vector bundle $\mathcal{E}$ on an algebraic space S, then a generalization of Matsushima's Theorem asserts that $G \to S$ is geometrically reductive if and only if the quotient $\mathrm{GL}(\mathcal{E})/G$ is affine [Alp14, Thm. 9.4.1].

If S is affine and $G \to S$ is embeddable and geometrically reductive, then any quotient stack $\mathcal{X} = [\operatorname{Spec} A/G]$ has the resolution property. Indeed, if G is a closed subgroup of $\mathrm{GL}(\mathcal{E})$ for some vector bundle $\mathcal{E}$ of rank n on S, then the $(\mathrm{GL}(\mathcal{E}), \mathrm{GL}_{n,S})$-bitorsor $\mathrm{Isom}_{\mathcal{O}_S}(\mathcal{E}, \mathcal{O}_S^n)$ induces an isomorphism $B_S GL(\mathcal{E}) \cong B_S\mathrm{GL}_n$, and the composition $\mathcal{X} = [\operatorname{Spec} A/G] \to B_S G \to B_S \mathrm{GL}(\mathcal{E}) \cong B_S \mathrm{GL}_n$ is affine, that is $\mathcal{X} \cong [\operatorname{Spec} B/\mathrm{GL}_{n,S}]$. By [Gro17, Thm. 1.1], $\mathcal{X}$ has the resolution property.

Remark 2.6 (Affineness). In contrast to [Alp13], we have only defined linear reductivity for affine group schemes $G \to S$. We will however prove that if $G \to S$ is a separated, flat group scheme of finite presentation with (affine) linearly reductive fibers, then $G \to S$ is necessarily quasi-affine. If in addition $BG \to S$ is cohomologically affine, then $G \to S$ is affine (Theorem 18.9).

2.2. Fundamental stacks.

In [AHR19], we dealt with stacks of the form $[\operatorname{Spec} A/G]$ where G is a linearly reductive group scheme over a field k. In this paper, we are working over an arbitrary base and it will be convenient to introduce the following classes of quotient stacks.

Definition 2.7. Let $\mathcal{X}$ be an algebraic stack. We say that $\mathcal{X}$ is:

(1) *fundamental* if $\mathcal{X}$ admits an affine morphism to $B\mathrm{GL}_{n,\mathbb{Z}}$ for some n, i.e. if $\mathcal{X} = [U/\mathrm{GL}_{n,\mathbb{Z}}]$ for an affine scheme U;
(2) *linearly fundamental* if $\mathcal{X}$ is fundamental and cohomologically affine; and
(3) *nicely fundamental* if $\mathcal{X}$ admits an affine morphism to $B_S Q$, where Q is a nice and embeddable group scheme over some affine scheme S.

Remark 2.8 (Relations between the notions). We have the obvious implications:

$$\text{nicely fundamental} \implies \text{linearly fundamental} \implies \text{fundamental}$$

If $\mathcal{X}$ is fundamental (resp. linearly fundamental), then $\mathcal{X}$ admits an adequate (resp. good) moduli space: $\operatorname{Spec} \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

[1]This can happen even if $s \in S$ is a closed point. For instance, let R be the localization $\Sigma^{-1}\mathbb{Z}[x]$ where Σ is the multiplicative set generated by the elements $p+x$ as p ranges over all primes. Then $S = \operatorname{Spec} R$ is a noetherian and excellent integral scheme, and $s = (x) \in S$ is a closed point with residue field $\mathbb{Q}$ which has no characteristic 0 neighborhood. Also see Appendix A.1.

In characteristic 0, an algebraic stack is linearly fundamental if and only if it is fundamental. We will show that in positive equicharacteristic, a linearly fundamental stack is nicely fundamental étale-locally over its good moduli space (Corollary 13.6 and Lemma 2.15).

The additional condition of a fundamental stack to be linearly fundamental is that $\mathcal{X} \cong [\operatorname{Spec} B/\mathrm{GL}_N]$ is cohomologically affine, which means that the adequate moduli space $\mathcal{X} \to \operatorname{Spec} B^{\mathrm{GL}_n}$ is a good moduli space. We will show that this happens precisely when the stabilizer of every closed point is linearly reductive (Corollary 13.7).

Remark 2.9 (Equivalences I). If G is a group scheme which is affine, flat and of finite presentation over an affine scheme S, then $G \to S$ is geometrically reductive (resp. linearly reductive, resp. nice) and embeddable if and only if $B_S G$ is fundamental (resp. linearly fundamental, resp. nicely fundamental). This follows from Remark 2.5 and the definitions for geometrically reductive and linearly reductive and an easy additional argument for the nicely fundamental case.

By definition, a fundamental (resp. nicely fundamental) stack is of the form $[U/G]$, where S is an affine scheme, $G \to S$ is a geometrically reductive (resp. nice) and embeddable group scheme, and $U \to S$ is affine; for fundamental, we may even take $G = \mathrm{GL}_{n,S}$. Note that we may replace S with the adequate moduli space $U/\!\!/G$.

The definition of linearly fundamental is not analogous. If G is a linearly reductive and embeddable group scheme over an affine scheme S and $U \to S$ is affine, then $[U/G]$ is linearly fundamental. The converse, that every linearly fundamental stack $\mathcal{X}$ is of the form $[U/G]$, is not true; see Appendix A.1. We will, however, show that under mild mixed characteristic hypotheses every linearly fundamental stack over S is, étale-locally over its good moduli space, of the form $[U/G]$ with $G \to S$ linearly reductive and embeddable, and $U \to S$ affine (Corollary 13.5).

Remark 2.10 (Equivalences II). An algebraic stack $\mathcal{X}$ is a *global quotient stack* if $\mathcal{X} \cong [U/\mathrm{GL}_n]$, where U is an algebraic space. Since adequately affine and representable morphisms are necessarily affine ([Alp14, Thm. 4.3.1]), we have the following equivalences for a quasi-compact and quasi-separated algebraic stack $\mathcal{X}$:

$$\begin{aligned} \text{fundamental} &\iff \text{adequately affine and a global quotient} \\ \text{linearly fundamental} &\iff \text{cohomologically affine and a global quotient} \end{aligned}$$

Remark 2.11 (Positive characteristic). Let $\mathcal{G}$ be a gerbe over a field k of characteristic $p > 0$. If $\mathcal{G}$ is cohomologically affine, then it is nicely fundamental. Indeed, since $\mathcal{G} \to \operatorname{Spec} k$ is smooth, there is a finite separable extension $k \subseteq k'$ that neutralizes the gerbe. Hence, $\mathcal{G}_{k'} \cong BQ'$, for some linearly reductive group scheme Q' over k'. By Remark 2.3, Q' is nice. Let Q be the Weil restriction of Q' along $\operatorname{Spec} k' \to \operatorname{Spec} k$; then Q is nice and there is an induced affine morphism $\mathcal{G} \to BQ$.

2.3. **Approximation.** Here we prove that the property of a stack being fundamental or nicely fundamental, or the property of an embeddable group scheme being geometrically reductive or nice, can be approximated. These results will be used to reduce from the situation of a complete local ring to an excellent henselian local ring (via Artin approximation), from a henselian local ring to an étale neighborhood, and from (non-)noetherian rings to excellent rings.

Lemma 2.12. *Let $\{S_\lambda\}_{\lambda \in \Lambda}$ be an inverse system of quasi-compact and quasi-separated algebraic spaces with affine transition maps and limit S. Let $\lambda_0 \in \Lambda$ and let $G_{\lambda_0} \to S_{\lambda_0}$ be a flat group algebraic space of finite presentation. For $\lambda \geq \lambda_0$, let G_λ be the pullback of G_{λ_0} along $S_\lambda \to S_{\lambda_0}$ and let G be the pullback of G_{λ_0} along*

$S \to S_{\lambda_0}$. *If* G *is geometrically reductive (resp. nice) and embeddable over* S*, then* G_λ *is geometrically reductive (resp. nice) and embeddable over* S_λ *for all* $\lambda \gg \lambda_0$.

Proof. Let $\mathcal{E}$ be a vector bundle on S and let $G \hookrightarrow \mathrm{GL}(\mathcal{E})$ be a closed embedding. By standard limit methods, there exists a vector bundle $\mathcal{E}_\lambda$ on S_λ and a closed embedding $G_\lambda \hookrightarrow \mathrm{GL}(\mathcal{E}_\lambda)$ for all sufficiently large λ. If G is geometrically reductive, then $\mathrm{GL}(\mathcal{E})/G$ is affine and so is $\mathrm{GL}(\mathcal{E}_\lambda)/G_\lambda$ for all sufficiently large λ; hence G_λ is geometrically reductive (Remark 2.5).

If $G^0 \subseteq G$ is an open and closed normal subgroup as in the definition of a nice group scheme, then by standard limit methods, we can find an open and closed normal subgroup $G^0_\lambda \subseteq G_\lambda$ for all sufficiently large λ satisfying the conditions in the definition of nice group schemes. □

Lemma 2.13. *An algebraic stack* $\mathcal{X}$ *is nicely fundamental if and only if there exists an affine scheme* S *of finite presentation over* $\operatorname{Spec}\mathbb{Z}$*, a nice and embeddable group scheme* $Q \to S$ *and an affine morphism* $\mathcal{X} \to B_S Q$.

Proof. The condition is sufficient by definition and the necessity is Lemma 2.12. □

Lemma 2.14. *Let* $\mathcal{X}$ *be a fundamental (resp. a nicely fundamental) stack. Then there exists an inverse system of fundamental (resp. nicely fundamental) stacks* $\mathcal{X}_\lambda$ *of finite type over* $\operatorname{Spec}\mathbb{Z}$ *with affine transition maps such that* $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$.

Proof. If $\mathcal{X}$ is fundamental, then we have an affine morphism $\mathcal{X} \to B\mathrm{GL}_{n,\mathbb{Z}}$ and can thus write $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$ where $\mathcal{X}_\lambda \to B\mathrm{GL}_{n,\mathbb{Z}}$ are affine and of finite type. Indeed, every quasi-coherent sheaf on the noetherian stack $B\mathrm{GL}_{n,\mathbb{Z}}$ is a union of its finitely generated subsheaves [LMB, Prop. 15.4]. If $\mathcal{X}$ is nicely fundamental, we argue analogously with $B_S Q$ of Lemma 2.13 instead of $B\mathrm{GL}_{n,\mathbb{Z}}$. □

If $\mathcal{X} \to X$ and $\mathcal{X}_\lambda \to X_\lambda$ denote the corresponding adequate moduli spaces, then in general $\mathcal{X} \to \mathcal{X}_\lambda \times_{X_\lambda} X$ is not an isomorphism. It is, however, true that $X = \varprojlim_\lambda X_\lambda$ (see Lemma 2.15 below). If $\mathcal{X} \to X$ is of finite presentation and $\mathcal{X}_\lambda$ is linearly fundamental for sufficiently large λ, then one can also arrange that $\mathcal{X} \to \mathcal{X}_\lambda \times_{X_\lambda} X$ is an isomorphism.

Lemma 2.15. *Let* $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$ *be an inverse limit of quasi-compact and quasi-separated algebraic stacks with affine transition maps.*

(1) *If* $\mathcal{X}$ *is fundamental (resp. nicely fundamental), then so is* $\mathcal{X}_\lambda$ *for all sufficiently large* λ.
(2) *If* $\mathcal{X} \to X$ *and* $\mathcal{X}_\lambda \to X_\lambda$ *are adequate moduli spaces, then* $X = \varprojlim_\lambda X_\lambda$.
(3) *Let* $x \in |\mathcal{X}|$ *be a point with image* $x_\lambda \in |\mathcal{X}_\lambda|$. *If* $\mathcal{G}_x$ *(resp.* $\overline{\{x\}}$*) is nicely fundamental, then so is* $\mathcal{G}_{x_\lambda}$ *(resp.* $\overline{\{x_\lambda\}}$*) for all sufficiently large* λ.

Proof. For the first statement, let $\mathcal{Y} = B\mathrm{GL}_{n,\mathbb{Z}}$ (resp. $\mathcal{Y} = B_S Q$ for Q as in Lemma 2.13). Then there is an affine morphism $\mathcal{X} \to \mathcal{Y}$ and hence an affine morphism $\mathcal{X}_\lambda \to \mathcal{Y}$ for all sufficiently large λ [Ryd15, Prop. B.1, Thm. C].

The second statement follows directly from the following two facts (a) pushforward of quasi-coherent sheaves along $\pi_\lambda \colon \mathcal{X}_\lambda \to X_\lambda$ preserves filtered colimits and (b) if $\mathcal{A}$ is a quasi-coherent sheaf of algebras, then the adequate moduli space of $\operatorname{Spec}_{\mathcal{X}_\lambda} \mathcal{A}$ is $\operatorname{Spec}_{X_\lambda} (\pi_\lambda)_* \mathcal{A}$.

The third statement follows from the first by noting that $\mathcal{G}_x = \varprojlim_\lambda \mathcal{G}_{x_\lambda}$ and $\overline{\mathcal{G}_x} = \varprojlim_\lambda \overline{\mathcal{G}_{x_\lambda}}$. □

Remark 2.16. The analogous statements of Lemma 2.12 (resp. Lemma 2.15) for linearly reductive and embeddable group schemes (resp. linearly fundamental stacks) are false in mixed characteristic. Indeed, $\mathrm{GL}_{2,\mathbb{Q}} = \varprojlim_m \mathrm{GL}_{2,\mathbb{Z}[\frac{1}{m}]}$ and $\mathrm{GL}_{2,\mathbb{Q}}$ is linearly reductive but $\mathrm{GL}_{2,\mathbb{Z}[\frac{1}{m}]}$ is never linearly reductive. Likewise, $B\mathrm{GL}_{2,\mathbb{Q}}$ is linearly fundamental but $B\mathrm{GL}_{2,\mathbb{Z}[\frac{1}{m}]}$ is never linearly fundamental.

The analogue of Lemma 2.14 for linearly fundamental stacks holds in equal characteristic and in certain mixed characteristics (Corollary 15.4) but not always (Appendix A).

3. Local, henselian, and coherently complete pairs

In this section, we define local, henselian and coherently complete pairs. We also state a general version of Artin approximation (Theorem 3.4) and establish some basic properties.

3.1. Preliminaries.

Definition 3.1. Fix a closed immersion of algebraic stacks $\mathcal{Z} \subseteq \mathcal{X}$. The pair $(\mathcal{X}, \mathcal{Z})$ is said to be

(1) *local* if every non-empty closed subset of $|\mathcal{X}|$ intersects $|\mathcal{Z}|$ non-trivially;
(2) *henselian* if for every finite morphism $\mathcal{X}' \to \mathcal{X}$, the restriction map

$$\mathrm{ClOpen}(\mathcal{X}') \to \mathrm{ClOpen}(\mathcal{Z} \times_{\mathcal{X}} \mathcal{X}'), \tag{3.1}$$

is bijective, where $\mathrm{ClOpen}(\mathcal{X})$ denotes the set of closed and open substacks of $\mathcal{X}$ [EGA, IV.18.5.5]; and
(3) *coherently complete* if $\mathcal{X}$ is noetherian and the functor

$$\mathsf{Coh}(\mathcal{X}) \to \varprojlim_n \mathsf{Coh}(\mathcal{X}_{\mathcal{Z}}^{[n]})$$

is an equivalence of abelian categories, where $\mathcal{X}_{\mathcal{Z}}^{[n]}$ denotes the nth nilpotent thickening of $\mathcal{Z}$ in $\mathcal{X}$.

In addition, we call a pair $(\mathcal{X}, \mathcal{Z})$ *affine* if $\mathcal{X}$ is affine and an affine pair $(\mathcal{X}, \mathcal{Z})$ *(quasi-)excellent* if $\mathcal{X}$ is (quasi-)excellent. Occasionally, we will also say $\mathcal{X}$ is local, henselian, or coherently complete along $\mathcal{Z}$ if the pair $(\mathcal{X}, \mathcal{Z})$ has the corresponding property.

Remark 3.2. For a pair $(\mathcal{X}, \mathcal{Z})$, we have the following sequence of implications:

$$\text{coherently complete} \implies \text{henselian} \implies \text{local}.$$

The second implication is trivial: if $\mathcal{W} \subseteq \mathcal{X}$ is a closed substack, then $\mathrm{ClOpen}(\mathcal{W}) \to \mathrm{ClOpen}(\mathcal{Z} \cap \mathcal{W})$ is bijective. For the first implication, note that we have bijections:

$$\mathrm{ClOpen}(\mathcal{X}) \simeq \varinjlim_n \mathrm{ClOpen}(\mathcal{X}_{\mathcal{Z}}^{[n]}) \simeq \mathrm{ClOpen}(\mathcal{Z})$$

whenever $(\mathcal{X}, \mathcal{Z})$ is coherently complete. The implication now follows from the elementary Lemma 3.5(1). It also follows from the main result of [Ryd16] that if $\mathcal{X}$ is quasi-compact and quasi-separated, then $(\mathcal{X}, \mathcal{Z})$ is a henselian pair if and only if (3.1) is bijective for every integral morphism $\mathcal{X}' \to \mathcal{X}$.

Remark 3.3 (Nakayama's lemma for stacks). Passing to a smooth presentation, it is not difficult to see that the following variants of Nakayama's lemma hold for local pairs $(\mathcal{X}, \mathcal{Z})$: (1) if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module of finite type and $\mathcal{F}|_{\mathcal{Z}} = 0$, then $\mathcal{F} = 0$; and (2) if $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules with $\mathcal{G}$ of finite type and $\varphi|_{\mathcal{Z}}$ is surjective, then φ is surjective.

The following theorem is well-known. When S is the henselization of a local ring essentially of finite type over a field or an excellent Dedekind domain, it is Artin's original approximation theorem [Art69, Cor. 2.2].

Theorem 3.4 (Artin approximation over henselian pairs)**.** *Let $(S, S_0) = (\operatorname{Spec} A, \operatorname{Spec} A/I)$ be an affine excellent henselian pair and let $\widehat{S} = \operatorname{Spec} \widehat{A}$ be its I-adic completion. Let $F\colon (\mathsf{Sch}/S)^{\mathrm{opp}} \to \mathsf{Sets}$ be a limit preserving functor. Given an element $\overline{\xi} \in F(\widehat{S})$ and an integer $n \geq 0$, there exists an element $\xi \in F(S)$ such that ξ and $\overline{\xi}$ have equal images in $F(S_n)$ where $S_n = \operatorname{Spec} A/I^{n+1}$.*

Proof. The completion map $\widehat{S} \to S$ is regular, hence by Néron–Popescu desingularization [Pop86, Thm. 1.8], there exists a smooth morphism $S' \to S$ and a section $\xi' \in F(S')$ such that $\xi'|_{\widehat{S}} = \overline{\xi}$. By Elkik [Elk73, Thm., p. 568], there is an element $\xi \in F(S)$ as requested. Also see [Pop86, Thm. 1.3]. □

3.2. **Permanence properties.** We now establish some techniques to verify that a pair $(\mathcal{X}, \mathcal{Z})$ is henselian or coherently complete. Analogous results for local pairs typically require far fewer hypotheses and will not be used in this article, so are omitted.

Let A be a noetherian ring and let $I \subseteq J \subseteq A$ be ideals. Assume that A is J-adically complete. Recall that A/I is then J-adically complete and A is also I-adically complete. This is analogous to parts (1) and (2), respectively, of the following result. We omit the proof.

Lemma 3.5. *Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed immersion of algebraic stacks. Assume that the pair $(\mathcal{X}, \mathcal{Z})$ is henselian or coherently complete.*

(1) *Let $f\colon \mathcal{X}' \to \mathcal{X}$ be a finite morphism and let $\mathcal{Z}' \subseteq \mathcal{X}'$ be the pullback of $\mathcal{Z}$. Then $(\mathcal{X}', \mathcal{Z}')$ is henselian or coherently complete, respectively.*
(2) *Let $\mathcal{W} \subseteq \mathcal{X}$ be a closed substack. If $|\mathcal{Z}| \subseteq |\mathcal{W}|$, then $(\mathcal{X}, \mathcal{W})$ is henselian or coherently complete, respectively.*

For henselian pairs, the analogue of Theorem 1.6 is straightforward.

Theorem 3.6. *Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack with adequate moduli space $\pi\colon \mathcal{X} \to X$. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack with $Z = \pi(\mathcal{Z})$. The pair $(\mathcal{X}, \mathcal{Z})$ is henselian if and only if the pair (X, Z) is henselian.*

Proof. The induced morphism $\mathcal{Z} \to Z$ factors as the composition of an adequate moduli space $\mathcal{Z} \to \widetilde{Z}$ and an adequate homeomorphism $\widetilde{Z} \to Z$ [Alp14, Lem. 5.2.11]. If $\mathcal{X}' \to \mathcal{X}$ is integral, then $\mathcal{X}'$ admits an adequate moduli space X' and $X' \to X$ is integral. Conversely, if $X' \to X$ is integral, then $\mathcal{X} \times_X X' \to X'$ factors as the composition of an adequate moduli space $\mathcal{X} \times_X X' \to \widetilde{X}'$ and an adequate homeomorphism $\widetilde{X}' \to X'$ [Alp14, Prop. 5.2.9(3)]. It is thus enough to show that

$$\mathrm{ClOpen}(\mathcal{X}) \to \mathrm{ClOpen}(\mathcal{Z})$$

is bijective if and only if

$$\mathrm{ClOpen}(X) \to \mathrm{ClOpen}(Z)$$

is bijective. But $\mathcal{X} \to X$ and $\mathcal{Z} \to Z$ are surjective and closed with connected fibers [Alp14, Thm. 5.3.1]. Thus we have identifications $\mathrm{ClOpen}(\mathcal{X}) = \mathrm{ClOpen}(X)$ and $\mathrm{ClOpen}(\mathcal{Z}) = \mathrm{ClOpen}(Z)$ that are compatible with the restriction maps. The result follows. □

One direction of Theorem 1.6 is also not difficult and holds more generally for adequate moduli spaces.

Proposition 3.7. *Let $\mathcal{X}$ be a noetherian algebraic stack with noetherian adequate moduli space $\pi\colon \mathcal{X} \to X$. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack with $Z = \pi(\mathcal{Z})$. If $X = \operatorname{Spec} A$ is affine and the pair $(\mathcal{X}, \mathcal{Z})$ is coherently complete, then the pair (X, Z) is coherently complete.*

Proof. Let $I \subseteq A$ be the ideal defining Z, let $A \to \widehat{A}$ be the I-adic completion and let $\widehat{X} = \operatorname{Spec} \widehat{A}$. The composition $\mathcal{X}_{\mathcal{Z}}^{[n]} \to \mathcal{X} \to X$ factors through $X_Z^{[n]}$, hence lifts uniquely to $\widehat{X}$. By Tannaka duality, we obtain a unique lift $\mathcal{X} \to \widehat{X}$. But, by definition, $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = A$, so we obtain a retraction $\widehat{A} \to A$. It follows that A is I-adically complete. $\square$

Remark 3.8. An alternative argument establishes that the conclusion of Proposition 3.7 still holds if the hypothesis that X is affine is replaced with the hypothesis that $\pi \colon \mathcal{X} \to X$ is a good moduli space.

3.3. **Examples.** We list some examples of henselian and coherently complete pairs.

Example 3.9. Let A be a noetherian ring and let $I \subseteq A$ be an ideal. Then $(\operatorname{Spec} A, \operatorname{Spec} A/I)$ is a coherently complete pair if and only if A is I-adically complete. The sufficiency is trivial. For the necessity, we note that $\varprojlim_n \mathsf{Coh}(A/I^{n+1}) \simeq \mathsf{Coh}(\widehat{A})$, where $\widehat{A}$ denotes the completion of A with respect to the I-adic topology. Hence, the natural functor $\mathsf{Coh}(A) \to \mathsf{Coh}(\widehat{A})$ is an equivalence of abelian tensor categories. It follows from Tannaka duality (see §1.7.6) that the natural map $A \to \widehat{A}$ is an isomorphism.

Example 3.10. Let A be a ring and let $I \subseteq A$ be an ideal. Let $f \colon \mathcal{X} \to \operatorname{Spec} A$ be a proper morphism of algebraic stacks. Let $\mathcal{Z} = f^{-1}(\operatorname{Spec} A/I)$.

(1) If A is I-adically complete, then $(\mathcal{X}, \mathcal{Z})$ is coherently complete. This is just the usual Grothendieck Existence Theorem, see [EGA, III.5.1.4] for the case of schemes and [Ols05, Thm. 1.4] for algebraic stacks.
(2) If A is henselian along I, then $(\mathcal{X}, \mathcal{Z})$ is henselian. This is part of the proper base change theorem in étale cohomology; the case where I is maximal is well-known, see [HR14, Rem. B.6] for further discussion.

3.4. **Characterization of henselian pairs.** A quasi-compact and quasi-separated pair of schemes (X, X_0) is henselian if and only if for every étale morphism $g \colon X' \to X$, every section of $g_0 \colon X' \times_X X_0 \to X_0$ extends to a section of g (for g separated see [EGA, IV.18.5.4] and in general see [SGA4$_3$, Exp. XII,Prop. 6.5]). This is also true for stacks:

Proposition 3.11. *Let $(\mathcal{X}, \mathcal{X}_0)$ be a pair of quasi-compact and quasi-separated algebraic stacks. Then the following are equivalent*

(1) $(\mathcal{X}, \mathcal{X}_0)$ *is henselian.*
(2) *For every* representable *étale morphism $g \colon \mathcal{X}' \to \mathcal{X}$, the induced map*
$$\Gamma(\mathcal{X}'/\mathcal{X}) \to \Gamma(\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_0/\mathcal{X}_0)$$
is bijective.

Proof. This is the equivalence between (1) and (3) of [HR16, Prop. 5.4]. $\square$

We will later prove that (2) holds for non-representable étale morphisms when $\mathcal{X}$ is a stack with a good moduli space and affine diagonal (Proposition 16.4). A henselian pair does not always satisfy (2) for general non-representable morphisms though, see Example 3.16.

3.5. **Application: Universality of adequate moduli spaces.** For noetherian algebraic stacks, good moduli spaces were shown in [Alp13, Thm. 6.6] to be universal for maps to quasi-separated algebraic spaces and adequate moduli spaces were shown in [Alp14, Thm. 7.2.1] to be universal for maps to algebraic spaces which are either locally separated or Zariski-locally have affine diagonal. We now establish this result unconditionally for adequate (and hence good) moduli spaces.

Theorem 3.12. *Let $\mathcal{X}$ be an algebraic stack. An adequate moduli space $\pi\colon \mathcal{X} \to X$ is universal for maps to algebraic spaces.*

Proof. We need to show that if Y is an algebraic space, then the natural map

$$\operatorname{Map}(X, Y) \to \operatorname{Map}(\mathcal{X}, Y) \tag{3.2}$$

is bijective. To see the injectivity of (3.2), suppose that $h_1, h_2\colon X \to Y$ are maps such that $h_1 \circ \pi = h_2 \circ \pi$. Let $E \to X$ be the equalizer of h_1 and h_2, that is, the pullback of the diagonal $Y \to Y \times Y$ along $(h_1, h_2)\colon X \to Y \times Y$. The equalizer is a monomorphism and locally of finite type. By assumption $\pi\colon \mathcal{X} \to X$ factors through E and it follows that $E \to X$ is universally closed, hence a closed immersion [Stacks, Tag 04XV]. Since $\mathcal{X} \to X$ is schematically dominant, so is $E \to X$, hence $E = X$.

The surjectivity of (3.2) is an étale-local property on X; indeed, the injectivity of (3.2) implies the gluing condition in étale descent. Thus, we may assume that X is affine. In particular, $\mathcal{X}$ is quasi-compact and since any map $\mathcal{X} \to Y$ factors through a quasi-compact open of Y, we may assume that Y is also quasi-compact. Let $g\colon \mathcal{X} \to Y$ be a map and $p\colon Y' \to Y$ be an étale presentation where Y' is an affine scheme. The pullback $f\colon \mathcal{X}' \to \mathcal{X}$ of $p\colon Y' \to Y$ along $g\colon \mathcal{X} \to Y$ is representable, étale, surjective and induces an isomorphism of stabilizer group schemes at all points.

Let $x \in X$ be a point, $q \in |\mathcal{X}|$ be the unique closed point over x and $q' \in |\mathcal{X}'|$ any point over q. Note that $\kappa(q)/\kappa(x)$ is a purely inseparable extension. After replacing X with an étale neighborhood of x (with a residue field extension), we may thus assume that $\kappa(q') = \kappa(q)$. Since f induces an isomorphism of stabilizer groups, the induced map $\mathcal{G}_{q'} \to \mathcal{G}_q$ on residual gerbes is an isomorphism. Theorem 3.6 implies that $(\mathcal{X} \times_X \operatorname{Spec} \mathcal{O}^h_{X,x}, \mathcal{G}_q)$ is a henselian pair and since f is locally of finite presentation, Proposition 3.11 implies that after replacing X with an étale neighborhood of x, there is a section $s\colon \mathcal{X} \to \mathcal{X}'$ of $f\colon \mathcal{X}' \to \mathcal{X}$. Thus, the map $g\colon \mathcal{X} \to Y$ factors as $\mathcal{X} \xrightarrow{s} \mathcal{X}' \xrightarrow{g'} Y' \xrightarrow{p} Y$. Since X and Y' are affine, the equality $\Gamma(X, \mathcal{O}_X) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ implies that the map $\mathcal{X} \xrightarrow{s} \mathcal{X}' \xrightarrow{g'} Y'$ factors through $\pi\colon \mathcal{X} \to X$. □

3.6. Application: Luna's fundamental lemma.

Definition 3.13. If $\mathcal{X}$ and $\mathcal{Y}$ are algebraic stacks admitting adequate moduli spaces $\mathcal{X} \to X$ and $\mathcal{Y} \to Y$, we say that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is *strongly étale* if the induced morphism $X \to Y$ is étale and $\mathcal{X} \cong X \times_Y \mathcal{Y}$.

The following result generalizes [Alp10, Thm. 6.10] from good moduli spaces to adequate moduli spaces and also removes noetherian and separatedness assumptions.

Theorem 3.14 (Luna's fundamental lemma)**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks with adequate moduli spaces $\pi_{\mathcal{X}}\colon \mathcal{X} \to X$ and $\pi_{\mathcal{Y}}\colon \mathcal{Y} \to Y$. Let $x \in |\mathcal{X}|$ be a closed point such that*

1. *f is étale and representable in a neighborhood of x,*
2. *$f(x) \in |\mathcal{Y}|$ is closed, and*
3. *f induces an isomorphism of stabilizer groups at x.*

Then there exists an open neighborhood $\mathcal{U} \subseteq \mathcal{X}$ of x such that $\pi_{\mathcal{X}}^{-1}(\pi_{\mathcal{X}}(\mathcal{U})) = \mathcal{U}$ and $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{Y}$ is strongly étale.

Remark 3.15. If G a smooth algebraic group over an algebraically closed field k such that G^0 is reductive and $\varphi\colon U \to V$ is a G-equivariant morphism of irreducible normal affine varieties over k, then [BR85, Thm. 4.1] (see also [MFK94, pg. 198]) established the result above for $f\colon [U/G] \to [V/G]$.

Proof. We may replace $\mathcal{X}$ with a saturated open neighborhood of x such that f becomes étale and representable. Let $y = f(x)$. The question is étale-local on Y so we can assume that Y is affine. Then $\mathcal{Y}$ is quasi-compact and quasi-separated by definition.

If Y is strictly henselian, then $(\mathcal{Y}, y)$ is a henselian pair (Theorem 3.6) and $\mathcal{G}_x \to \mathcal{G}_y$ is an isomorphism. We can thus find a section s of f such that $s(y) = x$ (Proposition 3.11). In general, since f is locally of finite presentation, we obtain a section s of f such that $s(y) = x$ after replacing Y with an étale neighborhood $(Y', y') \to (Y, y)$. The image of s is an open substack $\mathcal{U} \subseteq \mathcal{X}$ and $f|_{\mathcal{U}}$ is an isomorphism. Let $\mathcal{V} = \mathcal{X} \smallsetminus \pi^{-1}(\pi(\mathcal{X} \smallsetminus \mathcal{U})) \subseteq \mathcal{U}$. Then $\mathcal{V} \subseteq \mathcal{X}$ is a saturated open neighborhood of x and it is enough to prove the result after replacing $\mathcal{X}$ with $\mathcal{V}$. We can thus assume that f is separated. After repeating the argument we obtain a section s which is open and closed. Then $\mathcal{U} \subseteq \mathcal{X}$ is automatically saturated and we are done. □

The result is not true in general if f is not representable and $\mathcal{Y}$ does not have separated diagonal.

Example 3.16. Let k be a field and let S be the strict henselization of the affine line at the origin. Let $G = (\mathbb{Z}/2\mathbb{Z})_S$ and let $G' = G/H$ where $H \subseteq G$ is the open subgroup that is the complement of the non-trivial element over the origin. Let $\mathcal{X} = BG$ and $\mathcal{Y} = BG'$ which both have good moduli space S (adequate if $\operatorname{char} k = 2$). The induced morphism $f \colon \mathcal{X} \to \mathcal{Y}$ is étale and induces an isomorphism of the residual gerbes $B\mathbb{Z}/2\mathbb{Z}$ of the unique closed points but is not strongly étale and does not admit a section.

4. Theorem on formal functions

The following theorem on formal functions for good moduli spaces is an essential ingredient in our proof of Theorems 1.1 and 1.3 (and more specifically in the proof of the coherent completeness result of Theorem 1.6). This theorem is close in spirit to [EGA, III.4.1.5] and is a generalization of [Alp12, Thm. 1.1].

Theorem 4.1 (Formal functions, adequate version)**.** *Let $\mathcal{X}$ be an algebraic stack that is adequately affine. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack defined by a sheaf of ideals $\mathcal{I}$. Let $I = \Gamma(\mathcal{X}, \mathcal{I})$ be the corresponding ideal of $A = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. If A is noetherian and I-adically complete, and $\mathcal{X} \to \operatorname{Spec} A$ is of finite type, then for every $\mathcal{F} \in \mathsf{Coh}(\mathcal{X})$ the following natural map*

$$\Gamma(\mathcal{X}, \mathcal{F}) \to \varprojlim_n \Gamma(\mathcal{X}, \mathcal{F})/\Gamma(\mathcal{X}, \mathcal{I}^n\mathcal{F}) \tag{4.1}$$

is an isomorphism.

Proof. By [Alp14, Thm. 6.3.3], $\Gamma(\mathcal{X}, -)$ preserves coherence. Let $I_n = \Gamma(\mathcal{X}, \mathcal{I}^n)$ and $F_n = \Gamma(\mathcal{X}, \mathcal{I}^n\mathcal{F})$. Note that $\mathcal{I}^* := \bigoplus \mathcal{I}^n$ is a finitely generated $\mathcal{O}_{\mathcal{X}}$-algebra and $\mathcal{I}^*\mathcal{F} := \bigoplus \mathcal{I}^n\mathcal{F}$ is a finitely generated $\mathcal{I}^*$-module [AM69, Lem. 10.8]. If we let $I_* = \bigoplus I_n = \Gamma(\mathcal{X}, \mathcal{I}^*)$, then $\operatorname{Spec}_{\mathcal{X}} \mathcal{I}^* \to \operatorname{Spec} I_*$ is an adequate moduli space [Alp14, Lem. 5.2.11]. It follows that I_* is a finitely generated A-algebra and that $F_* := \bigoplus F_n = \Gamma(\mathcal{X}, \mathcal{I}^*\mathcal{F})$ is a finitely generated I_*-module [Alp14, Thm. 6.3.3].

Hence, there is a sufficiently divisible integer $N \geq 1$ (e.g., a common multiple of the degrees of a set of homogeneous A-algebra generators for I_*) such that $I_{Nk} = (I_N)^k$ for all $k \geq 1$. That is, the topology induced by the non-adic system I_n is equivalent to the I_N-adic topology. Without loss of generality, we can replace $\mathcal{I}$ with $\mathcal{I}^N$ so that $I_* = I^* = \bigoplus_{k \geq 0} I^k$.

Similarly, for sufficiently large n (e.g., larger than all degrees of a set of homogeneous generators), $F_{n+1} = IF_n$ [AM69, Lem. 10.8]; that is, (F_n) is an I-stable filtration on $F := \Gamma(\mathcal{X}, \mathcal{F})$. It follows that (F_n) induces the same topology on F as $(I^n F)$ [AM69, Lem. 10.6]. But F is a finite A-module, hence I-adically complete, hence complete with respect to (F_n). □

Corollary 4.2 (Formal functions, good version)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack that is cohomologically affine. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack defined by a sheaf of ideals $\mathcal{I}$. Let $I = \Gamma(\mathcal{X}, \mathcal{I})$ be the corresponding ideal of $A = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. If A is I-adically complete, then for every $\mathcal{F} \in \mathsf{Coh}(\mathcal{X})$ the following natural map*

$$\Gamma(\mathcal{X}, \mathcal{F}) \to \varprojlim_n \Gamma(\mathcal{X}, \mathcal{F}/\mathcal{I}^n \mathcal{F}) \tag{4.2}$$

is an isomorphism.

Proof. By [Alp13, Thm. 4.16(x)], the ring A is noetherian and by [AHR19, Thm. A.1], $\mathcal{X} \to \operatorname{Spec} A$ is of finite type so Theorem 4.1 applies. For good moduli spaces, the natural map $\Gamma(\mathcal{X}, \mathcal{F})/\Gamma(\mathcal{X}, \mathcal{I}^n \mathcal{F}) \to \Gamma(\mathcal{X}, \mathcal{F}/\mathcal{I}^n \mathcal{F})$ is an isomorphism by definition. □

Remark 4.3. The formal functions theorem generalizes the isomorphism of [AHR19, Eqn. (2.1)] from the case of $\mathcal{X} = [\operatorname{Spec} B/G]$ for G linearly reductive and $A = B^G$ complete local, all defined over a field k, to $\mathcal{X} = [\operatorname{Spec} B/\mathrm{GL}_n]$ and $A = B^{\mathrm{GL}_n}$ complete but not necessarily local. This also includes $[\operatorname{Spec} B/G]$ for G geometrically reductive and embeddable, see Remark 2.5.

Remark 4.4. In the setting of Theorem 4.1, if $H^i(\mathcal{X}, -)$ preserves coherence for all i, then it seems likely that (4.2) is an isomorphism with an argument similar to [EGA, III.4.1.5]. We note that if $\mathcal{X} = [\operatorname{Spec}(A)/G]$ where A is a finitely generated k-algebra and G is reductive group over k, then it follows from [TK10, Thm. 1.1] that $H^i(\mathcal{X}, -)$ preserves coherence.

5. Coherently complete pairs of algebraic stacks

The main result of this section is the the following important special case of Theorem 1.6.

Proposition 5.1 (Coherent completeness assuming resolution property)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal and good moduli space $\pi\colon \mathcal{X} \to X = \operatorname{Spec} A$. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack defined by a coherent sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ and let $I = \Gamma(\mathcal{X}, \mathcal{I})$. Assume that $\mathcal{X}$ has the resolution property. If A is I-adically complete, then $\mathcal{X}$ is coherently complete along $\mathcal{Z}$.*

Note that in this proposition, $\mathcal{X}$ is assumed to have the resolution property, whereas in Theorem 1.6 it is only assumed that $\mathcal{Z}$ has the resolution property. The proof of Theorem 1.6 will be completed in Section 10.

The following full faithfulness result uses arguments similar to those of [EGA, III.5.1.3] and [GZB15, Thm. 1.1(i)].

Lemma 5.2. *Let $\mathcal{X}$ be a noetherian algebraic stack that is cohomologically affine. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack defined by a sheaf of ideals $\mathcal{I}$. Let $I = \Gamma(\mathcal{X}, \mathcal{I})$ be the corresponding ideal of $A = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. If A is I-adically complete, then the functor*

$$\mathsf{Coh}(\mathcal{X}) \to \varprojlim_n \mathsf{Coh}(\mathcal{X}_{\mathcal{Z}}^{[n]}).$$

is fully faithful.

Proof. Following [Con, §1], let $\mathcal{O}_{\widehat{\mathcal{X}}}$ denote the sheaf of rings on the lisse-étale site of $\mathcal{X}$ that assigns to each smooth morphism $p\colon \operatorname{Spec} B \to \mathcal{X}$ the ring $\varprojlim_n B/\mathcal{I}^n B$. The sheaf of rings $\mathcal{O}_{\widehat{\mathcal{X}}}$ is coherent and the natural functor

$$\mathsf{Coh}(\widehat{\mathcal{X}}) \to \varprojlim_n \mathsf{Coh}(\mathcal{X}_{\mathcal{Z}}^{[n]})$$

is an equivalence of categories [Con, Thm. 2.3]. Let $c\colon \widehat{\mathcal{X}} \to \mathcal{X}$ denote the induced morphism of ringed topoi and let $\mathcal{F}, \mathcal{G} \in \mathsf{Coh}(\mathcal{X})$; then it remains to prove that the map

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_{\widehat{\mathcal{X}}}}(c^*\mathcal{F}, c^*\mathcal{G})$$

is bijective. Now we have the following commutative square, whose vertical arrows are isomorphisms:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{O}_{\widehat{\mathcal{X}}}}(c^*\mathcal{F}, c^*\mathcal{G}) \arrow[d] \\
\Gamma(\mathcal{X}, \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G})) \arrow[r] & \Gamma(\widehat{\mathcal{X}}, \mathcal{H}om_{\mathcal{O}_{\widehat{\mathcal{X}}}}(c^*\mathcal{F}, c^*\mathcal{G})).
\end{tikzcd}$$

Since c is flat and $\mathcal{F}$ is coherent the natural morphism

$$c^*\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_{\widehat{\mathcal{X}}}}(c^*\mathcal{F}, c^*\mathcal{G})$$

is an isomorphism [GZB15, Lem. 3.2]. Thus, it remains to prove that the map

$$\Gamma(\mathcal{X}, \mathcal{Q}) \to \Gamma(\widehat{\mathcal{X}}, c^*\mathcal{Q})$$

is an isomorphism whenever $\mathcal{Q} \in \mathsf{Coh}(\mathcal{X})$. But there are natural isomorphisms:

$$\Gamma(\widehat{\mathcal{X}}, c^*\mathcal{Q}) \cong \varprojlim_n \Gamma(\widehat{\mathcal{X}}, \mathcal{Q}/\mathcal{I}^{n+1}\mathcal{Q}) \cong \varprojlim_n \Gamma(\mathcal{X}_{\mathcal{Z}}^{[n]}, \mathcal{Q}/\mathcal{I}^{n+1}\mathcal{Q}) \cong \varprojlim_n \Gamma(\mathcal{X}, \mathcal{Q}/\mathcal{I}^{n+1}\mathcal{Q}).$$

The result now follows from Corollary 4.2. □

Proof of Proposition 5.1. By Lemma 5.2 it remains to show that if $\{\mathcal{F}_n\} \in \varprojlim_n \mathsf{Coh}(\mathcal{X}_{\mathcal{Z}}^{[n]})$, then there exists a coherent $\mathcal{F}$ on $\mathcal{X}$ with $(i^{[n]})^*\mathcal{F} \simeq \mathcal{F}_n$ for all n. Now $\mathcal{X}$ has the resolution property, so there is a vector bundle $\mathcal{E}$ on $\mathcal{X}$ together with a surjection $\phi_0\colon \mathcal{E} \to \mathcal{F}_0$. We claim that ϕ_0 lifts to a compatible system of morphisms $\phi_n\colon \mathcal{E} \to \mathcal{F}_n$ for every $n > 0$. Indeed, since $\mathcal{E}^\vee \otimes \mathcal{F}_{n+1} \to \mathcal{E}^\vee \otimes \mathcal{F}_n$ is surjective and $\Gamma(\mathcal{X}, -)$ is exact, it follows that the natural map $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{F}_{n+1}) \to \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{F}_n)$ is surjective. By Nakayama's Lemma (see Remark 3.3), each ϕ_n is surjective.

It follows that we obtain an induced morphism of systems $\{\phi_n\}\colon \{\mathcal{E}_n\} \to \{\mathcal{F}_n\}$, which is surjective. Applying this procedure to the kernel of $\{\phi_n\}$, there is another vector bundle $\mathcal{H}$ and a morphism of systems $\{\psi_n\}\colon \{\mathcal{H}_n\} \to \{\mathcal{E}_n\}$ such that $\operatorname{coker}\{\psi_n\} \cong \{\mathcal{F}_n\}$. By the full faithfulness (Lemma 5.2), the morphism $\{\psi_n\}$ arises from a unique morphism $\psi\colon \mathcal{H} \to \mathcal{E}$. Let $\widetilde{\mathcal{F}} = \operatorname{coker} \psi$; then the universal property of cokernels proves that there is an isomorphism of systems $\{\widetilde{\mathcal{F}}_n\} \cong \{\mathcal{F}_n\}$ and the result follows. □

We conclude this section with the following key example.

Example 5.3. Let $S = \operatorname{Spec} B$ where B is a noetherian ring. Let $G \subseteq \mathrm{GL}_{n,S}$ be a linearly reductive closed subgroup scheme acting on a noetherian affine scheme $X = \operatorname{Spec} A$. Then $[\operatorname{Spec} A/G]$ satisfies the resolution property; see Remark 2.5. If $(A^G, \mathfrak{m})$ is an $\mathfrak{m}$-adically complete local ring, then it follows from Proposition 5.1 that $[\operatorname{Spec} A/G]$ is coherently complete along the unique closed point. When S is the spectrum of a field and the unique closed G-orbit is a fixed point, this is [AHR19, Thm. 1.3].

6. Effectivity I: general setup and characteristic zero

In this section, we consider an adic sequence $\{\mathcal{X}_n\}_{n \geq 0}$ of noetherian algebraic stacks (see Definition 1.8). A classical result states that if each $\mathcal{X}_i$ is affine, then $A = \varprojlim_n \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ is a noetherian ring and $\mathcal{X}_i$ is the ith infinitesimal neighborhood of $\mathcal{X}_0$ in $\operatorname{Spec} A$ [EGA, 0_{I}.7.2.8]. One of our main results (Theorem 1.10) is that an analogous result also holds when $\mathcal{X}_0$ is linearly fundamental. In this section, we will prove this result in characteristic 0 and lay the groundwork for the general case.

6.1. **Preliminary lemmas.**

Setup 6.1. Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence of noetherian algebraic stacks and let $\mathcal{I}_{(j)}$ be the coherent sheaf of ideals defining the closed immersion $u_{0j}\colon \mathcal{X}_0 \hookrightarrow \mathcal{X}_j$. Let $A_n = \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$, $X_n = \operatorname{Spec} A_n$, $A = \varprojlim_n A_n$, $I_n = \ker(A \to A_{n-1})$, and $X = \operatorname{Spec} A$.

A key observation here is that the sequence of closed immersions of affine schemes:

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots . \tag{6.1}$$

is not adic (this is just as in the proof of Theorem 4.1). The following lemma shows that the sequence (6.1) is equivalent to an adic one, however.

Lemma 6.2. *Assume Setup 6.1. If $\mathcal{X}_0$ is cohomologically affine, then A is noetherian and I_1-adically complete.*

Proof. Let $\mathcal{A}_{(i)} = \operatorname{Gr}_{\mathcal{I}_{(i)}} \mathcal{O}_{\mathcal{X}_i} = \bigoplus_{j=0}^{i} \mathcal{I}_{(i)}^j/\mathcal{I}_{(i)}^{j+1}$. This is a graded $\mathcal{O}_{\mathcal{X}_0}$-algebra that is finitely generated in degree 1. If $i \leq k$, then $\mathcal{A}_{(i)} = \mathcal{A}_{(k)}^{\leq i}$. In particular, if $\mathcal{F}_i := \mathcal{I}_{(i)}^i$, then $\mathcal{F}_i = \mathcal{I}_{(k)}^i/\mathcal{I}_{(k)}^{i+1}$ for every $k \geq i$ and $\mathcal{A}^\bullet = \bigoplus_{j=0}^{\infty} \mathcal{F}_j$ is an $\mathcal{O}_{\mathcal{X}_0}$-algebra that is finitely generated in degree 1. Moreover, $I_n/I_{n+1} = \ker(A_n \to A_{n-1}) = \Gamma(\mathcal{X}_0, \mathcal{F}_n)$ (here we use cohomological affineness). Thus, $\Gamma(\mathcal{X}_0, \mathcal{A}^\bullet) = \operatorname{Gr}_{I_*} A := \bigoplus I_n/I_{n+1}$.

Now by [AHR19, Lem. A.2], $\operatorname{Gr}_{I_*} A$ is a finitely generated and graded A_0-algebra. That is, for the filtration $\{I_n\}_{n\geq 0}$ on the ring A, the associated graded ring is a noetherian A_0-algebra. It follows from [God56, Thm. 4] that A is noetherian.

Since A is noetherian and complete with respect to the topology defined by $\{I_n\}_{n\geq 0}$, it is also complete with respect to the I_1-adic topology. Indeed, if $\widehat{A}$ denotes the I_1-adic completion of A, then there is a natural factorization

$$A \to \widehat{A} = \varprojlim_n A/(I_1)^n \to A = \varprojlim_n A/I_n$$

of the identity. Since $\widehat{A} \to A$ is surjective and $\widehat{A}$ is noetherian and complete with respect to the I_1-adic topology, so is A. $\square$

The following lemma generalizes [AHR19, Prop. A.8(1)] to the non-local situation.

Lemma 6.3. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{I}$ be a nilpotent quasi-coherent sheaf of ideals of $\mathcal{O}_{\mathcal{X}}$. Let $\mathcal{X}_1 \subseteq \mathcal{X}$ be the closed immersion defined by $\mathcal{I}^2$. If the composition $\mathcal{X}_1 \to \mathcal{X} \xrightarrow{f} \mathcal{Y}$ is a closed immersion, then f is a closed immersion.*

Proof. The statement is local on $\mathcal{Y}$ for the smooth topology, so we may assume that $\mathcal{Y} = \operatorname{Spec} A$. Since $\mathcal{X}_1$ is affine and $\mathcal{X}$ is an infinitesimal thickening of $\mathcal{X}_1$, it follows that $\mathcal{X}$ is also affine [Ryd15, Cor. 8.2]. Hence, we may assume that $\mathcal{X} = \operatorname{Spec} B$ and $\mathcal{I} = \widetilde{I}$ for some nilpotent ideal I of B. Let $\phi\colon A \to B$ be the induced morphism.

The assumptions are that the composition $A \to B \to B/I^2$ is surjective and that $I^{n+1} = 0$ for some $n \geq 0$. Let $K = \ker(A \to B/I)$. Since $KB \to I \to I/I^2$ is surjective and $I^{n+1} = 0$, it follows that $KB \to I$ is surjective by Nakayama's Lemma for B-modules ($I = KB + I^2 = KB + I^4 = \cdots = KB$). That is, $KB = I$.

Further, since $A \to B \to B/KB = B/I$ is surjective and $K^{n+1}B = I^{n+1} = 0$, it follows that $\phi\colon A \to B$ is surjective by Nakayama's Lemma for A-modules ($B = \operatorname{im}\phi + KB = \operatorname{im}\phi + K^2B = \cdots = \operatorname{im}\phi$). $\square$

The following lifting lemma is a key result.

Lemma 6.4. *Let S be an affine scheme. Let $\mathcal{Z} \hookrightarrow \mathcal{Z}'$ be a closed immersion of algebraic S-stacks defined by a nilpotent quasi-coherent sheaf of ideals $\mathcal{I}$. Let $f\colon \mathcal{Z} \to \mathcal{W}$ be a representable morphism of algebraic S-stacks. If $\mathcal{W} \to S$ is smooth and $\mathcal{Z}$ is cohomologically affine, then there is lift of f to a S-morphism $f'\colon \mathcal{Z}' \to \mathcal{W}$.*

Proof. By induction, we immediately reduce to the situation where $\mathcal{I}^2 = 0$. The obstruction to lifting f now belongs to the group $\operatorname{Ext}^1_{\mathcal{O}_\mathcal{Z}}(\mathsf{L}f^*L_{\mathcal{W}/S}, \mathcal{I})$ [Ols06, Thm. 1.5]. Since $\mathcal{W} \to S$ is smooth, the cotangent complex $L_{\mathcal{W}/S}$ is perfect of amplitude $[0,1]$. The assumption that $\mathcal{Z}$ is cohomologically affine now proves that this obstruction group vanishes. Hence, there is an S-lift $f'\colon \mathcal{Z}' \to \mathcal{Y}$ as claimed. $\square$

We now come to a general embedding lemma. We state it in greater generality than strictly needed now, so we can use it later in the paper.

Lemma 6.5. *Assume Setup 6.1. Let $\mathcal{Y} \to X$ be smooth and fundamental. If $\mathcal{X}_0$ is cohomologically affine and there is a representable morphism $\mathcal{X}_0 \to \mathcal{Y}$, then there exists*

1. *an affine morphism $\mathcal{H} \to \mathcal{Y}$; and*
2. *compatible closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{H}$*

such that the natural morphism $H \to X$, where H is the adequate moduli space of $\mathcal{H}$, is finite, adequate, and admits a section. In particular, H is noetherian and $\mathcal{H}$ has linearly reductive stabilizers at closed points.

Remark 6.6. Once we establish Theorem 9.3 in Section 9, it will follow that $\mathcal{H}$ is necessarily cohomologically affine.

Proof of Lemma 6.5. By [AHR19, Thm. A.1], $\mathcal{X}_0 \to X_0$ is of finite type. Hence, $\mathcal{X}_0 \to X$ is of finite type and cohomologically affine. But the diagonal of $\mathcal{Y} \to X$ is affine and of finite type, so $\phi_0\colon \mathcal{X}_0 \to \mathcal{Y}$ is cohomologically affine and of finite type. By assumption, it is representable, so Serre's Theorem (e.g., [Alp13, Prop. 3.3]) tells us that $\phi_0\colon \mathcal{X}_0 \to \mathcal{Y}$ is also affine. By Lemma 6.4, there is a lift of ϕ_0 to $\phi_1\colon \mathcal{X}_1 \to \mathcal{Y}$.

Since $\mathcal{Y}$ has the resolution property, there exists a vector bundle of finite rank $\mathcal{E}$ on $\mathcal{Y}$ and a surjection of quasi-coherent $\mathcal{O}_\mathcal{Y}$-algebras $\operatorname{Sym}_{\mathcal{O}_\mathcal{Y}}(\mathcal{E}) \to (\phi_1)_*\mathcal{O}_{\mathcal{X}_1}$. Let $\widetilde{\mathcal{H}}$ be the relative spectrum of $\operatorname{Sym}_{\mathcal{O}_\mathcal{Y}}(\mathcal{E})$; then there is an induced closed immersion $i_1\colon \mathcal{X}_1 \hookrightarrow \widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}} \to X$ is smooth. Using Lemma 6.4, we can produce compatible X-morphisms $i_n\colon \mathcal{X}_n \to \widetilde{\mathcal{H}}$ lifting i_1. By Lemma 6.3, the i_n are all closed immersions.

Let $\widetilde{H} = \operatorname{Spec}\Gamma(\widetilde{\mathcal{H}}, \mathcal{O}_{\widetilde{\mathcal{H}}})$ be the adequate moduli space of $\widetilde{\mathcal{H}}$. Since $\widetilde{\mathcal{H}} \to X$ is of finite type and X is noetherian (Lemma 6.2), $\widetilde{H} \to X$ is of finite type [Alp14, Thm. 6.3.3] and so $\widetilde{\mathcal{H}} \to \widetilde{H}$ is of finite type and $\widetilde{H}$ is noetherian. Since $\mathcal{X}_n \to X_n$ is a good moduli space, there are uniquely induced morphisms $X_n \to \widetilde{H}$. Passing to limits, we produce a unique morphism $x\colon X \to \widetilde{H}$; moreover, the composition $X \to \widetilde{H} \to X$ is the identity. Take $\mathcal{H}$ to be the base change of $\widetilde{\mathcal{H}} \to \widetilde{H}$ along $X \to \widetilde{H}$. We now take $H = \operatorname{Spec}\Gamma(\mathcal{H}, \mathcal{O}_\mathcal{H})$; then arguing as before we see that $H \to X$ is now an adequate universal homeomorphism of finite type, which is finite. Since $\mathcal{H} \to X$ is universally closed, the statement about stabilizers only needs to be verified when X is a field. But $H \to X$ is a finite universal homeomorphism, so $\mathcal{H}$ has a unique closed point and this is in the image of $\mathcal{X}_0$. The claim follows. $\square$

6.2. **Effectivity.**

Proposition 6.7. *In the situation of Lemma 6.5, if $\mathcal{H}$ is cohomologically affine (e.g., if $\mathcal{Y}$ is cohomologically affine), then the completion of the sequence $\{\mathcal{X}_n\}_{n\geq 0}$ exists and is a closed substack of $\mathcal{H}$.*

Proof. By pulling back $\mathcal{H} \to H$ along the section $X \to H$, we may further assume in Lemma 6.5 that $H = X$. Let $\mathcal{H}_0 = \mathcal{X}_0$ and for $n > 0$ let $\mathcal{H}_n$ be the nth infinitesimal neighborhood of $\mathcal{H}_0$ in $\mathcal{H}$. Then the closed immersions $i_n \colon \mathcal{X}_n \to \mathcal{H}$ factor uniquely through closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{H}_n$. Since the system $\{\mathcal{X}_n\}_{n\geq 0}$ is adic, the 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}_{n-1} \arrow[r] \arrow[d] & \mathcal{X}_n \arrow[d] \\ \mathcal{H}_{n-1} \arrow[r] & \mathcal{H}_n \end{tikzcd}$$

is 2-cartesian. Indeed, since $\mathcal{X}_n \to \mathcal{H}$ is a closed immersion, $\mathcal{X}_n \times_{\mathcal{H}} \mathcal{H}_0 = \mathcal{X}_0$. If we let $\mathcal{K}$ be the sheaf of ideals defining the closed immersion $\mathcal{H}_0 \to \mathcal{H}$, this means that $\mathcal{K}\mathcal{O}_{\mathcal{X}_n} = \mathcal{I}_{(n)}$ and hence that $\mathcal{K}^n\mathcal{O}_{\mathcal{X}_n} = \mathcal{I}_{(n)}^n$ which shows that the diagram is 2-cartesian.

But $\mathcal{H}$ is linearly fundamental, so $\mathcal{H}$ is coherently complete along $\mathcal{H}_0$ (Proposition 5.1) and so there exists a closed immersion $\widehat{\mathcal{X}} \hookrightarrow \mathcal{H}$ that induces the $\mathcal{X}_n$. □

Corollary 6.8 (Effectivity in characteristic zero)**.** *Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence of noetherian algebraic $\mathbb{Q}$-stacks. If $\mathcal{X}_0$ is linearly fundamental, then the completion of the sequence exists and is linearly fundamental.*

Proof. Since $\mathcal{X}_0$ is linearly fundamental, it admits an affine morphism to $B\mathrm{GL}_{N,\mathbb{Q}}$ for some $N > 0$. This gives an affine morphism $\mathcal{X}_0 \to \mathcal{Y} := B\mathrm{GL}_{N,X}$. Note that X is a $\mathbb{Q}$-scheme, so $\mathcal{Y}$ is cohomologically affine. Since $\mathcal{H} \to \mathcal{Y}$ is affine in Lemma 6.5, we conclude that $\mathcal{H}$ also is cohomologically affine. The result now follows from Proposition 6.7. □

To prove effectivity in positive and mixed characteristic (Theorem 1.10), we will need to make a better choice of group than $\mathrm{GL}_{N,\mathbb{Q}}$. To do this, we will study the deformations of nice group schemes in Section 7. This neatly handles effectivity in the "local case", i.e., when $\mathcal{X}_0$ is a gerbe over a field so that the completion is a local stack with residual gerbe $\mathcal{X}_0$, see Section 8. The local case is used to prove Theorem 9.3 in Section 9, which in turn is used to prove the general effectivity theorem in Section 10.

7. Deformation of nice group schemes

In this section, we will prove Proposition 7.1 which asserts that a nice and embeddable group scheme (see Definition 2.1) can be deformed along an affine henselian pair (Definition 3.1). This will be used to prove the effectivity theorem for a local ring in positive or mixed characteristic (Proposition 8.1). After we have established the general effectivity result, we will prove the corresponding result for linearly reductive group schemes (Proposition 16.8).

Proposition 7.1 (Deformation of nice group schemes)**.** *Let (S, S_0) be an affine henselian pair. If $G_0 \to S_0$ is a nice and embeddable group scheme, then there exists a nice and embeddable group scheme $G \to S$ whose restriction to S_0 is isomorphic to G_0.*

Proof. Let $(S, S_0) = (\operatorname{Spec} A, \operatorname{Spec} A/I)$. By standard reductions (using Lemma 2.12), we may assume that S is the henselization of an affine scheme of finite type over $\operatorname{Spec} \mathbb{Z}$. Let $S_n = \operatorname{Spec} A/I^{n+1}$. Also, let R be the I-adic completion of A and let $\widehat{S} = \operatorname{Spec} R$.

Let $F \colon (\mathsf{Sch}/S)^{\mathrm{opp}} \to \mathsf{Sets}$ be the functor that assigns to each S-scheme T the set of isomorphism classes of nice and embeddable group schemes over T. By

Lemma 2.12, F is limit preserving. Suppose that we have a nice embeddable group scheme $G_{\widehat{S}} \in F(\widehat{S})$ restricting to G_0. By Artin Approximation (Theorem 3.4), there exists $G_S \in F(S)$ that restricts to G_0. We can thus replace S by $\widehat{S}$ and assume that A is complete.

Fix a closed immersion of S_0-group schemes $i\colon G_0 \to \mathrm{GL}_{n,S_0}$. By definition, there is an open and closed subgroup $(G_0)^0 \subseteq G_0$ of multiplicative type. By [SGA3$_{\mathrm{II}}$, Exp. XI, Thm. 5.8], there is a lift of i to a closed immersion of group schemes $i_S\colon G_S^0 \to \mathrm{GL}_{n,S}$, where G_S^0 is of multiplicative type. Let $N = \mathrm{Norm}_{\mathrm{GL}_{n,S}}(G_S^0)$ be the normalizer, which is a smooth S-group scheme and closed S-subgroup scheme of $\mathrm{GL}_{n,S}$ [SGA3$_{\mathrm{II}}$, Exp. XI, 5.3 bis].

Since $(G_0)^0$ is a normal S_0-subgroup scheme of G_0, it follows that G_0 is a closed S_0-subgroup scheme of $N \times_S S_0$. In particular, there is an induced closed immersion $q_{S_0}\colon (G_0)/(G_0)^0 \to (N/G_S^0) \times_S S_0$ of group schemes over S_0. Since G_0 is nice, the locally constant group scheme $(G_0)/(G_0)^0$ has order prime to p. Since R is complete, there is a unique locally constant group scheme H over S such that $H \times_S S_0 = (G_0)/(G_0)^0$. Note that H is finite and linearly reductive over S.

Since N/G_S^0 is a smooth and affine group scheme over S, there are compatible closed immersions of S_n-group schemes $q_{S_n}\colon H \times_S S_n \to (N/G_S^0) \times_S S_n$ lifting q_{S_0}, which are unique up to conjugation [SGA3$_{\mathrm{I}}$, Exp. III, Cor. 2.8]. Since H is finite, these morphisms effectivize to a morphism of group schemes $q_S\colon H \to N/G_S^0$. We now define G_S to be the preimage of H under the quotient map $N \to N/G_S^0$. Then G_S is nice and embeddable, and $G_S \times_S S_0 \cong G$. ☐

8. Effectivity II: local case in positive characteristic

In this short section, we apply the results of the previous section on nice group schemes to establish the next level of generality for our effectivity theorem (Theorem 1.10). The main result of this section is the following proposition which uses the terminology of nicely fundamental stacks introduced in Definition 2.7.

Proposition 8.1 (Effectivity for nice stacks)**.** *Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence of noetherian algebraic stacks. If $\mathcal{X}_0$ is nicely fundamental, then the completion of the sequence exists and is nicely fundamental.*

Proof. Let X_0 be the good moduli space of $\mathcal{X}_0$. Since $\mathcal{X}_0$ is nicely fundamental, it admits an affine morphism to $B_{X_0}Q_0$, for some nice and embeddable group scheme $Q_0 \to X_0$. Now let $X = \operatorname{Spec}\big(\varprojlim_n \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})\big)$ as in Setup 6.1. By Lemma 6.2, X is complete along X_0. It follows from Proposition 7.1 that there is a nice and embeddable group scheme $Q \to X$ lifting $Q_0 \to X_0$. Let $\mathcal{Y} = B_X Q$; then $\mathcal{Y} \to X$ is smooth and linearly fundamental. The result now follows immediately from Proposition 6.7. ☐

The following corollary will shortly be subsumed by Theorem 1.10. We include it here, however, because it is an essential step in the proof of Theorem 9.3, which features in the full proof of Theorem 1.10. We expect Corollary 8.2 to be sufficient for many applications.

Corollary 8.2 (Effectivity for local stacks)**.** *Let $\{\mathcal{X}_n\}_{n\geq 0}$ be an adic sequence of noetherian algebraic stacks. Assume that $\mathcal{X}_0$ is a gerbe over a field k. If $\mathcal{X}_0$ is linearly fundamental (i.e., has linearly reductive stabilizer), then the completion of the sequence exists and is linearly fundamental.*

Proof. If $\mathcal{X}$ is a $\mathbb{Q}$-stack, then we are already done by Corollary 6.8. If not, then k has characteristic $p > 0$ and $\mathcal{X}_0$ is nicely fundamental by Remark 2.11. Proposition 8.1 completes the proof. ☐

9. Adequate moduli spaces with linearly reductive stabilizers are good

In this section we prove that adequate moduli spaces of stacks with linearly reductive stabilizers at closed points are good (Theorem 9.3). This uses the adequate version of the formal function theorem (Theorem 4.1) and the effectivity theorem in the form of Corollary 8.2. This theorem is fundamental in proving the general effectivity result (Theorem 1.10) and therefore in the proof of Theorem 1.1.

Lemma 9.1. *Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed substack defined by the sheaf of ideals $\mathcal{I}$. Assume that $\mathcal{X}$ has an adequate moduli space $\pi\colon \mathcal{X} \to \operatorname{Spec} A$ of finite type, where A is noetherian and I-adically complete along $I = \Gamma(\mathcal{X}, \mathcal{I})$. Let $B_n = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1})$ for $n \geq 0$ and $B = \varprojlim_n B_n$. If $\mathcal{Z}$ is cohomologically affine with affine diagonal, then the induced homomorphism $A \to B$ is finite.*

Proof. Let $I_n = \Gamma(\mathcal{X}, \mathcal{I}^n)$. By Theorem 4.1, A is complete with respect to the filtration given by (I_n), that is, $A = \varprojlim_n A/I_n$. We note that $A/I_{n+1} \to B_n$ is injective and adequate for all n. In particular, the homomorphism $A \to B$ is an injective continuous map between complete topological rings.

Since $\mathcal{Z} = X^{[0]}$ is cohomologically affine with affine diagonal, so are its infinitesimal neighborhoods $X^{[n]}$. It follows that $B_n \to B_{n-1}$ is surjective with kernel $\Gamma(\mathcal{X}, \mathcal{I}^n/\mathcal{I}^{n+1})$ for all n. Thus, if we let $J_{n+1} = \ker(B \to B_n)$, then $J_n/J_{n+1} = \Gamma(\mathcal{X}, \mathcal{I}^n/\mathcal{I}^{n+1})$ and the topology on B is given by the filtration (J_n).

The surjection $\mathcal{I}^n \to \mathcal{I}^n/\mathcal{I}^{n+1}$ induces an injective map $I_n/I_{n+1} \to J_n/J_{n+1}$. Taking direct sums gives a surjection of algebras $\bigoplus \mathcal{I}^n \to \operatorname{Gr}_{\mathcal{I}}(\mathcal{O}_{\mathcal{X}})$, hence an injective adequate map $\operatorname{Gr}_{I_*} A = \bigoplus I_n/I_{n+1} \to \operatorname{Gr}_{J_*} B = \bigoplus J_n/J_{n+1}$.

We further note that $\operatorname{Gr}_{\mathcal{I}}(\mathcal{O}_{\mathcal{X}})$ is a finitely generated algebra. Since $\operatorname{Spec}\big(\operatorname{Gr}_{J_*} B\big)$ is the adequate moduli space of $\operatorname{Spec}_{\mathcal{X}}\big(\operatorname{Gr}_{\mathcal{I}}(\mathcal{O}_{\mathcal{X}})\big)$, it follows that $\operatorname{Gr}_{J_*} B$ is a finitely generated A-algebra [Alp14, Thm. 6.3.3]. Thus $\operatorname{Gr}_{I_*} A \to \operatorname{Gr}_{J_*} B$ is an injective adequate map of finite type, hence finite. It follows that $A \to B$ is finite [God56, Lem. on p. 6]. ☐

Remark 9.2. It is, a priori, not clear that $A \to B$ is adequate. Consider the following example: $A = \mathbb{F}_2[\![x]\!]$, $B = A[y]/(y^2 - x^2y - x)$. Then $\operatorname{Spec} B \to \operatorname{Spec} A$ is a ramified, generically étale, finite flat cover of degree 2, so not adequate. But the induced map on graded rings $\mathbb{F}_2[x] \to \mathbb{F}_2[x,y]/(y^2 - x)$ is adequate. Nevertheless, it follows from Theorem 9.3, proven below, that $A = B$ in Lemma 9.1. If the formal functions theorem (Corollary 4.2) holds for stacks with adequate moduli spaces, then $A = B$ without assuming that $\mathcal{Z}$ is cohomologically affine.

Theorem 9.3. *Let S be a noetherian algebraic space. Let $\mathcal{X}$ be an algebraic stack of finite type over S with an adequate moduli space $\pi\colon \mathcal{X} \to X$. Assume that π has affine diagonal. Then π is a good moduli space if and only if every closed point of $\mathcal{X}$ has linearly reductive stabilizer.*

Remark 9.4. See Corollary 13.7 and Corollary 13.11 for non-noetherian versions.

Proof. We begin by noting that X is of finite type over S [Alp14, Thm. 6.3.3]. We can thus replace S with X. If π is a good moduli space, then every closed point has linearly reductive stabilizer [Alp13, Prop. 12.14]. For the converse, we need to prove that π_* is exact. This can be verified after replacing $X = S$ with the completion at every closed point. We may thus assume that $X = S$ is a complete local scheme.

By Corollary 8.2, the adic sequence $\mathcal{X}_0 \hookrightarrow \mathcal{X}_1 \hookrightarrow \dots$ has completion $\widehat{\mathcal{X}}$ that has a good moduli space X'. By Tannaka duality (see §1.7.6), there is a natural map

$f\colon \widehat{\mathcal{X}} \to \mathcal{X}$. This induces a map $g\colon X' \to X$ of adequate moduli spaces. In the notation of Lemma 9.1, $X' = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, and we conclude that $X' \to X$ is finite. In particular, $f\colon \widehat{\mathcal{X}} \to \mathcal{X}$ is also of finite type since the good moduli map $\widehat{\mathcal{X}} \to X'$ is of finite type [AHR19, Thm. A.1]. The morphism $f\colon \widehat{\mathcal{X}} \to \mathcal{X}$ is formally étale, hence étale, and also affine [AHR19, Prop. 3.2], hence representable. Moreover, $f\colon \widehat{\mathcal{X}} \to \mathcal{X}$ induces an isomorphism of stabilizer groups at the unique closed points so we may apply Luna's fundamental lemma (Theorem 3.14) to conclude that $X' \times_X \mathcal{X} = \widehat{\mathcal{X}}$ and thus $f\colon \widehat{\mathcal{X}} \to \mathcal{X}$ is finite. But f is an isomorphism over the unique closed point of $\mathcal{X}$, hence f is a closed immersion. But f is also étale, hence a closed and open immersion, hence an isomorphism. We conclude that $X' = X$ and thus that π_* is exact. □

As an immediate corollary, we obtain:

Corollary 9.5. *Let S be a noetherian algebraic space and let $G \to S$ be an affine flat group scheme of finite presentation. Then $G \to S$ is linearly reductive if and only if $G \to S$ is geometrically reductive and every closed fiber is linearly reductive.* □

The corollary also holds in the non-noetherian case by Corollary 13.11.

10. Effectivity III: the general case

We now finally come to the proof of the general effectivity result for adic systems of algebraic stacks..

Proof of Theorem 1.10. Let X be as in Setup 6.1. Since $\mathcal{X}_0$ is linearly fundamental, it admits an affine morphism to $\mathcal{Y} = B\mathrm{GL}_{N,X}$. By Lemma 6.5, there is an affine morphism $\mathcal{H} \to \mathcal{Y}$ and compatible closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{H}$ such that the induced morphism $H \to X$ (where H is the adequate moduli space of $\mathcal{H}$) is finite, adequate, and admits a section. In particular, H is noetherian and $\mathcal{H}$ has linearly reductive stabilizers at closed points. By Theorem 9.3, $\mathcal{H}$ is cohomologically affine. The result now follows from Proposition 6.7. □

We can now finish the general coherent completeness theorem:

Proof of Theorem 1.6. The necessity of the condition follows from Proposition 3.7. By effectivity (Theorem 1.10), the completion $\widehat{\mathcal{X}}$ of $\{\mathcal{X}^{[n]}_{\mathcal{Z}}\}$ exists and is linearly fundamental. By formal functions (Corollary 4.2), the good moduli space of $\widehat{\mathcal{X}}$ is X. By Tannaka duality, there is an induced morphism $f\colon \widehat{\mathcal{X}} \to \mathcal{X}$ and it is affine [AHR19, Prop. 3.2], cf. Proposition 12.5(1). The composition $\widehat{\mathcal{X}} \to \mathcal{X} \to X$ is a good moduli space and hence of finite type [AHR19, Thm. A.1]. It follows that f is of finite type. Since f is formally étale, it is thus étale. Luna's fundamental lemma (Theorem 3.14) now implies that $f\colon \widehat{\mathcal{X}} \to \mathcal{X}$ is an isomorphism. In particular, $\mathcal{X}$ is linearly fundamental, i.e. has the resolution property. □

We are now in position to prove Formal GAGA (Corollary 1.7).

Proof of Corollary 1.7. The first case follows from the second since if $I \subseteq R$ is a maximal ideal, $\mathcal{X} \times_{\operatorname{Spec} R} \operatorname{Spec}(R/I)$ necessarily has the resolution property [AHR19, Cor. 4.14]. The corollary then follows from applying Theorem 1.6 with $\mathcal{Z} = \mathcal{X} \times_{\operatorname{Spec} R} \operatorname{Spec}(R/I)$. □

11. Formally syntomic neighborhoods

In this section, we prove Theorem 1.11, which establishes the existence of formally syntomic neighborhoods of locally closed substacks. We then use this theorem to prove Theorem 1.12 establishing the existence of completions at points with linearly reductive stabilizers. These two results are stated and proved more generally for *pro-unramified* morphisms; see Theorems 11.1 and 11.2.

If $\mathcal{X}$ is a noetherian algebraic stack, then a morphism $\mathcal{V} \to \mathcal{X}$ is *pro-unramified* (resp. a *pro-immersion*) if it can be written as a composition $\mathcal{V} \hookrightarrow \mathcal{V}' \to \mathcal{X}$, where $\mathcal{V} \hookrightarrow \mathcal{V}'$ is a flat quasi-compact monomorphism and $\mathcal{V}' \hookrightarrow \mathcal{X}$ is unramified and of finite type (resp. a closed immersion). Clearly, pro-immersions are pro-unramified. Note that residual gerbes on quasi-separated algebraic stacks are pro-immersions [Ryd11b, Thm. B.2]. Moreover, every monomorphism of finite type is pro-unramified.

11.1. **Existence of formally syntomic neighborhoods.** As promised, we now establish the following generalization of Theorem 1.11.

Theorem 11.1 (Formal neighborhoods)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack. Let $\mathcal{X}_0 \to \mathcal{X}$ be pro-unramified. Let $h_0\colon \mathcal{W}_0 \to \mathcal{X}_0$ be a syntomic (e.g., smooth) morphism. Assume that $\mathcal{W}_0$ is linearly fundamental. If either*

1. *$\mathcal{X}$ has quasi-affine diagonal; or*
2. *$\mathcal{X}$ has affine stabilizers and $\Gamma(\mathcal{W}_0, \mathcal{O}_{\mathcal{W}_0})$ is quasi-excellent;*

then there is a flat morphism $h\colon \widehat{\mathcal{W}} \to \mathcal{X}$, where $\widehat{\mathcal{W}}$ is noetherian, linearly fundamental, $h|_{\mathcal{X}_0} \simeq h_0$, and $\widehat{\mathcal{W}}$ is coherently complete along $\mathcal{W}_0 = h^{-1}(\mathcal{X}_0)$. Moreover if h_0 is smooth (resp. étale), then h is unique up to non-unique 1-isomorphism (resp. unique up to unique 2-isomorphism).

Proof. Since $\mathcal{X}_0 \to \mathcal{X}$ is pro-unramified, it factors as $\mathcal{X}_0 \xrightarrow{j} \mathcal{V}_0 \xrightarrow{u} \mathcal{X}$, where j is a flat quasi-compact monomorphism and u is unramified and of finite type. Note that j is schematic [Stacks, Tag 0B8A] and even quasi-affine [Ray68, Prop. 1.5] and that $\mathcal{X}_0$ is noetherian [Ray68, Prop. 1.2]. By [Ryd11a, Thm. 1.2], there is a further factorization $\mathcal{V}_0 \xrightarrow{i} \mathcal{X}' \xrightarrow{p} \mathcal{X}$, where i is a closed immersion and p is étale and finitely presented. Since p has quasi-affine diagonal, we may replace $\mathcal{X}$ by $\mathcal{X}'$.

Let $g_0 = j \circ h_0\colon \mathcal{W}_0 \to \mathcal{V}_0 = \mathcal{X}^{[0]}_{\mathcal{V}_0}$. We claim that it suffices to prove, using induction on $n \geq 1$, that there are compatible cartesian diagrams:

$$\begin{tikzcd} \mathcal{W}_{n-1} \arrow[r] \arrow[d, "g_{n-1}"'] & \mathcal{W}_n \arrow[d, "g_n"] \\ \mathcal{X}^{[n-1]}_{\mathcal{V}_0} \arrow[r] & \mathcal{X}^{[n]}_{\mathcal{V}_0}, \end{tikzcd}$$

where each g_n is flat and the $\mathcal{W}_n$ are noetherian. Indeed, the flatness of the g_n implies that the resulting system $\{\mathcal{W}_n\}_{n\geq 0}$ is adic. By Theorem 1.10, the completion $\widehat{\mathcal{W}}$ of the sequence $\{\mathcal{W}_n\}_{n\geq 0}$ exists and is noetherian and linearly fundamental. If $\mathcal{X}$ has quasi-affine diagonal, then the morphisms $\mathcal{W}_n \to \mathcal{X}$ induce a unique morphism $\widehat{\mathcal{W}}$ by Tannaka duality (case (b) of §1.7.6). If $\mathcal{X}$ only has affine stabilizers, however, then Tannaka duality (case (a) of §1.7.6) has the additional hypothesis that $\widehat{\mathcal{W}}$ is locally the spectrum of a G-ring, so we prove that the quasi-excellency of $\Gamma(\mathcal{W}_0, \mathcal{O}_{\mathcal{W}_0})$ implies this. But $A = \Gamma(\widehat{\mathcal{W}}, \mathcal{O}_{\widehat{\mathcal{W}}})$ is a J-adically complete noetherian ring, where $J = \ker\big(\Gamma(\widehat{\mathcal{W}}, \mathcal{O}_{\widehat{\mathcal{W}}}) \to \Gamma(\mathcal{W}_0, \mathcal{O}_{\mathcal{W}_0})\big)$. Since $\widehat{\mathcal{W}}$ is linearly fundamental, $A/J = \Gamma(\mathcal{W}_0, \mathcal{O}_{\mathcal{W}_0})$. Hence, A is quasi-excellent by the Gabber–Kurano–Shimomoto Theorem [KS16, Main Thm. 1]. But $\widehat{\mathcal{W}} \to \operatorname{Spec} A$ is of finite type, so $\widehat{\mathcal{W}}$ is locally quasi-excellent. The flatness of $\widehat{\mathcal{W}} \to \mathcal{X}$ is just the local criterion for flatness [EGA, 0_{III}.10.2.1].

We now get back to solving the lifting problem. By [Ols06, Thm. 1.4], the obstruction to lifting g_{n-1} to g_n belongs to the group $\mathrm{Ext}^2_{\mathcal{O}_{\mathcal{W}_0}}(L_{\mathcal{W}_0/\mathcal{V}_0}, g_0^*(\mathcal{I}^n/\mathcal{I}^{n+1}))$, where $\mathcal{I}$ is the coherent ideal sheaf defining the closed immersion $i\colon \mathcal{V}_0 \hookrightarrow \mathcal{X}$. Note that Olsson's paper requires that g_0 is representable. To work around this, we may choose an affine morphism $\mathcal{W}_0 \to B\mathrm{GL}_N$ for some N and replace $\mathcal{X}$ with $\mathcal{X} \times B\mathrm{GL}_N$; since $B\mathrm{GL}_N$ has smooth diagonal, the induced *representable* morphism $\mathcal{W}_0 \to \mathcal{X}_0 \times B\mathrm{GL}_N$ is syntomic.

Now since $\mathcal{X}_0 \to \mathcal{V}_0$ is a flat monomorphism, it follows immediately that $L_{\mathcal{X}_0/\mathcal{V}_0} \simeq 0$ [LMB, Prop. 17.8]. Hence, $L_{\mathcal{W}_0/\mathcal{V}_0} \simeq L_{\mathcal{W}_0/\mathcal{X}_0}$. But $\mathcal{W}_0 \to \mathcal{X}_0$ is syntomic, so $L_{\mathcal{W}_0/\mathcal{X}_0}$ is perfect of amplitude $[-1, 0]$ and $\mathcal{W}_0$ is cohomologically affine. Thus, the Ext-group vanishes, and we have the required lift. That $\mathcal{W}_n$ is noetherian is clear: it is a thickening of a noetherian stack by a coherent sheaf of ideals.

For the uniqueness statement: Let $h\colon \widehat{\mathcal{W}} \to \mathcal{X}$ and $h'\colon \widehat{\mathcal{W}'} \to \mathcal{X}$ be two different morphisms as in the theorem. Let $g_n = j_n \circ h_n\colon \mathcal{W}_n \to \mathcal{X}^{[n]}_{\mathcal{V}_0}$ and $g'_n = j_n \circ h'_n\colon \mathcal{W}'_n \to \mathcal{X}^{[n]}_{\mathcal{V}_0}$ be the induced nth infinitesimal neighborhoods. By Tannaka duality, it is enough to show that an isomorphism $f_{n-1}\colon \mathcal{W}_{n-1} \to \mathcal{W}'_{n-1}$ lifts (resp. lifts up to a unique 2-isomorphism) to an isomorphism $f_n\colon \mathcal{W}_n \to \mathcal{W}'_n$. The obstruction to a lift lies in $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{W}_0}}(f_0^* L_{\mathcal{W}'_0/\mathcal{V}_0}, g_0^*(\mathcal{I}^n/\mathcal{I}^{n+1}))$, which vanishes if $h_0 = h'_0$ is smooth. The obstruction to the existence of a 2-isomorphism between two lifts lies in $\mathrm{Ext}^0_{\mathcal{O}_{\mathcal{W}_0}}(L_{\mathcal{W}'_0/\mathcal{V}_0}, g_0^*(\mathcal{I}^n/\mathcal{I}^{n+1}))$ and the 2-automorphisms of a lift lies in $\mathrm{Ext}^{-1}_{\mathcal{O}_{\mathcal{W}_0}}(L_{\mathcal{W}'_0/\mathcal{V}_0}, g_0^*(\mathcal{I}^n/\mathcal{I}^{n+1}))$. All three groups vanish if h_0 is étale. $\square$

11.2. **Existence of completions.** If $\mathcal{X}_0 \to \mathcal{X}$ is a morphism of algebraic stacks, we say that a morphism of pairs $(\mathcal{W}, \mathcal{W}_0) \to (\mathcal{X}, \mathcal{X}_0)$ (that is, compatible maps $\mathcal{W} \to \mathcal{X}$ and $\mathcal{W}_0 \to \mathcal{X}_0$) is the *completion of* $\mathcal{X}$ *along* $\mathcal{X}_0$ if $(\mathcal{W}, \mathcal{W}_0)$ is a coherently complete pair (Definition 3.1) and $(\mathcal{W}, \mathcal{W}_0) \to (\mathcal{X}, \mathcal{X}_0)$ is final among morphisms from coherently complete pairs. That is, if $(\mathcal{Z}, \mathcal{Z}_0) \to (\mathcal{X}, \mathcal{X}_0)$ is any other morphism of pairs from a coherently complete pair, there exists a morphism $(\mathcal{Z}, \mathcal{Z}_0) \to (\mathcal{W}, \mathcal{W}_0)$ over $\mathcal{X}$ unique up to unique 2-isomorphism. In particular, the pair $(\mathcal{W}, \mathcal{W}_0)$ is unique up to unique 2-isomorphism.

We prove the following generalization of Theorem 1.12.

Theorem 11.2 (Existence of completions)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack. Let $\mathcal{X}_0 \to \mathcal{X}$ be a pro-immersion such that $\mathcal{X}_0$ is linearly fundamental, e.g., the residual gerbe at a point with linearly reductive stabilizer. If either*

(1) *$\mathcal{X}$ has quasi-affine diagonal; or*
(2) *$\mathcal{X}$ has affine stabilizers and $\Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0})$ is quasi-excellent;*

then the completion of $\mathcal{X}$ along $\mathcal{X}_0$ exists and is linearly fundamental.

Proof. Let $\widehat{\mathcal{X}} \to \mathcal{X}$ be the flat morphism extending the pro-immersion $\mathcal{X}_0 \to \mathcal{X}$ of Theorem 11.1 applied to $\mathcal{W}_0 = \mathcal{X}_0$. Let $(\mathcal{Z}, \mathcal{Z}_0)$ be any other coherently complete stack with a morphism $\varphi\colon \mathcal{Z} \to \mathcal{X}$ such that $\varphi|_{\mathcal{Z}_0}$ factors through $\mathcal{X}_0$. Let $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ be the sheaf of ideals defining the closure of $\mathcal{X}_0$. Then $\mathcal{X}_n = V(\mathcal{I}^{n+1}\mathcal{O}_{\widehat{\mathcal{X}}})$ and $\mathcal{Z}_n \subseteq V(\mathcal{I}^{n+1}\mathcal{O}_{\mathcal{Z}})$. Since $\mathcal{X}_n \to V(\mathcal{I}^{n+1})$ is a flat monomorphism, it follows that $\mathcal{Z}_n \to \mathcal{X}$ factors uniquely through $\mathcal{X}_n$. By coherent completeness of $\mathcal{Z}$ and Tannaka duality (using that $\widehat{\mathcal{X}}$ has affine diagonal), there is a unique morphism $\mathcal{Z} \to \widehat{\mathcal{X}}$. $\square$

If $\mathcal{X}$ is a noetherian algebraic stack, then we let $\widehat{\mathcal{X}}_x$ denote the completion at a point x with linearly reductive stabilizer. Note that when $x = V(\mathcal{I})$ is a closed point, then $\widehat{\mathcal{X}}_x = \varinjlim_n V(\mathcal{I}^{n+1})$ in the category of algebraic stacks with affine stabilizers.

12. The local structure of algebraic stacks

In this section, we prove a slightly more general version of the local structure theorem (Theorem 1.1).

Theorem 12.1 (Local structure). *Suppose that:*

- *S is a quasi-separated algebraic space;*
- *$\mathcal{X}$ is an algebraic stack, locally of finite presentation and quasi-separated over S, with affine stabilizers;*
- *$x \in |\mathcal{X}|$ is a point with residual gerbe $\mathcal{G}_x$ and image $s \in |S|$ such that the residue field extension $\kappa(x)/\kappa(s)$ is finite; and*
- *$h_0 \colon \mathcal{W}_0 \to \mathcal{G}_x$ is a smooth (resp., étale) morphism where $\mathcal{W}_0$ is linearly fundamental and $\Gamma(\mathcal{W}_0, \mathcal{O}_{\mathcal{W}_0})$ is a field.*

Then there exists a cartesian diagram of algebraic stacks

$$\begin{tikzcd} & \mathcal{W}_0 \arrow[r, "h_0"] \arrow[d] & \mathcal{G}_x \arrow[d] \\ {[\operatorname{Spec} A/\mathrm{GL}_n]} \arrow[r, phantom, "="] & \mathcal{W} \arrow[r, "h"] & \mathcal{X} \end{tikzcd}$$

where $h \colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ is a smooth (resp., étale) pointed morphism and w is closed in its fiber over s. Moreover, if $\mathcal{X}$ has separated (resp., affine) diagonal and h_0 is representable, then h can be arranged to be representable (resp., affine).

Remark 12.2. Theorem 1.1 is the special case when in addition $\mathcal{W}_0$ is a gerbe over the spectrum of a field.

Remark 12.3. In Theorems 1.1 and 12.1, the condition that $\kappa(x)/\kappa(s)$ is finite is equivalent to the condition that the morphism $\mathcal{G}_x \to \mathcal{X}_s$ is of finite type. In particular, it holds if x is closed in its fiber $\mathcal{X}_s = \mathcal{X} \times_S \operatorname{Spec} \kappa(s)$.

Proof of Theorem 1.1. **Step 1: Reduction to S an excellent scheme.** It is enough to find a solution $(\mathcal{W}, w) \to (\mathcal{X}, x)$ after replacing S with an étale neighborhood of s so we can assume that S is affine. We can also replace $\mathcal{X}$ with a quasi-compact neighborhood of x and assume that $\mathcal{X}$ is of finite presentation.

Write S as a limit of affine schemes S_λ of finite type over $\operatorname{Spec} \mathbb{Z}$. For sufficiently large λ, we can find $\mathcal{X}_\lambda \to S_\lambda$ of finite presentation such that $\mathcal{X} = \mathcal{X}_\lambda \times_{S_\lambda} S$. Let $w_0 \in |\mathcal{W}_0|$ be the unique closed point and let $x_\lambda \in |\mathcal{X}_\lambda|$ be the image of x. Since $\mathcal{G}_x$ is the limit of the $\mathcal{G}_{x_\lambda}$, we can, for sufficiently large λ, also find a smooth (or étale if h_0 is étale) morphism $h_{0,\lambda} \colon (\mathcal{W}_{0,\lambda}, w_{0,\lambda}) \to (\mathcal{G}_{x_\lambda}, x_\lambda)$ with pull-back h_0. For sufficiently large λ:

1. $\mathcal{X}_\lambda$ has affine stabilizers [HR15, Thm. 2.8];
2. if $\mathcal{X}$ has separated (resp. affine) diagonal, then so has $\mathcal{X}_\lambda$;
3. $\operatorname{Stab}(x_\lambda) = \operatorname{Stab}(x)$ (because $\operatorname{Stab}(x_\mu) \to \operatorname{Stab}(x_\lambda)$ is a closed immersion for every $\mu > \lambda$); and
4. $\mathcal{W}_{0,\lambda}$ is fundamental (Lemma 2.15).

That $\mathcal{G}_x \to \mathcal{G}_{x_\lambda}$ is stabilizer-preserving implies that $\mathcal{G}_x = \mathcal{G}_{x_\lambda} \times_{\operatorname{Spec} \kappa(x_\lambda)} \operatorname{Spec} \kappa(x)$ and, in particular, $\mathcal{W}_0 = \mathcal{W}_{0,\lambda} \times_{\operatorname{Spec} \kappa(x_\lambda)} \operatorname{Spec} \kappa(x)$. It follows, by flat descent, that $\mathcal{W}_{0,\lambda}$ is cohomologically affine and that $\Gamma(\mathcal{W}_{0,\lambda}, \mathcal{O}_{\mathcal{W}_{0,\lambda}})$ is the spectrum of a field. We can thus replace S, $\mathcal{X}$, $\mathcal{W}_0$ with S_λ, $\mathcal{X}_\lambda$, $\mathcal{W}_{0,\lambda}$ and assume that S is an excellent scheme. By standard limit arguments, it is also enough to find a solution after replacing S with $\operatorname{Spec} \mathcal{O}_{S,s}$. We can thus assume that s is closed.

Step 2: An effective formally smooth solution. Since $\mathcal{W}_0$ is linearly fundamental, we can find a formal neighborhood of $\mathcal{W}_0 \to \mathcal{X}_0 := \mathcal{G}_x \hookrightarrow \mathcal{X}$, that is, deform the smooth morphism $\mathcal{W}_0 \to \mathcal{X}_0$ to a flat morphism $\widehat{\mathcal{W}} \to \mathcal{X}$ where $\widehat{\mathcal{W}}$ is a linearly fundamental stack which is coherently complete along $\mathcal{W}_0$ (Theorem 1.11). Since $\mathcal{W}_n \to \mathcal{X}_n$ is smooth, $\widehat{\mathcal{W}} \to \mathcal{X}$ is formally smooth at $\mathcal{W}_0$ [AHR19, Prop. A.14].

Step 3: Algebraization. We now apply equivariant Artin algebraization (Theorem 12.4 below) to obtain a fundamental stack $\mathcal{W}$, a closed point $w \in \mathcal{W}$, a morphism $h\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ smooth at w, and an isomorphism $\widehat{\mathcal{W}}_w \cong \widehat{\mathcal{W}}$ over $\mathcal{X}$. Let $\widetilde{\mathcal{W}}_0 = h^{-1}(\overline{\mathcal{X}}_0)$. Then $(\widetilde{\mathcal{W}}_0)\widehat{\ }_w \cong (\mathcal{W}_0)\widehat{\ }_{w_0} = \mathcal{W}_0$ since $\mathcal{W}_0$ is complete along w_0. It follows that after shrinking $\mathcal{W}$, the adequate moduli space of $\widetilde{\mathcal{W}}_0$ is a point and $\widetilde{\mathcal{W}}_0 = \mathcal{W}_0 = h^{-1}(\mathcal{X}_0)$.

If $h_0\colon \mathcal{W}_0 \to \mathcal{X}_0$ is étale, then h is étale at w. After shrinking $\mathcal{W}$, we can assume that h is smooth (resp. étale). If $\mathcal{X}$ has separated (resp. affine) diagonal, then we can shrink $\mathcal{W}$ so that h becomes representable (resp. affine), see Proposition 12.5 below.

To keep $\mathcal{W}$ adequately affine during these shrinkings we proceed as follows. If $\pi\colon \mathcal{W} \to W$ is the adequate moduli space, then when shrinking to an open neighborhood U of w, we shrink to the smaller open neighborhood $\pi^{-1}(V)$ where V is an open affine neighborhood of $\pi(w)$ contained in $W \smallsetminus \pi(\mathcal{W} \smallsetminus U)\big)$. □

In the proof we used the following version of equivariant Artin algebraization:

Theorem 12.4 (Equivariant Artin algebraization)**.** *Let S be an excellent scheme. Let $\mathcal{X}$ be an algebraic stack, locally of finite presentation over S. Let $\mathcal{Z}$ be a noetherian fundamental stack with adequate moduli space map $\pi\colon \mathcal{Z} \to Z$ of finite type (automatic if $\mathcal{Z}$ is linearly fundamental). Let $z \in |\mathcal{Z}|$ be a closed point such that $\mathcal{G}_z \to S$ is of finite type. Let $\eta\colon \mathcal{Z} \to \mathcal{X}$ be a morphism over S that is formally versal at z. Then there exists*

1. *an algebraic stack $\mathcal{W}$ which is fundamental and of finite type over S;*
2. *a closed point $w \in \mathcal{W}$;*
3. *a morphism $\xi\colon \mathcal{W} \to \mathcal{X}$ over S; and*
4. *isomorphisms $\varphi^{[n]}\colon \mathcal{W}^{[n]} \to \mathcal{Z}^{[n]}$ over $\mathcal{X}$ for every n;*
5. *if $\operatorname{Stab}(z)$ is linearly reductive, an isomorphism $\widehat{\varphi}\colon \widehat{\mathcal{W}} \to \widehat{\mathcal{Z}}$ over $\mathcal{X}$, where $\widehat{\mathcal{W}}$ and $\widehat{\mathcal{Z}}$ denote the completions of $\mathcal{W}$ at w and $\mathcal{Z}$ at z which exist by Theorem 1.12.*

In particular, ξ is formally versal at w.

Proof. We apply [AHR19, Thm. A.18] with $T = Z$ and $\mathcal{X}_1 = \mathcal{X}$ and $\mathcal{X}_2 = B\mathrm{GL}_n$ for a suitable n such that there exists an affine morphism $\mathcal{Z} \to \mathcal{X}_2$. This gives (1)–(4) and (5) is an immediate consequence of (4). □

We also used the following generalization of [AHR19, Prop. 3.2 and Prop. 3.4], which also answers part of [AHR19, Question 1.10].

Proposition 12.5. *Let $f\colon \mathcal{W} \to \mathcal{X}$ be a morphism of algebraic stacks such that $\mathcal{W}$ is adequately affine with affine diagonal (e.g., fundamental). Suppose $\mathcal{W}_0 \subseteq \mathcal{W}$ is a closed substack such that $f|_{\mathcal{W}_0}$ is representable.*

1. *If $\mathcal{X}$ has affine diagonal, then there exists an adequately affine open neighborhood $\mathcal{U} \subseteq \mathcal{W}$ of $\mathcal{W}_0$ such that $f|_{\mathcal{U}}$ is affine.*
2. *If $\mathcal{X}$ has separated diagonal and $\mathcal{W}$ is fundamental, then there exists an adequately affine open neighborhood $\mathcal{U} \subseteq \mathcal{W}$ of $\mathcal{W}_0$ such that $f|_{\mathcal{U}}$ is representable.*

Proof. Since $f|_{\mathcal{W}_0}$ is representable, we can after replacing $\mathcal{W}$ with an open, adequately affine, neighborhood of $\mathcal{W}_0$, assume that f has quasi-finite diagonal (or in fact, even unramified diagonal). For (1) we argue exactly as in [AHR19, Prop. 3.2] but replace [Alp13, Prop. 3.3] with [Alp14, Cor. 4.3.2].

For (2), we note that the subgroup $G := I_{\mathcal{W}/\mathcal{X}} \hookrightarrow I_{\mathcal{W}}$ is closed because $\mathcal{X}$ has separated diagonal and is quasi-finite over $\mathcal{W}$ because f has quasi-finite diagonal. We conclude by Lemma 12.6 below and Nakayama's lemma. □

Lemma 12.6. *Let $\mathcal{W}$ be a fundamental stack and let $G \hookrightarrow I_{\mathcal{W}}$ be a closed subgroup. If $G \to \mathcal{W}$ is quasi-finite, then $G \to \mathcal{W}$ is finite.*

Proof. Note that $I_{\mathcal{W}} \to \mathcal{W}$ is affine so $G \to \mathcal{W}$ is also affine. If $h \in |G|$ is a point, then the order of h is finite. It is thus enough to prove the following: if $h \in |I_{\mathcal{W}}|$ is a point of finite order such that $\mathcal{Z} := \overline{\{h\}} \to \mathcal{W}$ is quasi-finite, then $\mathcal{Z} \to \mathcal{W}$ is finite. Using approximation of fundamental stacks (Lemma 2.14) we reduce this question to the case that $\mathcal{W}$ is of finite presentation over $\operatorname{Spec} \mathbb{Z}$.

By [Alp14, Lem. 8.3.1], it is enough to prove that $\mathcal{Z} \to \mathcal{W}$ takes closed points to closed points and that the morphism on their adequate moduli spaces $Z \to W$ is universally closed. This can be checked using DVRs as follows: for every DVR R with fraction field K, every morphism $f\colon \operatorname{Spec} R \to W$ and every lift $h\colon \operatorname{Spec} K \to \mathcal{Z}$, there exists a lift $\widetilde{h}\colon \operatorname{Spec} R \to \mathcal{Z}$ such that the closed point $0 \in \operatorname{Spec} R$ maps to a point in $\mathcal{W}$ that is closed in the fiber over $f(0)$.

Since $\mathcal{W} \to W$ is universally closed, we can start with a lift $\xi\colon \operatorname{Spec} R \to \mathcal{W}$, such that $\xi(0)$ is closed in the fiber over $f(0)$. We can then identify h with an automorphism $h \in \operatorname{Aut}_{\mathcal{W}}(\xi)(K)$ of finite order. Applying [AHH18, Prop. 5.7 and Lem. 5.14][2] gives us an extension of DVRs $R \hookrightarrow R'$ and a new lift $\xi'\colon \operatorname{Spec} R' \to \mathcal{W}$ such that $\xi'(0) = \xi(0)$ together with an automorphism $\widetilde{h} \in \operatorname{Aut}_{\mathcal{W}}(\xi')(R')$. Since $\mathcal{Z}$ is closed in $I_{\mathcal{W}}$, this is a morphism $\widetilde{h}\colon \operatorname{Spec} R \to \mathcal{Z}$ as requested. □

Remark 12.7. If $h \in |I_{\mathcal{W}}|$ is any element of finite order, then every element of $\mathcal{Z} = \overline{\{h\}}$ is of finite order but $\mathcal{Z}$ is not always quasi-finite. For an example see [AHH18, Ex. 3.54].

13. Applications to good moduli spaces and linearly reductive groups

In this section, we prove that if $\pi\colon \mathcal{X} \to X$ is a good moduli space, with affine stabilizers and separated diagonal, then $\mathcal{X}$ has the resolution property étale-locally on X (Theorem 13.1). This generalizes [AHR19, Thm. 4.12] to the relative case. We also give a version for adequate moduli spaces (Theorem 13.10).

Theorem 13.1. *Let $\mathcal{X}$ be an algebraic stack with good moduli space $\pi\colon \mathcal{X} \to X$. Assume that $\mathcal{X}$ has affine stabilizers, separated diagonal and is of finite presentation over a quasi-separated algebraic space S. Then*

1. *there is a Nisnevich covering $X' \to X$ such that the pull-back $\mathcal{X}' = \mathcal{X} \times_X X'$ is linearly fundamental,*
2. *$\pi\colon \mathcal{X} \to X$ has affine diagonal, and*
3. *$X \to S$ is of finite presentation,*
4. *$\pi_* \mathcal{F}$ is finitely presented if $\mathcal{F}$ is a finitely presented $\mathcal{O}_{\mathcal{X}}$-module.*

Moreover, if every closed point $x \in |X|$ either has $\operatorname{char} \kappa(x) > 0$ or has an open neighborhood of characteristic zero, then we can arrange that $\mathcal{X}' \cong [\operatorname{Spec} A/G]$ where $G \to X'$ is linearly reductive and embeddable.

If there are closed points of characteristic zero without characteristic zero neighborhoods, then it is sometimes impossible to find a linearly reductive G; see Appendix A.

[2] While the paper [AHH18] cites this paper on several occasions, the proofs of [AHH18, Prop. 5.7 and Lem. 5.14] do not rely on results of this paper.

Corollary 13.2. *Let S be a quasi-separated algebraic space and let $G \to S$ be a linearly reductive group scheme (or merely a separated group algebraic space, flat of finite presentation, with affine fibers such that $BG \to S$ is a good moduli space). Then there exists a Nisnevich covering $S' \to S$ such that $G' = G \times_S S'$ is embeddable.* □

Remark 13.3. A consequence of Corollary 13.2 is that in the definition of a tame group scheme given in [Hoy17, Defn. 2.26], if we assume that $G \to B$ is separated with affine fibers, then the condition on having the G-resolution property Nisnevich-locally is automatic.

Before proving Theorem 13.1, we study the structure around points of positive characteristic.

13.1. **Niceness is étale local.** We show that fundamental stacks (resp. geometrically reductive and embeddable group schemes) are étale-locally nicely fundamental (resp. nice) near points of positive characteristic.

Proposition 13.4. *Let $\mathcal{X}$ be a fundamental algebraic stack with adequate moduli space $\mathcal{X} \to X$. Let $x \in |X|$ be a point and let $y \in |\mathcal{X}|$ be the unique closed point in the fiber of x. If the stabilizer of y is nice, then there exists an étale neighborhood $(X', x') \to (X, x)$, with $\kappa(x') = \kappa(x)$, such that $\mathcal{X} \times_X X'$ is nicely fundamental.*

Proof. Since nicely fundamental stacks can be approximated (Lemma 2.15(1)), we may assume that X is henselian with closed point x. Then y is the unique closed point of $\mathcal{X}$. Note that the residual gerbe $\mathcal{G}_y = \overline{\{y\}}$ is nicely fundamental (cf. Remark 2.11).

We can write $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$ where the $\mathcal{X}_\lambda$ are fundamental and of finite type over $\operatorname{Spec} \mathbb{Z}$ with adequate moduli space X_λ of finite type over $\operatorname{Spec} \mathbb{Z}$ (Lemma 2.14). Let $x_\lambda \in X_\lambda$ be the image of x and let $y_\lambda \in |\mathcal{X}_\lambda|$ be the unique closed point above x_λ. Then y_λ is contained in the closure of the image of y. Thus, for sufficiently large λ, we can assume that y_λ has nice stabilizer (Lemma 2.15(3)).

Let X^h_λ denote the henselization of X_λ at x_λ and $\mathcal{X}^h_\lambda = \mathcal{X}_\lambda \times_{X_\lambda} X^h_\lambda$. Then the canonical map $X \to X_\lambda$ factors uniquely through X^h_λ and the induced map $\mathcal{X} \to \mathcal{X}^h_\lambda$ is affine. It is thus enough to prove that $\mathcal{X}^h_\lambda$ is nicely fundamental. By Theorem 9.3, the adequate moduli space $\mathcal{X}^h_\lambda \to X^h_\lambda$ is good, that is, $\mathcal{X}^h_\lambda$ is linearly fundamental.

We can thus assume that X is excellent and that $\mathcal{X}$ is linearly fundamental. Let $\mathcal{X}_n$ be the nth infinitesimal neighborhood of x. Let $Q_0 \to \operatorname{Spec} \kappa(x)$ be a nice group scheme such that there exists an affine morphism $f_0 \colon \mathcal{X}_0 \to B_{\kappa(x)} Q_0$. By the existence of deformations of nice group schemes (Proposition 7.1), there exists a nice and embeddable group scheme $Q \to X$. Let $\mathcal{I} \subseteq \mathcal{X}$ denote the sheaf of ideals defining $\mathcal{X}_0$. By [Ols06, Thm. 1.5], the obstruction to lifting a morphism $\mathcal{X}_{n-1} \to B_X Q$ to $\mathcal{X}_n \to B_X Q$ is an element of $\operatorname{Ext}^1_{\mathcal{O}_{\mathcal{X}_0}}(Lf_0^* L_{B_X Q/X}, \mathcal{I}^n/\mathcal{I}^{n+1})$. The obstruction vanishes because the cotangent complex $L_{B_X Q/X}$ is perfect of amplitude $[0, 1]$, since $B_X Q \to X$ is smooth, and $\mathcal{X}_0$ is cohomologically affine.

Let $\widehat{X} = \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{X}} = \mathcal{X} \times_X \widehat{X}$. Since $\widehat{\mathcal{X}}$ is linearly fundamental, it is coherently complete along $\mathcal{X}_0$ (Proposition 5.1). By Tannaka duality (see §1.7.6), we may thus extend $\mathcal{X}_0 \to B_{X_0} Q_0$ to a morphism $\widehat{\mathcal{X}} \to B_X Q$, which is affine by Proposition 12.5(1). Applying Artin approximation (Theorem 3.4) to the functor $\operatorname{Hom}_X(\mathcal{X} \times_X -, B_X Q) \colon (\mathsf{Sch}/X)^{\mathrm{opp}} \to \mathsf{Sets}$ yields an affine morphism $\mathcal{X} \to B_X Q$. □

Note that if $\mathcal{X}$ is linearly fundamental and $\operatorname{char} \kappa(x) > 0$, then y has nice stabilizer. We thus have the following corollaries:

Corollary 13.5. *Let $\mathcal{X}$ be a linearly fundamental algebraic stack with good moduli space $\mathcal{X} \to X$ and let $x \in |X|$ be a point. If either $\operatorname{char} \kappa(x) > 0$ or x has an open neighborhood of characteristic zero, then there exists an étale neighborhood $(X', x') \to (X, x)$, with $\kappa(x') = \kappa(x)$, such that $\mathcal{X} \times_X X' = [\operatorname{Spec} A/G]$ where $G \to X'$ is a linearly reductive embeddable group scheme.* □

Corollary 13.6. *Let (S, s) be a Henselian local scheme such that $\operatorname{char} \kappa(s) > 0$.*

(1) *If $\mathcal{X}$ is a linearly fundamental algebraic stack with good moduli space $\mathcal{X} \to S$, then $\mathcal{X}$ is nicely fundamental.*
(2) *If $G \to S$ is a linearly reductive and embeddable group scheme, then $G \to S$ is nice.* □

We also obtain the following non-noetherian variant of Theorem 9.3 at the expense of assuming that $\mathcal{X}$ has the resolution property. Also see Corollary 13.11 for a different variant.

Corollary 13.7. *Let $\mathcal{X}$ be a fundamental algebraic stack. Then the following are equivalent.*

(1) *$\mathcal{X}$ is linearly fundamental.*
(2) *Every closed point of $\mathcal{X}$ has linearly reductive stabilizer.*
(3) *Every closed point of $\mathcal{X}$ with positive characteristic has nice stabilizer.*

Proof. The only non-trivial implication is (3) $\implies$ (1). Let $\pi\colon \mathcal{X} \to X$ be the adequate moduli space. It is enough to prove that π is a good moduli space after base change to the henselization at a closed point. We may thus assume that X is the spectrum of a henselian local ring. If X is a $\mathbb{Q}$-scheme, then the notions of adequate and good coincide. If not, then the closed point of X has positive characteristic, hence the unique closed point of $\mathcal{X}$ has nice stabilizer. We conclude that $\mathcal{X}$ is nicely fundamental by Proposition 13.4. □

Corollary 13.8. *Let $\mathcal{X}$ be a fundamental stack with adequate moduli space $\pi\colon \mathcal{X} \to S$. Let $g\colon S' \to S$ be a morphism of algebraic spaces such that $\mathcal{X}' := \mathcal{X} \times_S S'$ has a good moduli space. Then $\pi'\colon \mathcal{X}' \to S'$ is its good moduli space and the natural transformation $g^*\pi_* \to \pi'_* g'^*$ is an isomorphism on all quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules.*

Proof. Both claims can be checked on stalks so we may assume that $S' = \operatorname{Spec} A'$ and $S = \operatorname{Spec} A$ are spectra of local rings and that the closed point $s' \in S'$ maps to the closed point $s \in S$. Since $\mathcal{X}'$ has a good moduli space, it follows that the unique closed point of $\mathcal{X}$ has linearly reductive stabilizer. Hence $\mathcal{X}$ is linearly fundamental (Corollary 13.7) and the result follows from [Alp13, Prop. 4.7]. □

Corollary 13.9. *Let $\mathcal{X}$ be a linearly fundamental stack of finite presentation over a quasi-separated algebraic space S with good moduli space $\pi\colon \mathcal{X} \to X$. Then X is of finite presentation over S and π_* takes finitely presented $\mathcal{O}_{\mathcal{X}}$-modules to finitely presented $\mathcal{O}_X$-modules.*

Proof. We may assume that S is quasi-compact and can thus write S as an inverse limit of algebraic spaces S_λ of finite presentation over $\operatorname{Spec} \mathbb{Z}$ with affine transition maps [Ryd15, Thm. D]. For sufficiently large λ, we can find $\mathcal{X}_\lambda \to S_\lambda$ of finite presentation that pulls back to $\mathcal{X} \to S$. After increasing λ, we can assume that $\mathcal{X}_\lambda$ is fundamental by Lemma 2.15(1) and that a given $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ of finite presentation is the pull-back of a coherent $\mathcal{O}_{\mathcal{X}_\lambda}$-module $\mathcal{F}_\lambda$. Then $\mathcal{X}_\lambda$ has an adequate moduli space X_λ of finite presentation over S_λ and the push-forward of $\mathcal{F}_\lambda$ is a coherent $\mathcal{O}_{X_\lambda}$-module [Alp14, Thm. 6.3.3]. The result now follows from Corollary 13.8. In particular, $X = X_\lambda \times_{S_\lambda} S$ is the good moduli space of $\mathcal{X}$. □

13.2. **Étale-local structure of stacks with adequate moduli spaces around points with linearly reductive stabilizer.**

Theorem 13.10. *Let $\mathcal{X}$ be an algebraic stack with adequate moduli space $\pi\colon \mathcal{X} \to X$ and let $x \in X$ be a point. Assume that*

1. *$\mathcal{X}$ has affine stabilizers and separated diagonal,*
2. *$\mathcal{X}$ is of finite presentation over a quasi-separated algebraic space, and*
3. *the unique closed point in $\pi^{-1}(x)$ has linearly reductive stabilizer.*

Then there exists an étale neighborhood $(X', x') \to (X, x)$ with $\kappa(x') = \kappa(x)$ such that the pull-back $\mathcal{X}'$ of $\mathcal{X}$ is fundamental. That is, there is a cartesian diagram

$$\begin{tikzcd} {[\operatorname{Spec} A/\mathrm{GL}_n] = \mathcal{X}'} \arrow[r, "f"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} B = X' \arrow[r] & X. \end{tikzcd}$$

where π' is an adequate moduli space (i.e. $B = A^{\mathrm{GL}_n}$). In particular, π has affine diagonal in an open neighborhood of x.

Proof. Applying Theorem 1.1 with $h_0\colon \mathcal{W}_0 \to \mathcal{G}_x$ an isomorphism yields an étale representable morphism $f\colon ([\operatorname{Spec} A/\mathrm{GL}_n], w) \to (\mathcal{X}, x)$ inducing an isomorphism $\mathcal{G}_w \to \mathcal{G}_x$. The result follows from Luna's fundamental lemma (Theorem 3.14). $\square$

As a consequence, we may remove the noetherian hypothesis from Theorem 9.3.

Corollary 13.11. *Let $\mathcal{X}$ be an algebraic stack of finite presentation over a quasi-compact and quasi-separated algebraic space S. Suppose that there exists an adequate moduli space $\pi\colon \mathcal{X} \to X$. Then π is a good moduli space with affine diagonal if and only if*

1. *$\mathcal{X}$ has separated diagonal and affine stabilizers, and*
2. *every closed point of $\mathcal{X}$ has linearly reductive stabilizer.*

Proof. The conditions are clearly necessary. If they are satisfied, then it follows that π has affine diagonal from Theorem 13.10. To verify that π is a good moduli space, we may replace X with the henselization at a closed point. Then $\mathcal{X}$ is fundamental by Theorem 13.10 and the result follows from Corollary 13.7. $\square$

Corollary 13.12. *Let S be a quasi-separated algebraic space. Let $G \to S$ be a flat and separated group algebraic space of finite presentation with affine fibers such that $BG \to S$ is adequately affine (e.g., $G \to S$ is geometrically reductive). If $s \in S$ is a point such that G_s is linearly reductive, then there exists an étale neighborhood $(S', s') \to (S, s)$, with trivial residue field extension, such that $G' = G \times_S S'$ is embeddable.*

Proof. This follows from Theorem 13.10 since G' is embeddable if and only if BG' is fundamental (Remark 2.9). $\square$

Remark 13.13. If $G \to S$ is a reductive group scheme (i.e., geometrically reductive, smooth, and with connected fibers) then $G \to S$ is étale-locally split reductive. A split reductive group is a pull-back from $\operatorname{Spec} \mathbb{Z}$ [$\mathrm{SGA3}_{\mathrm{II}}$, Exp. XXV, Thm. 1.1, Cor. 1.2], hence embeddable.

Proof of Theorem 13.1. Parts (1) and (2) follow from Theorem 13.10. Parts (3) and (4) then follow from Corollary 13.9 and descent. The final claim follows from Corollary 13.5. $\square$

14. Applications to compact generation and algebraicity

14.1. **Compact generation of derived categories.** Here we prove a variant of [AHR19, Thm. 5.1] in the mixed characteristic situation.

Proposition 14.1. *Let $\mathcal{X}$ be a quasi-compact algebraic stack with good moduli space $\pi\colon \mathcal{X} \to X$. If $\mathcal{X}$ has affine stabilizers, separated diagonal and is of finite presentation over a quasi-separated algebraic space S, then $\mathcal{X}$ has the Thomason condition; that is,*

(1) *$\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ is compactly generated by a countable set of perfect complexes; and*
(2) *for every quasi-compact open immersion $\mathcal{U} \subseteq \mathcal{X}$, there exists a compact and perfect complex $P \in \mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ with support precisely $\mathcal{X} \setminus \mathcal{U}$.*

Proof. By Theorem 13.1, there exists a surjective, étale, separated and representable morphism $p\colon \mathcal{W} \to \mathcal{X}$ such that $\mathcal{W}$ has the form $[\operatorname{Spec} C/\mathrm{GL}_n]$; in particular, $\mathcal{W}$ has the resolution property. Moreover, since $\mathcal{X}$ and p are concentrated (i.e., quasi-compact, quasi-separated and of finite cohomological dimension [HR17, §2]), it follows that $\mathcal{W}$ is concentrated. In particular, $\mathcal{W}$ is $\aleph_0$-crisp [HR17, Prop. 8.4]. By [HR17, Thm. C], the result follows. □

14.2. **Algebraicity results.** Here we generalize the algebraicity results of [AHR19, §5.3] to the setting of mixed characteristic. We will do this using the formulation of Artin's criterion in [Hal17, Thm. A]. This requires us to prove that certain deformation and obstruction functors are coherent, in the sense of [Aus66].

In this subsection, we will assume that we are in the following situation:

Setup 14.2. Fix an excellent algebraic space X and an algebraic stack $\mathcal{X}$ with affine diagonal over X, such that $\mathcal{X} \to X$ is a good moduli space. Note that $\mathcal{X} \to X$ is automatically of finite type [AHR19, Thm. A.1].

Remark 14.3. The results of this section also hold when X is non-excellent, provided that $\mathcal{X}$ satisfies one of the conditions (FC), (PC) or (N) (see Corollary 15.5).

The following result generalizes [AHR19, Prop 5.14] to the setting of mixed characteristic.

Proposition 14.4. *Assume Setup 14.2 and that X is affine. If $\mathcal{F} \in \mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ and $\mathcal{G} \in \mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X})$, then the functor*

$$\operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G} \otimes^{\mathsf{L}}_{\mathcal{O}_\mathcal{X}} \mathsf{L}\pi^*(-))\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(X)$$

is coherent.

Proof. The proof is identical to [AHR19, Prop. 5.14]: by Proposition 14.1, $\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ is compactly generated. Also, the restriction of $\mathsf{R}(f_{\mathrm{qc}})_*\colon \mathsf{D}_{\mathrm{qc}}(\mathcal{X}) \to \mathsf{D}_{\mathrm{qc}}(X)$ to $\mathsf{D}^+_{\mathsf{Coh}}(\mathcal{X})$ factors through $\mathsf{D}^+_{\mathsf{Coh}}(X)$ [Alp13, Thm. 4.16(x)]. By [HR17, Cor. 4.19], the result follows. □

The following corollary is a mixed characteristic variant of [AHR19, Cor. 5.15]. The proof is identical, so is omitted (also see [Hal14, Thm. D]).

Corollary 14.5. *Assume Setup 14.2. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module. Let $\mathcal{G}$ be a finitely presented $\mathcal{O}_\mathcal{X}$-module. If $\mathcal{G}$ is flat over X, then the X-presheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{X}/X}(\mathcal{F}, \mathcal{G})$, whose objects over $T \xrightarrow{\tau} X$ are homomorphisms $\tau_\mathcal{X}^*\mathcal{F} \to \tau_\mathcal{X}^*\mathcal{G}$ of $\mathcal{O}_{\mathcal{X}\times_X T}$-modules (where $\tau_\mathcal{X}\colon \mathcal{X} \times_X T \to \mathcal{X}$ is the projection), is representable by an affine X-scheme.* □

Theorem 14.6 (Stacks of coherent sheaves)**.** *Assume Setup 14.2. The X-stack $\underline{\mathsf{Coh}}_{\mathcal{X}/X}$, whose objects over $T \to X$ are finitely presented quasi-coherent sheaves on $\mathcal{X} \times_X T$ flat over T, is an algebraic stack, locally of finite presentation over X, with affine diagonal over X.*

Proof. The proof is identical to [AHR19, Thm. 5.7], which is a small modification of [Hal17, Thm. 8.1]: the formal GAGA statement of Corollary 1.7 implies that formally versal deformations are effective and Proposition 14.4 implies that the automorphism, deformation and obstruction functors are coherent. Therefore, Artin's criterion (as formulated in [Hal17, Thm. A]) is satisfied and the result follows. Corollary 14.5 implies that the diagonal is affine. □

Just as in [AHR19], the following corollaries follow immediately from Theorem 14.6 appealing to the observation that Corollary 14.5 implies that $\underline{\mathsf{Quot}}_{\mathcal{X}/X}(\mathcal{F}) \to \underline{\mathsf{Coh}}_{\mathcal{X}/X}$ is quasi-affine.

Corollary 14.7 (Quot schemes)**.** *Assume Setup 14.2. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module, then the X-sheaf $\underline{\mathsf{Quot}}_{\mathcal{X}/X}(\mathcal{F})$, whose objects over $T \xrightarrow{\tau} X$ are quotients $\tau_X^*\mathcal{F} \to \mathcal{G}$ (where $\tau_X \colon \mathcal{X} \times_X T \to \mathcal{X}$ is the projection) such that $\mathcal{G}$ is a finitely presented $\mathcal{O}_{\mathcal{X}\times_X T}$-module that is flat over T, is a separated algebraic space over X. If $\mathcal{F}$ is finitely presented, then $\underline{\mathsf{Quot}}_{\mathcal{X}/X}(\mathcal{F})$ is locally of finite presentation over X.* □

Corollary 14.8 (Hilbert schemes)**.** *Assume Setup 14.2. The X-sheaf $\underline{\mathrm{Hilb}}_{\mathcal{X}/X}$, whose objects over $T \to X$ are closed substacks $\mathcal{Z} \subseteq \mathcal{X} \times_X T$ such that $\mathcal{Z}$ is flat and of finite presentation over T, is a separated algebraic space locally of finite presentation over X.* □

We now establish algebraicity of Hom stacks. Related results were established in [HP14] under other hypotheses.

Theorem 14.9 (Hom stacks)**.** *Assume Setup 14.2. Let $\mathcal{Y}$ be an algebraic stack, quasi-separated and locally of finite presentation over X with affine stabilizers. If $\mathcal{X} \to X$ is flat, then the X-stack $\underline{\mathrm{Hom}}_X(\mathcal{X}, \mathcal{Y})$, whose objects are pairs consisting of a morphism $T \to X$ of algebraic spaces and a morphism $\mathcal{X} \times_X T \to \mathcal{Y}$ of algebraic stacks over X, is an algebraic stack, locally of finite presentation over X with quasi-separated diagonal. If $\mathcal{Y} \to X$ has affine (resp. quasi-affine, resp. separated) diagonal, then the same is true for $\underline{\mathrm{Hom}}_X(\mathcal{X}, \mathcal{Y}) \to X$.*

Proof. This is also identical to the proof of [AHR19, Thm. 5.10], which is a variant of [HR19, Thm. 1.2], so is omitted. □

Corollary 14.10 (G-equivariant Hom stacks)**.** *Let S be an excellent algebraic space. Let Z be an algebraic space of finite type over S and $\mathcal{W}$ a quasi-separated Deligne–Mumford stack, locally of finite type over S. Let $G \to S$ be a linearly reductive affine group scheme acting on Z and $\mathcal{Y}$. Suppose that $Z \to S$ is flat and a good GIT quotient (i.e. $[Z/G] \to S$ is a good moduli space). Then the S-stack $\underline{\mathrm{Hom}}_S^G(Z, \mathcal{W})$, whose objects over $T \to S$ are G-equivariant S-morphisms $Z \times_S T \to \mathcal{W}$, is a Deligne–Mumford stack, locally of finite type over S. In addition, if $\mathcal{W}$ has separated diagonal (resp. is an algebraic space), then $\underline{\mathrm{Hom}}_S^G(Z, \mathcal{W})$ is quasi-separated (resp. is a quasi-separated algebraic space).*

Proof. This is also identical to the proof of [AHR19, Cor. 5.11], so is omitted. □

15. Approximation of linearly fundamental stacks

In this section we use the results of Section 13 to extend the approximation results for fundamental and nicely fundamental stacks in Section 2.3 to linearly fundamental stacks (Theorem 15.3) and good moduli spaces (Corollary 15.5). This will be crucial in Section 16 to reduce from the henselian case to the excellent henselian case. To this end, we introduce the following mild mixed characteristic assumptions on an algebraic stack $\mathcal{W}$:

(FC) There is only a finite number of different characteristics in $\mathcal{W}$.
(PC) Every closed point of $\mathcal{W}$ has positive characteristic.
(N) Every closed point of $\mathcal{W}$ has nice stabilizer.

Remark 15.1. Note that if $\eta \rightsquigarrow s$ is a specialization in $\mathcal{W}$, then the characteristic of η is 0 or agrees with that of s. In particular, if $(\mathcal{W}, \mathcal{W}_0)$ is a local pair (Definition 3.1), then conditions (FC), (PC) and (N) for $\mathcal{W}_0$ and $\mathcal{W}$ are equivalent.

Let $\mathcal{X}$ be a fundamental stack with adequate moduli space $\pi\colon \mathcal{X} \to X$. Let $X_{\mathrm{nice}} \subseteq |X|$ be the locus of points $x \in |X|$ such that the unique closed point in the fiber $\pi^{-1}(x)$ has nice stabilizer. If $x \in X_{\mathrm{nice}}$, then there exists an étale neighborhood $X' \to X$ of x such that $\mathcal{X} \times_X X'$ is nicely fundamental (Proposition 13.4). It follows that X_{nice} is open and that $\mathcal{X} \times_X X_{\mathrm{nice}} \to X_{\mathrm{nice}}$ is a good moduli space.

Lemma 15.2. *Let $\mathcal{X}$ be a fundamental stack with adequate moduli space X. Let $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$ be an inverse limit of fundamental stacks with affine transition maps. Then*

1. *$X \times_{X_\lambda} (X_\lambda)_{\mathrm{nice}} \subseteq X_{\mathrm{nice}}$ for every λ, and*
2. *if $V \subseteq X_{\mathrm{nice}}$ is a quasi-compact open subset, then $V \subseteq X \times_{X_\lambda} (X_\lambda)_{\mathrm{nice}}$ for every sufficiently large λ.*

Proof. Note that the map $\mathcal{X} \to \mathcal{X}_\lambda \times_{X_\lambda} X$ is affine but not an isomorphism (if it was, the result would follow immediately).

For $x \in (X_\lambda)_{\mathrm{nice}}$, let $U_\lambda \to X_\lambda$ be an étale neighborhood of x such that $\mathcal{X}_\lambda \times_{X_\lambda} U_\lambda$ is nicely fundamental (Proposition 13.4). Then $\mathcal{X} \times_{X_\lambda} U_\lambda$ is also nicely fundamental as it is affine over the former. Thus $X \times_{X_\lambda} (X_\lambda)_{\mathrm{nice}} \subseteq X_{\mathrm{nice}}$. This proves (1).

For (2), let $U \to V$ be an étale surjective morphism such that $\mathcal{X} \times_X U$ is nicely fundamental (Proposition 13.4). Since $X = \varprojlim_\lambda X_\lambda$ (Lemma 2.15(2)) and $U \to X$ is affine, we can for all sufficiently large λ find $U_\lambda \to X_\lambda$ affine étale such that $U = U_\lambda \times_{X_\lambda} X$. Since $\mathcal{X} \times_X U = \varprojlim_\lambda \mathcal{X}_\lambda \times_{X_\lambda} U_\lambda$ is nicely fundamental, so is $\mathcal{X}_\lambda \times_{X_\lambda} U_\lambda$ for all sufficiently large λ (Lemma 2.15(1)). It follows that $(X_\lambda)_{\mathrm{nice}}$ contains the image of U_λ so $V \subseteq X \times_{X_\lambda} (X_\lambda)_{\mathrm{nice}}$. ∎

The main theorem of this section is the following variant of Lemma 2.15 for linearly fundamental stacks.

Theorem 15.3 (Approximation of linearly fundamental)**.** *Let $\mathcal{Y}$ be a quasi-compact and quasi-separated algebraic stack. Let $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$ where $\mathcal{X}_\lambda$ is an inverse system of quasi-compact and quasi-separated algebraic stacks over $\mathcal{Y}$ with affine transition maps. Assume that* (1) *$\mathcal{Y}$ is (FC), or* (2) *$\mathcal{X}$ is (PC), or* (3) *$\mathcal{X}$ is (N). Then, if $\mathcal{X}$ is linearly fundamental, so is $\mathcal{X}_\lambda$ for all sufficiently large λ.*

Proof. By Lemma 2.15 we can assume that the $\mathcal{X}_\lambda$ are fundamental. Since $\mathcal{X}$ is linearly fundamental, (PC) $\implies$ (N). If $\mathcal{X}$ satisfies (N), then $X_{\mathrm{nice}} = X$ and it follows from Lemma 15.2 that $(X_\lambda)_{\mathrm{nice}} = X_\lambda$ for all sufficiently large λ; hence that $\mathcal{X}_\lambda$ is linearly fundamental. Thus, it remains to prove the theorem when $\mathcal{Y}$ satisfies (FC). In this case, $\mathcal{Y}_{\mathbb{Q}} := \mathcal{Y} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Q}$ is open in $\mathcal{Y}$. Similarly for the other stacks. In particular, if X denotes the good moduli space of $\mathcal{X}$, then X is the union of the two open subschemes X_{nice} and $X_{\mathbb{Q}}$. In addition, since $X \smallsetminus X_{\mathbb{Q}}$ is closed, hence quasi-compact, we may find a quasi-compact open subset $V \subseteq X_{\mathrm{nice}}$ such that $X = V \cup X_{\mathbb{Q}}$. For sufficiently large λ, we have that $V \subseteq (X_\lambda)_{\mathrm{nice}} \times_{X_\lambda} X$ (Lemma 15.2(2)) and thus, after possibly increasing λ, that $X_\lambda = (X_\lambda)_{\mathrm{nice}} \cup (X_\lambda)_{\mathbb{Q}}$. It follows that $\mathcal{X}_\lambda$ is linearly fundamental. □

Corollary 15.4. *Let $\mathcal{X}$ be a linearly fundamental stack. Assume that $\mathcal{X}$ satisfies (FC), (PC) or (N). Then we can write $\mathcal{X} = \varprojlim_\lambda \mathcal{X}_\lambda$ as an inverse limit of linearly fundamental stacks, with affine transition maps, such that each $\mathcal{X}_\lambda$ is essentially of finite type over* $\operatorname{Spec} \mathbb{Z}$.

Proof. If $\mathcal{X}$ satisfies (FC), let S be the semi-localization of $\operatorname{Spec}\mathbb{Z}$ in all characteristics that appear in S. Then there is a canonical map $\mathcal{X} \to S$. If $\mathcal{X}$ satisfies (PC) or (N), let $S = \operatorname{Spec}\mathbb{Z}$. Since $\mathcal{X}$ is fundamental, we can write $\mathcal{X}$ as an inverse limit of algebraic stacks $\mathcal{X}_\lambda$ that are fundamental and of finite presentation over S. The result then follows from Theorem 15.3. □

Corollary 15.4 is not true unconditionally, even if we merely assume that the $\mathcal{X}_\lambda$ are noetherian, see Appendix A.

Corollary 15.5 (Approximation of good moduli spaces)**.** *Let $X = \varprojlim_\lambda X_\lambda$ be an inverse system of quasi-compact algebraic spaces with affine transition maps. Let α be an index, let $f_\alpha \colon \mathcal{X}_\alpha \to X_\alpha$ be a morphism of finite presentation and let $f_\lambda \colon \mathcal{X}_\lambda \to X_\lambda$, for $\lambda \geq \alpha$, and $f \colon \mathcal{X} \to X$ denote its base changes. Assume that X_α satisfies (FC) or $\mathcal{X}$ satisfies (PC) or (N). Then if $\mathcal{X} \to X$ is a good moduli space, so is $\mathcal{X}_\lambda \to X_\lambda$ for all sufficiently large λ.*

Proof. Theorem 13.1 gives an étale and surjective morphism $X' \to X$ such that $\mathcal{X}' = \mathcal{X} \times_X X'$ is linearly fundamental. For sufficiently large λ, we can find an étale surjective morphism $X'_\lambda \to X_\lambda$ that pulls back to $X' \to X$. For sufficiently large λ, we have that $\mathcal{X}'_\lambda := \mathcal{X}_\lambda \times_{X_\lambda} X'_\lambda$ is linearly fundamental by Theorem 15.3. Its good moduli space $\overline{X}'_\lambda$ is of finite presentation over X'_λ (Corollary 13.9). It follows that $\overline{X}'_\lambda \to X'_\lambda$ is an isomorphism for all sufficiently large λ. By descent, it follows that $\mathcal{X}_\lambda \to X_\lambda$ is a good moduli space for all sufficiently large λ □

16. Deformation of linearly fundamental stacks

In this section, we will be concerned with deforming objects over henselian pairs (Definition 3.1). For the majority of this section, we will be in the following situation.

Setup 16.1. Let $\mathcal{X}$ be a quasi-compact algebraic stack with affine diagonal and affine good moduli space X. Let $\mathcal{X}_0 \hookrightarrow \mathcal{X}$ be a closed substack with good moduli space X_0. Assume that (X, X_0) is an affine henselian pair and one of the following conditions holds:

(a) $\mathcal{X}_0$ has the resolution property, $\mathcal{X}$ is noetherian and (X, X_0) is complete; or
(b) $\mathcal{X}_0$ has the resolution property, $\mathcal{X}$ is noetherian and (X, X_0) is excellent; or
(c) $\mathcal{X}$ has the resolution property and $\mathcal{X}_0$ satisfies (FC), (PC), or (N).

In Section 16.6, we will deform objects over étale neighborhoods instead of over henselian pairs.

Remark 16.2. Note that (FC) and (PC) for $\mathcal{X}_0$ are clearly equivalent to the corresponding properties for X_0. Since the pair (X, X_0) is henselian and so local, it follows that these are equivalent to the corresponding properties for X (Remark 15.1) and so $\mathcal{X}$. Similarly, (N) for $\mathcal{X}_0$ is equivalent to (N) for $\mathcal{X}$. Also, Corollary 15.5 permits "$\mathcal{X}$ has the resolution property" to be weakened to "$\mathcal{X}_0$ has the resolution property" in Setup 16.1(c) if $\mathcal{X} \to X$ is of finite presentation (e.g., $\mathcal{X}$ noetherian).

16.1. **Deformation of the resolution property.** The first result of this section is the following remarkable proposition. It is a simple consequence of some results proved several sections ago.

Proposition 16.3 (Deformation of the resolution property)**.** *Assume Setup 16.1*(a) *or* (b)*. Then $\mathcal{X}$ has the resolution property; in particular, $\mathcal{X}$ is linearly fundamental.*

Proof. Case (a) is part of the coherent completeness result (Theorem 1.6). For (b), let $\widehat{X}$ denote the completion of X along X_0 and $\widehat{\mathcal{X}} = \mathcal{X} \times_X \widehat{X}$. By the complete case, $\widehat{\mathcal{X}}$ has the resolution property. Equivalently, there is a quasi-affine morphism $\widehat{\mathcal{X}} \to B\mathrm{GL}_n$ for some n. The functor parametrizing quasi-affine morphisms to $B\mathrm{GL}_n$ is locally of finite presentation [Ryd15, Thm. C] so by Artin approximation (Theorem 3.4), there exists a quasi-affine morphism $\mathcal{X} \to B\mathrm{GL}_n$. □

16.2. **Deformation of sections.** If $f\colon \mathcal{X}' \to \mathcal{X}$ is a morphism of algebraic stacks, we will denote the groupoid of sections $s\colon \mathcal{X} \to \mathcal{X}'$ of f as $\Gamma(\mathcal{X}'/\mathcal{X})$.

Proposition 16.4 (Deformation of sections)**.** *Assume Setup 16.1. If $f\colon \mathcal{X}' \to \mathcal{X}$ is a quasi-separated and smooth (resp. smooth gerbe, resp. étale) morphism with affine stabilizers, then $\Gamma(\mathcal{X}'/\mathcal{X}) \to \Gamma(\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_0/\mathcal{X}_0)$ is essentially surjective (resp. essentially surjective and full, resp. an equivalence of groupoids).*

Proof. Any section s_0 of $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_0 \to \mathcal{X}_0$ has quasi-compact image. In particular, we may immediately reduce to the situation where f is finitely presented.

We first handle case (a): By Theorem 1.6, $\mathcal{X}$ is coherently complete along $\mathcal{X}_0$. Let $\mathcal{I}$ be the ideal sheaf defining $\mathcal{X}_0 \subseteq \mathcal{X}$ and let $\mathcal{X}_n := \mathcal{X}_{\mathcal{X}_0}^{[n]}$ be its nilpotent thickenings. Set $\mathcal{X}'_n = \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_n$. Let $s_0\colon \mathcal{X}_0 \to \mathcal{X}'_0$ be a section of $\mathcal{X}'_0 \to \mathcal{X}_0$. Given a section s_{n-1} of $\mathcal{X}'_{n-1} \to \mathcal{X}_{n-1}$, lifting s_0, the obstruction to deforming s_{n-1} to a section s_n of $\mathcal{X}'_n \to \mathcal{X}_n$ is an element of $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}_0}}(Ls_0^* L_{\mathcal{X}'/\mathcal{X}}, \mathcal{I}^n/\mathcal{I}^{n+1})$ by [Ols06, Thm. 1.5].[3] Since $\mathcal{X}' \to \mathcal{X}$ is smooth (resp. a smooth gerbe, resp. étale), the cotangent complex $L_{\mathcal{X}'/\mathcal{X}}$ is perfect of amplitude $[0,1]$ (resp. perfect of amplitude 1, resp. zero). Further $\mathcal{X}_0$ is cohomologically affine, so there exists a lift (resp. a unique lift up to non-unique 2-isomorphism, resp. a unique lift up to unique 2-isomorphism). By Tannaka duality (see §1.7.6), these sections lift to a unique section $s\colon \mathcal{X} \to \mathcal{X}'$.

We now handle case (b). Let $\widehat{X}$ be the completion of X along X_0 and set $\widehat{\mathcal{X}} = \mathcal{X} \times_S \widehat{S}$ and $\widehat{\mathcal{X}}' = \mathcal{X}' \times_S \widehat{S}$. Case (a) yields a section $\widehat{s}\colon \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}'$ extending s_0. The functor assigning an S-scheme T to the set of sections $\Gamma(\mathcal{X}' \times_S T/\mathcal{X} \times_S T)$ is limit preserving, and we may apply Artin approximation (Theorem 3.4) to obtain a section of $s\colon \mathcal{X}' \to \mathcal{X}$ restricting to s_0.

Finally, we handle case (c). Fix a section $s_0\colon \mathcal{X}_0 \to \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_0$ to $f_0\colon \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_0 \to \mathcal{X}_0$. Then there is a factorization $\mathcal{X}_0 \hookrightarrow \widetilde{\mathcal{X}}_0 \hookrightarrow \mathcal{X}$, where $\widetilde{\mathcal{X}}_0 \hookrightarrow \mathcal{X}$ is a finitely presented closed immersion ($\mathcal{X}$ has the resolution property, making the approximation trivial) and s_0 extends to a section $\widetilde{s}_0$ to $\mathcal{X}' \times_{\mathcal{X}} \widetilde{\mathcal{X}}_0 \to \widetilde{\mathcal{X}}_0$. By Remark 16.2, $\mathcal{X}$ inherits the properties (FC), (PC) or (N). We may now approximate $(\widetilde{\mathcal{X}}_0, \mathcal{X})$ by $(\widetilde{\mathcal{X}}_{\lambda,0}, \mathcal{X}_\lambda)$, where $\mathcal{X}_\lambda$ is linearly fundamental and essentially of finite type over $\operatorname{Spec} \mathbb{Z}$ (Corollary 15.4). Since $\mathcal{X}' \to \mathcal{X}$ is smooth (resp. a smooth gerbe, resp. étale) and finitely presented, after possibly increasing λ it descends to $\mathcal{X}'_\lambda \to \mathcal{X}_\lambda$ and retains its properties of being smooth (resp. a smooth gerbe, resp. étale) [Ryd15, Prop. B.3]. After further increasing λ, $\widetilde{s}_0$ descends. Now pull all of the

[3] Note that [Ols06, Thm. 1.5] only treats the case of embedded deformations over a base *scheme*. In the case of a relatively flat target morphism, however, this can be generalized to a base algebraic stack by deforming the graph and employing [Ols06, Thm. 1.1], together with the tor-independent base change properties properties of the cotangent complex. In the situation at hand we may also simply apply [Ols06, Thm. 1.1] to $s_n\colon \mathcal{X}_n \to \mathcal{X}'$ and $\mathcal{X}_n \hookrightarrow \mathcal{X}$.

descended objects back along the henselization of the good moduli space $\mathcal{X}_\lambda$ along the good moduli space of $\widetilde{\mathcal{X}}_{\lambda,0}$. The claim now follows from (b).

The uniqueness statements can be argued using similar methods—the complete case is clear, the excellent case can be reduced to the complete case using Artin approximation, and the others can be also reduced to the excellent case. $\square$

16.3. **Deformation of morphisms.** A simple application of Proposition 16.4 yields a deformation result of morphisms.

Proposition 16.5. *Assume Setup 16.1. If $\mathcal{Y} \to X$ is a quasi-separated and smooth (resp. smooth gerbe, resp. étale) morphism with affine stabilizers, then any morphism $\mathcal{X}_0 \to \mathcal{Y}$ can be extended (resp. extended uniquely up to non-unique 2-isomorphism, resp. extended uniquely up to unique 2-isomorphism) to a morphism $\mathcal{X} \to \mathcal{Y}$. In particular,*

1. *The natural functor $\mathsf{F\acute{E}T}(\mathcal{X}) \to \mathsf{F\acute{E}T}(\mathcal{X}_0)$ between the categories of finite étale covers is an equivalence.*
2. *The natural functor $\mathsf{VB}(\mathcal{X}) \to \mathsf{VB}(\mathcal{X}_0)$ between the categories of vector bundles is essentially surjective and full.*
3. *Let $G \to X$ be a flat group scheme of finite presentation. If $\mathcal{X}_0 = [\operatorname{Spec} A/G]$, then $\mathcal{X} = [\operatorname{Spec} B/G]$.*
4. *If $\mathcal{X}_0$ is nicely fundamental, then so is $\mathcal{X}$.*

Proof. For the main statement, apply Proposition 16.4 with $\mathcal{X}' = \mathcal{X} \times_X \mathcal{Y}$. For (1), apply the result to $\mathcal{Y} = \coprod_n BS_{n,X}$ noting that BS_n classifies finite étale covers of degree n. Similarly, for (2), apply the result to $\mathcal{Y} = \coprod_n B\mathrm{GL}_{n,X}$. For (3), apply the result to $\mathcal{Y} = BG$ together with Proposition 12.5(1) to ensure that the induced morphism $\mathcal{X} \to BG$ is affine. For (4), note that, by definition, $\mathcal{X}_0 = [\operatorname{Spec} A/G_0]$ where $G_0 \to S_0$ is nice and embeddable. We next deform G_0 to a nice and embeddable group scheme $G \to S$ (Proposition 7.1) and then apply (3). $\square$

16.4. **Deformation of linearly fundamental stacks.** If (S, S_0) is an affine complete noetherian pair and $\mathcal{X}_0$ is a linearly fundamental stack with a syntomic morphism $\mathcal{X}_0 \to S_0$ that is a good moduli space, Theorem 1.11 constructs a noetherian and linearly fundamental stack $\mathcal{X}$ that is flat over S, such that $\mathcal{X}_0 = \mathcal{X} \times_S S_0$ and $\mathcal{X}$ is coherently complete along $\mathcal{X}_0$. The following lemma shows that $\mathcal{X} \to S$ is also a good moduli space. We also consider non-noetherian generalizations.

Lemma 16.6. *Let $\mathcal{X}$ be a quasi-compact algebraic stack with affine diagonal and affine good moduli space X. Let $\pi\colon \mathcal{X} \to S$ be a flat morphism. Let $S_0 \hookrightarrow S$ be a closed immersion. Let $\mathcal{X}_0 = \mathcal{X} \times_S S_0$ and assume $\pi_0\colon \mathcal{X}_0 \to S_0$ is a good moduli space and $(\mathcal{X}, \mathcal{X}_0)$ is a local pair. In addition, assume that (S, S_0) is an affine local pair and*

(a) *$\mathcal{X}$ is noetherian and (S, S_0) is complete; or*
(b) *$\mathcal{X}$ is noetherian and π is of finite type; or*
(c) *$\mathcal{X}$ has the resolution property, π is of finite presentation and $\mathcal{X}_0$ satisfies (FC), (PC), or (N).*

Then π is a good moduli space morphism of finite presentation. Moreover,

1. *if π_0 is syntomic (resp. smooth, resp. étale), then so is π; and*
2. *if π_0 is an fppf gerbe (resp. a smooth gerbe, resp. an étale gerbe), then so is π.*

Proof. We first show that π is a good moduli space morphism of finite presentation.

Let $S = \operatorname{Spec} A$, $S_0 = \operatorname{Spec}(A/I)$ and $X = \operatorname{Spec} B$. Since $(\mathcal{X}, \mathcal{X}_0)$ is a local pair, it follows that $(\operatorname{Spec} B, \operatorname{Spec} B/IB)$ is a local pair. In particular, IB is contained in the Jacobson radical of B. Note that if $\mathcal{X}$ is noetherian, then $\mathcal{X} \to \operatorname{Spec} B$ is of finite type [AHR19, Thm. A.1]. Moreover, in the commuting diagram:

$$\begin{tikzcd} \mathcal{X}_0 \arrow[r] \arrow[d, hook] & \operatorname{Spec}(B/IB) \arrow[r] \arrow[d, hook] & S_0 \arrow[d, hook] \\ \mathcal{X} \arrow[r] & X \arrow[r] & S, \end{tikzcd}$$

the outer rectangle is cartesian, as is the right square, so it follows that the left square is cartesian. Since the formation of good moduli spaces is compatible with arbitrary base change, it follows that the morphism $A/I \to B/IB$ is an isomorphism.

Case (a): let $A_n = A/I^{n+1}$ and $\mathcal{X}_n = V(I^{n+1}\mathcal{O}_{\mathcal{X}})$. Since $\mathcal{X}_n$ is noetherian and $\pi_n \colon \mathcal{X}_n \to S_n := \operatorname{Spec} A/I^{n+1}$ is flat, it follows that $B_n = \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}) = B/I^{n+1}B$ is a noetherian and flat $A_n = A/I^{n+1}$-algebra [Alp13, Thm. 4.16(ix)]. But $B_n/IB_n = A/I$ so $A_n \to B_n$ is surjective and hence an isomorphism. Let $\widehat{B}$ be the IB-adic completion of B; then the composition $A \to B \to \widehat{B}$ is an isomorphism and $B \to \widehat{B}$ is faithfully flat because IB is contained in the Jacobson radical of B. It follows immediately that $A \to B$ is an isomorphism.

Case (b): now the image of π contains S_0 and by flatness is stable under generizations; it follows immediately that π is faithfully flat. Since $\mathcal{X}$ is noetherian, it follows that S is noetherian.

We may now base change everything along the faithfully flat morphism $\operatorname{Spec} \widehat{A} \to \operatorname{Spec} A$, where $\widehat{A}$ is the I-adic completion of A. By faithfully flat descent of good moduli spaces, we are now reduced to Case (a).

Case (c): this follows from (b) using an approximation argument similar to that employed in the proof of Proposition 16.4.

Now claim (1) is immediate: every closed point of $\mathcal{X}$ lies in $\mathcal{X}_0$. For claim (2), since $\mathcal{X} \to S$ and $\mathcal{X} \times_S \mathcal{X} \to S$ are flat and $\mathcal{X}_0$ contains all closed points, the fiberwise criterion of flatness shows that $\Delta_{\mathcal{X}/S}$ is flat if and only if $\Delta_{\mathcal{X}_0/S_0}$ is flat. It then follows that $\Delta_{\mathcal{X}/S}$ is smooth (resp. étale) if $\Delta_{\mathcal{X}_0/S_0}$ is so. □

Combining Theorem 1.11 and Lemma 16.6 with Artin approximation yields the following result.

Proposition 16.7 (Deformation of linearly fundamental stacks)**.** *Let $\pi_0 \colon \mathcal{X}_0 \to S_0$ be a good moduli space, where $\mathcal{X}_0$ is linearly fundamental. Let (S, S_0) be an affine henselian pair and assume one of the following conditions:*

(a) *(S, S_0) is a noetherian complete pair;*
(b) *S is excellent;*
(c) *$\mathcal{X}_0$ satisfies (FC), (PC), or (N).*

If π_0 is syntomic, then there exists a syntomic morphism $\pi \colon \mathcal{X} \to S$ that is a good moduli space such that:

(1) *$\mathcal{X} \times_S S_0 \cong \mathcal{X}_0$;*
(2) *$\mathcal{X}$ is linearly fundamental; and*
(3) *$\mathcal{X}$ is coherently complete along $\mathcal{X}_0$ if (S, S_0) is a noetherian complete pair.*
(4) *π is smooth (resp. étale) if π_0 is smooth (resp. étale).*
(5) *π is an fppf (resp. smooth, resp. étale) gerbe if π_0 is such a gerbe.*

Moreover, if π_0 is smooth (resp. a smooth gerbe, resp. étale), then π is unique up to non-unique isomorphism (resp. non-unique 2-isomorphism, resp. unique 2-isomorphism).

Proof. In case (a): the existence of a flat morphism $\mathcal{X} \to S$ satisfying (1)–(3) is immediate from Theorem 1.11 applied to $\mathcal{X}_0 \to S_0 \to S$. Lemma 16.6(a) implies that $\mathcal{X} \to S$ is syntomic and a good moduli space, as well as the other conditions. If $\mathcal{X}' \to S$ is another lift, the uniqueness statements follow by applying Proposition 16.5 with $\mathcal{Y} = \mathcal{X}'$.

In case (b): consider the functor assigning an S-scheme T to the set of isomorphism classes of fundamental stacks $\mathcal{Y}$ over T such that $\pi\colon \mathcal{Y} \to T$ is syntomic. This functor is limit preserving by Lemma 2.15, so we may use the construction in the complete case and Artin approximation (Theorem 3.4) to obtain a fundamental stack $\mathcal{X}$ over S such that $\mathcal{X} \times_S S_0 = \mathcal{X}_0$ and $\mathcal{X} \to S$ is syntomic. An application of Lemma 16.6(b) completes the argument again.

Case (c) follows from case (b) by approximation (similar to that used in the proof of Proposition 16.4). ☐

16.5. **Deformation of linearly reductive groups.** As a direct consequence of Proposition 16.7, we can prove the following result, cf. Proposition 7.1.

Proposition 16.8 (Deformation of linearly reductive group schemes). *Let (S, S_0) be an affine henselian pair and $G_0 \to S_0$ a linearly reductive and embeddable group scheme. Assume one of the following conditions:*

(a) *(S, S_0) is a noetherian complete pair;*
(b) *S is excellent; or*
(c) *G_0 has nice fibers at closed points or S_0 satisfies (PC) or (FC).*

Then there exists a linearly reductive and embeddable group scheme $G \to S$ such that $G_0 = G \times_S S_0$. If, in addition, $G_0 \to S_0$ is smooth (resp. étale), then $G \to S$ is smooth (resp. étale) and unique up to non-unique (resp. unique) isomorphism.

Proof. Applying Proposition 16.7 to $BG_0 \to S_0$ yields a linearly fundamental and fppf gerbe $\mathcal{X} \to S$ such that $BG_0 = \mathcal{X} \times_S S_0$. By Proposition 16.4, we may extend the canonical section $S_0 \to BG_0$ to a section $S \to \mathcal{X}$ with the stated uniqueness property. We conclude that $\mathcal{X}$ is isomorphic to BG for an fppf affine group scheme $G \to S$ extending G_0. Since BG is linearly fundamental, $G \to S$ is linearly reductive and embeddable (see Remark 2.9). ☐

Remark 16.9. When $G_0 \to S_0$ is a split reductive group scheme, then the existence of $G \to S$ follows from the classification of reductive groups: $G_0 \to S_0$ is the pullback of a split reductive group over $\operatorname{Spec} \mathbb{Z}$ [SGA3_{II}, Exp. XXV, Thm. 1.1, Cor. 1.2]. Our methods require linearly reductivity but also work for non-connected, non-split and non-smooth group schemes.

16.6. **Extension over étale neighborhoods.** In this last subsection, we consider the problem of extending objects over étale neighborhoods. Recall that if $\pi\colon \mathcal{X} \to X$ is an adequate moduli space, then a morphism $\mathcal{X}' \to \mathcal{X}$ is *strongly étale* if $\mathcal{X}' = \mathcal{X} \times_X X'$ for some étale morphism $X' \to X$ (Definition 3.13).

Proposition 16.10 (Extension of gerbes). *Let (S, S_0) be an affine pair. Let $\pi_0\colon \mathcal{X}_0 \to S_0$ be an fppf gerbe (resp. smooth gerbe, resp. étale gerbe). Suppose that $\mathcal{X}_0$ is linearly fundamental and satisfies (PC), (N) or (FC). Then, there exists an étale neighborhood $S' \to S$ of S_0 and a fundamental fppf gerbe (resp. smooth gerbe, resp. étale gerbe) $\pi\colon \mathcal{X}' \to S'$ extending π_0.*

Proof. The henselization S^h of (S, S_0) is the limit of the affine étale neighborhoods $S' \to S$ of S_0 so the result follows from Proposition 16.7 and Lemma 2.15(1). ☐

Proposition 16.11 (Extension of groups). *Let (S, S_0) be an affine pair. Let $G_0 \to S_0$ be a linearly reductive and embeddable group scheme. Suppose that G_0 has nice fibers or that S_0 satisfies (PC) or (FC). Then, there exists an étale neighborhood $S' \to S$ of S_0 and a geometrically reductive embeddable group $G' \to S'$ extending G_0.*

Proof. Argue as before, using Proposition 16.8 and Lemma 2.12. □

Proposition 16.12 (Extension of morphisms)**.** *Let $(\mathcal{X}, \mathcal{X}_0)$ be a fundamental pair over an algebraic stack S. Suppose that $\mathcal{X}_0$ is linearly fundamental and satisfies (PC), (N) or (FC). Let $\mathcal{Y} \to S$ be a smooth morphism, that is quasi-separated with affine stabilizers (resp. affine diagonal) and let $f_0 \colon \mathcal{X}_0 \to \mathcal{Y}$ be an S-morphism (resp. an affine S-morphism). Then there exists a strongly étale neighborhood $\mathcal{X}' \to \mathcal{X}$ of $\mathcal{X}_0$ such that f_0 extends to an S-morphism (resp. an affine S-morphism) $f' \colon \mathcal{X}' \to \mathcal{Y}$.*

Proof. Let X be the adequate moduli space of $\mathcal{X}$ and $X_0 \subseteq X$ the image of $\mathcal{X}_0$. Then (X, X_0) is an affine pair and its henselization X^h is the limit of étale neighborhoods $X' \to X$ of X_0. Since $X_0 \hookrightarrow X^h$ contains all closed points, it follows that $\mathcal{X}^h := \mathcal{X} \times_X X^h$ is linearly fundamental by Corollary 13.7. The result follows from Proposition 16.5, Proposition 12.5(1) and standard limit methods. □

Proposition 16.13 (Extension of nicely fundamental)**.** *Let $(\mathcal{X}, \mathcal{X}_0)$ be a fundamental pair. If $\mathcal{X}_0$ is nicely fundamental, then there exists a strongly étale neighborhood $\mathcal{X}' \to \mathcal{X}$ of $\mathcal{X}_0$ such that $\mathcal{X}'$ is nicely fundamental.*

Proof. As in the previous proof, it follows that $\mathcal{X}^h$ is linearly fundamental, hence nicely fundamental by Proposition 16.5(4). By Lemma 2.15(1), there exists an étale neighborhood $X' \to X$ of X_0 such that $\mathcal{X}' := \mathcal{X} \times_X X'$ is nicely fundamental. □

Proposition 16.14 (Extension of linearly fundamental)**.** *Let $(\mathcal{X}, \mathcal{X}_0)$ be a fundamental pair. Suppose that $\mathcal{X}_0$ satisfies (PC), or (N), or that $\mathcal{X}$ satisfies (FC) in an open neighborhood of $\mathcal{X}_0$. If $\mathcal{X}_0$ is linearly fundamental, then there exists a saturated open neighborhood $\mathcal{X}' \subseteq \mathcal{X}$ of $\mathcal{X}_0$ such that $\mathcal{X}'$ is linearly fundamental.*

Proof. Let X be the adequate moduli space of $\mathcal{X}$ and X_0 the image of $\mathcal{X}_0$. The Zariskification X^Z of X is the limit of all affine open neighborhoods $X' \to X$ of X_0. Since $X_0 \hookrightarrow X^Z$ contains all closed points, the stack $\mathcal{X}^Z := \mathcal{X} \times_X X^Z$ is linearly fundamental (Corollary 13.7). By Theorem 15.3, there exists an open neighborhood $X' \to X$ of X_0 such that $\mathcal{X}' := \mathcal{X} \times_X X'$ is linearly fundamental. □

Remark 16.15. Note that when S_0 is a single point, then (FC) always holds for S_0 and for objects over S_0. In the results of this subsection, the substacks $S_0 \subseteq S$ and $\mathcal{X}_0 \subseteq \mathcal{X}$ are by definition closed substacks. The results readily generalize to the following situation: $S_0 = \{s\}$ is any point and $\mathcal{X}_0 = \mathcal{G}_x$ is the residual gerbe of a point x closed in its fiber over the adequate moduli space.

17. Refinements on the local structure theorem

In the section, we detail refinements of Theorem 1.1. These follows from the extension results of Section 16.

Proposition 17.1 (Gerbe refinement)**.** *Let S be a quasi-separated algebraic space. Let $\mathcal{W}$ be a fundamental stack of finite presentation over S. Let $w \in |\mathcal{W}|$ be a point with linearly reductive stabilizer and image $s \in |S|$ such that w is closed in its fiber $\mathcal{W}_s$. Then there exists a commutative diagram of algebraic stacks*

$$\begin{tikzcd} \mathcal{W}' \arrow[rr, "h"] \arrow[d, "t"'] & & \mathcal{W} \arrow[d] \\ \mathcal{H} \arrow[r] & S' \arrow[r, "g"] & S \end{tikzcd}$$

where

(1) *$g \colon (S', s') \to (S, s)$ is a smooth (étale if $\kappa(w)/\kappa(s)$ is separable) morphism such that there is a $\kappa(s)$-isomorphism $\kappa(w) \cong \kappa(s')$;*

(2) *$\mathcal{H} \to S'$ is a fundamental gerbe such that $\mathcal{H}_{s'} \cong \mathcal{G}_w$;*
(3) *$h\colon (\mathcal{W}', w') \to (\mathcal{W}, w)$ is a strongly étale (see Definition 3.13) neighborhood of w such that $\mathcal{G}_{w'} \to \mathcal{G}_w$ is an isomorphism; and*
(4) *$t\colon \mathcal{W}' \to \mathcal{H}$ is an affine morphism extending the inclusion $\mathcal{G}_{w'} \cong \mathcal{G}_w \to \mathcal{H}$.*

Moreover, we can arrange so that

(5) *if w has nice stabilizer (e.g. $\operatorname{char} \kappa(w) > 0$), then $\mathcal{H}$ is nicely fundamental;*
(6) *if w has nice stabilizer or admits an open neighborhood of characteristic zero, then $\mathcal{H}$ is linearly fundamental.*

Proof. We can replace (S, s) with an étale neighborhood and assume that S is an affine scheme. To obtain g as in (1), we may then take $S' = S \times \mathbb{A}^n$ for a suitable n or as an étale extension of $\operatorname{Spec} \kappa(w) \to \operatorname{Spec} \kappa(s)$ if the field extension is separable.

After replacing S' with an étale neighborhood of s', we obtain a fundamental gerbe $\mathcal{H} \to S'$ extending $\mathcal{G}_w$ by Proposition 16.10 (and Remark 16.15). Since $\mathcal{H} \to S' \to S$ is smooth, we may apply Proposition 16.12 to obtain the morphisms $h\colon \mathcal{W}' \to \mathcal{W}$ and $t\colon \mathcal{W}' \to \mathcal{H}$ satisfying (3) and (4). Finally, (5) and (6) follow from Proposition 16.13 and Proposition 16.14 respectively. $\square$

Proposition 17.2 (Group refinement)**.** *Let S be an affine scheme, let $\mathcal{H} \to S$ be a fundamental gerbe and let $s \in S$ be a point. Then after replacing S with an étale neighborhood of s, there exists a geometrically reductive and embeddable group scheme $G \to S$ and an affine S-morphism $\mathcal{H} \to BG$. Moreover, we can arrange so that*

(1) *if $\mathcal{H}_s = BG_0$, then $G_s \cong G_0$ and $\mathcal{H} \to BG$ is an isomorphism;*
(2) *if $\mathcal{H}$ is nicely fundamental, then G is nice; and*
(3) *if $\mathcal{H}$ is linearly fundamental, then G is linearly fundamental.*

Proof. If $\mathcal{H}_s = BG_0$, i.e., has a section σ_0 with automorphism group G_0, then after replacing S with an étale neighborhood, we obtain a section σ of $\mathcal{H}$ (Proposition 16.12) and the result follows with $G = \operatorname{Aut}(\sigma)$.

In general, there exists, after replacing S with an étale neighborhood of s, a finite étale surjective morphism $S' \to S$ such that $\mathcal{H} \times_S S' \to S'$ has a section σ'. The group scheme $H' = \operatorname{Aut}(\sigma') \to S'$ is geometrically reductive and embeddable. We let G be the Weil restriction of H' along $S' \to S$. It comes equipped with a morphism $\mathcal{H} \to BG$ which is representable, hence affine by Proposition 12.5(1). It can be seen that $G \to S$ is geometrically reductive and embeddable and also linearly reductive (resp. nice) if $\mathcal{H}$ is linearly fundamental (resp. nicely fundamental). $\square$

Proposition 17.3 (Smooth refinement)**.** *Let S be a quasi-separated algebraic space. Let $\mathcal{H} \to S$ be a fundamental gerbe and let $t\colon \mathcal{W} \to \mathcal{H}$ be an affine morphism of finite presentation. Let $w \in |\mathcal{W}|$ be a point with linearly reductive stabilizer and image $s \in |S|$ such that w is closed in its fiber $\mathcal{W}_s$. Suppose that the induced map $\mathcal{G}_w \to \mathcal{H}_s$ is an isomorphism. If $\mathcal{W} \to S$ is smooth, then after replacing S with an étale neighborhood of s, there exists*

(1) *a section $\sigma\colon \mathcal{H} \to \mathcal{W}$ of t such that $\sigma(s) = w$; and*
(2) *a morphism $q\colon \mathcal{W} \to \mathbb{V}(\mathcal{N}_\sigma)$, where $\mathcal{N}_\sigma = t_*(\mathcal{I}/\mathcal{I}^2)$ and $\mathcal{I}$ is the sheaf of ideals in $\mathcal{W}$ defining σ, which is strongly étale in an open neighborhood of σ and such that $q \circ \sigma$ is the zero-section.*

Proof. The existence of the section σ follows from Proposition 16.12. Note that since t is affine and smooth, the section σ is a regular closed immersion. An easy approximation argument allows us to replace S by the henselization at s. Then $\mathcal{H}$ is linearly fundamental (Corollary 13.7). Let $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{W}}$ be the ideal sheaf defining σ. Since $\mathcal{N}_\sigma = t_*(\mathcal{I}/\mathcal{I}^2)$ is locally free and $\mathcal{H}$ is cohomologically affine, the surjection

$t_*\mathcal{I} \to \mathcal{N}_\sigma$ of $\mathcal{O}_{\mathcal{H}}$-modules admits a section. The composition $\mathcal{N}_\sigma \to t_*\mathcal{I} \to t_*\mathcal{O}_{\mathcal{W}}$ gives a morphism $q\colon \mathcal{W} \to \mathbb{V}(\mathcal{N}_\sigma)$. By definition, q maps σ to the zero-section and induces an isomorphism of normal spaces along σ, hence is étale along σ, hence is strongly étale in a neighborhood by Luna's fundamental lemma (Theorem 3.14). $\square$

Corollary 17.4. *In the setting of Theorem 1.1, we can arrange that there is a commutative diagram of algebraic stacks*

$$\begin{tikzcd} \mathcal{W} \arrow[rrr, "h"] \arrow[d, "t"'] & & & \mathcal{X} \arrow[d] \\ \mathcal{H} \arrow[r, "r"] & BG \arrow[r] & S' \arrow[r, "g"] & S \end{tikzcd}$$

where

(1) *$g\colon (S', s') \to (S, s)$ is a smooth (étale if $\kappa(w)/\kappa(s)$ is separable) morphism such that there is a $\kappa(s)$-isomorphism $\kappa(w) \cong \kappa(s')$;*
(2) *$\mathcal{H} \to S'$ is a fundamental gerbe such that $\mathcal{H}_{s'} \cong \mathcal{W}_0$; and*
(3) *$G \to S'$ is a geometrically reductive embeddable group scheme; and*
(4) *$t\colon \mathcal{W} \to \mathcal{H}$ and $r\colon \mathcal{H} \to BG$ are affine morphisms, so $\mathcal{W} = [\operatorname{Spec} B/G]$.*

Moreover, we can arrange so that:

(5) *if $\mathcal{W}_0 = BG_0$, then $G_{s'} \cong G_0$ and $\mathcal{H} = BG$,*
(6) *if w has nice stabilizer, then $\mathcal{H}$ is nicely fundamental and G is nice,*
(7) *if $\operatorname{char} \kappa(s) > 0$ or s has an open neighborhood of characteristic zero, then $\mathcal{H}$ is linearly fundamental and G is linearly reductive,*
(8) *if $\mathcal{X} \to S$ is smooth at x and $\kappa(w)/\kappa(s)$ is separable, then there exists a commutative diagram*

$$\begin{tikzcd} \mathbb{V}(\mathcal{N}_\sigma) \arrow[dr] & \mathcal{W} \arrow[l, "q"'] \arrow[d, "t"', shift right] \\ & \mathcal{H} \arrow[ul, bend left, "0"] \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

where q is strongly étale and σ is a section of t such that $\sigma(s') = w$.

Proof. Theorem 1.1 produces a morphism $h\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$. We apply Proposition 17.1 to $(\mathcal{W}, w)$ and replace $(\mathcal{W}, w)$ with $(\mathcal{W}', w')$. Then we apply Proposition 17.2 to $(\mathcal{H}, s')$. Finally, if $\mathcal{X} \to S$ is smooth at x and $S' \to S$ is étale, then $\mathcal{W} \to S'$ is smooth and we can apply Proposition 17.3. $\square$

Proof of Theorem 1.3. Theorem 1.1 gives an étale neighborhood $(\mathcal{W}, w) \to (\mathcal{X}, x)$ inducing an isomorphism $\mathcal{G}_w \to \mathcal{G}_x$. Since $\mathcal{G}_x \to \operatorname{Spec} k$ is smooth, Proposition 16.12 shows that after replacing $\mathcal{W}$ with a strongly étale neighborhood, there is an affine morphism $\mathcal{W} \to \mathcal{G}_x$. $\square$

18. Structure of linearly reductive groups

Recall from Definition 2.1 that a linearly reductive (resp. geometrically reductive) group scheme $G \to S$ is flat, affine and of finite presentation such that $BG \to S$ is a good moduli space (resp. an adequate moduli space). In this section we will show that a group algebraic space is linearly reductive if and only if it is flat, separated, of finite presentation, has linearly reductive fibers, and has a finite component group (Theorem 18.9).

18.1. **Extension of closed subgroups.**

Lemma 18.1 (Anantharaman)**.** *Let S be the spectrum of a DVR. If $G \to S$ is a separated group algebraic space of finite type, then G is a scheme. If in addition $G \to S$ has affine fibers or is flat with affine generic fiber G_η, then G is affine.*

Proof. The first statement is [Ana73, Thm. 4.B]. For the second statement, it is enough to show that the flat group scheme $\overline{G} = \overline{G_\eta}$ is affine. This is [Ana73, Prop. 2.3.1]. □

Proposition 18.2. *Let $G \to S$ be a geometrically reductive group scheme that is embeddable fppf-locally on S (we will soon see that this is automatic if G is linearly reductive).*

(1) *If $N \subseteq G$ is a closed normal subgroup such that $N \to S$ is quasi-finite, then $N \to S$ is finite.*

Let $H \to S$ be a separated group algebraic space of finite presentation and let $u\colon G \to H$ be a homomorphism.

(2) *If u is a monomorphism, then u is a closed immersion.*
(3) *If $u_s\colon G_s \to H_s$ is a monomorphism for a point $s \in S$, then $u_U\colon G_U \to H_U$ is a closed immersion for some open neighborhood U of s.*

Proof. The questions are local on S so we can assume that G is embeddable. For (1) we note that a normal closed subgroup $N \subseteq G$ gives rise to a closed subgroup $[N/G]$ of the inertia stack $[G/G] = I_{BG}$ (where G acts on itself via conjugation). The result thus follows from Lemma 12.6.

For (2), it is enough to prove that u is proper. After noetherian approximation, we can assume that S is noetherian. By the valuative criterion for properness, we can further assume that S is the spectrum of a DVR. We can also replace H with the closure of $u(G_\eta)$. Then H is an affine group scheme (Lemma 18.1) so $H/G \to BG \to S$ is adequately affine, hence affine. It follows that u is a closed immersion.

For (3), we apply (1) to $\ker(u)$ which is quasi-finite, hence finite, in an open neighborhood of s. By Nakayama's lemma u is thus a monomorphism in an open neighborhood and we conclude by (2). □

Remark 18.3. If $H \to S$ is flat, then (2) says that any representable morphism $BG \to BH$ is separated. When G is of multiplicative type then Proposition 18.2 is [SGA3_{II}, Exp. IX, Thm. 6.4 and Exp. VIII, Rmq. 7.13b]. When G is reductive (i.e., smooth with connected reductive fibers) it is [SGA3_{II}, Exp. XVI, Prop. 6.1 and Cor. 1.5a].

Proposition 18.4. *Let (S,s) be a henselian local ring, let $G \to S$ be a flat group scheme of finite presentation with affine fibers and let $i_s\colon H_s \hookrightarrow G_s$ be a closed subgroup. If H_s is linearly reductive and G_s/H_s is smooth, then there exists a linearly reductive and embeddable group scheme $H \to S$ and a homomorphism $i\colon H \to G$ extending i_s.*

(1) *If G_s/H_s is étale (i.e., if i_s is open and closed), then i is étale and the pair (H,i) is unique.*
(2) *If $G \to S$ is separated, then i is a closed immersion.*

Proof. Note that since (S,s) is local, condition (FC) is satisfied. By Proposition 16.7, the gerbe BH_s extends to a unique linearly fundamental gerbe $\mathcal{H} \to S$.

Since $BG \to S$ is smooth, we can extend the morphism $\varphi_0\colon BH_s \to BG_s$ to a morphism $\varphi\colon \mathcal{H} \to BG$ (Proposition 16.5). The morphism φ is flat and the special fiber φ_0 is smooth since G_s/H_s is smooth. Thus φ is smooth. Similarly, if G_s/H_s is étale, then φ is étale and also unique by Proposition 16.4.

The tautological section $S \to BG$ restricted to the special fiber is compatible with the tautological section $f_s\colon s \to BH_s$ so we obtain a lift $f\colon S \to \mathcal{H}$ compatible with these by Proposition 16.4. The lift f is unique if φ is étale.

We let $H = \mathrm{Aut}(f)$ and let $i\colon H \to G = \mathrm{Aut}(\varphi \circ f)$ be the induced morphism, extending i_s. Finally, if G is separated, then i is a closed immersion (Proposition 18.2). $\square$

Remark 18.5. Note that even if G_s/H_s is not smooth the tautological section of BH_s extends to a section of $\mathcal{H} \to S$ so $\mathcal{H} = B\widetilde{H}$ where $\widetilde{H}$ is an extension of H_s and φ induces a homomorphism $\widetilde{H} \to \widetilde{G}$ where $\widetilde{G}$ is a twisted form of G. If H_s is smooth, then H is unique but not i.

18.2. **The smooth identity component of linearly reductive groups.** Recall that if $G \to S$ is a smooth group scheme, then there is an open subgroup $G^0 \subseteq G$ such that $G^0 \to S$ is smooth with connected fibers [SGA3$_{\mathrm{II}}$, Exp. 6B, Thm. 3.10]. This is also true when $G \to S$ is a smooth group algebraic space [LMB, 6.8]. For a (not necessarily smooth) group scheme of finite type over a field, the identity component exists and is open and closed. When (S, s) is henselian and $(G_s)^0$ is linearly reductive but not smooth, so of multiplicative type, then Proposition 18.4 gives the existence of a unique $i\colon G^0 \to G$ extending $i_s\colon (G_s)^0 \hookrightarrow G_s$. The group scheme G^0 has connected fibers in equal characteristic p but not necessarily in mixed characteristic. Also if G is not separated then i need not be injective. The latter phenomenon can also happen if G is smooth but not separated and then G^0 of Proposition 18.4 does not agree with the usual G^0.

Example 18.6. We give two examples in mixed characteristic and one in equal characteristic:

(1) Let $G = \boldsymbol{\mu}_{p,\mathbb{Z}_p} \to \operatorname{Spec} \mathbb{Z}_p$ which is a finite linearly reductive group scheme. Then $G^0 = G$ but the generic geometric fiber is not connected. If we let G' be the gluing of G and a finite group over $\mathbb{Q}_p$ containing $\boldsymbol{\mu}_p$ as a non-normal subgroup, then $G'^0 = G^0 \subseteq G'$ is not normal.

(2) Let G be as in the previous example and consider the étale group scheme $H \to \operatorname{Spec} \mathbb{Z}_p$ given as extension by zero from $\boldsymbol{\mu}_{p,\mathbb{Q}_p} \to \operatorname{Spec} \mathbb{Q}_p$. Then we have a bijective monomorphism $H \to G$ which is not an immersion and $G' = G/H$ is a quasi-finite group algebraic space with connected fibers which is not locally separated. Note that $(G')^0 = G^0$ and the étale morphism $(G')^0 \to G'$ is not injective.

(3) Let $G = \mathbb{G}_m \times S \to S = \operatorname{Spec} k[\![t]\!]$ and let $H \to S$ be μ_r over the generic point extended by zero. Let $G' = G/H$. Then G' is a smooth locally separated algebraic space, $G'^0 = G$ and $G'^0 \to G'$ is not injective.

From now on, we only consider separated group schemes. Then $G^0 \to G$ is a closed subgroup and the second phenomenon does not occur. The subgroup G^0 exists over the henselization but not globally in mixed characteristic. We remedy this by considering a slightly smaller subgroup which is closed but not open.

Lemma 18.7 (Identity component: nice case)**.** *Let S be an algebraic space and let $G \to S$ be a flat and separated group algebraic space of finite presentation with affine fibers.*

(1) *The locus of $s \in S$ such that $(G_s)^0$ is nice is open in S.*

Now assume that $(G_s)^0$ is nice for all $s \in S$.

(2) *There exist a unique characteristic closed subgroup $G^0_{\mathrm{sm}} \hookrightarrow G$ smooth over S that restricts to $(G_s)^0_{\mathrm{red}}$ on fibers.*

(3) *$G^0_{\mathrm{sm}} \to S$ is a torus, $G/G^0_{\mathrm{sm}} \to S$ is quasi-finite and separated, and $G \to S$ is quasi-affine.*

Now assume in addition that S has equal characteristic.

(4) *There exist a unique characteristic open and closed subgroup $G^0 \hookrightarrow G$ that restricts to $(G_s)^0$ on fibers.*
(5) *$G^0 \to S$ is of multiplicative type with connected fibers and $G/G^0 \to S$ is étale and separated.*

Proof. The questions are étale-local on S. For (1), if $(G_s)^0$ is nice, i.e., of multiplicative type, then over the henselization at s we can find an open and closed subgroup $G^0 \subseteq G$ such that G^0 is of multiplicative type Proposition 18.4. After replacing S with an étale neighborhood of s, we can thus find an open and closed subgroup $H \subseteq G$ where H is embeddable and of multiplicative type. It follows that $(G_s)^0$ is of multiplicative type for all s in S.

For an H as above, we have a characteristic closed subgroup $H_{\mathrm{sm}} \hookrightarrow H$ such that H_{sm} is a torus and H/H_{sm} is finite. Indeed, the Cartier dual of H is an étale sheaf of abelian groups and its torsion is a characteristic subgroup. It follows that G/H_{sm} is quasi-finite and separated and that G is quasi-affine.

It remains to prove that H_{sm} is characteristic and independent on the choice of H so that it glues to a characteristic subgroup G^0_{sm}. This can be checked after base change to henselian local schemes. If (S,s) is henselian, then $G^0 \subseteq H$ and since these are group schemes of multiplicative type of the same dimension, it follows that $G^0_{\mathrm{sm}} = H_{\mathrm{sm}}$. Since any automorphism of G leaves G^0 fixed, any automorphism leaves G^0_{sm} fixed as well.

If S has equal characteristic, then H is an open and closed subgroup with connected fibers, hence clearly unique. □

Lemma 18.8 (Identity component: smooth case)**.** *Let S be an algebraic space and let $G \to S$ be a flat and separated group algebraic space of finite presentation with affine fibers. Suppose that $G \to S$ is smooth and that $(G_s)^0$ is linearly reductive for all s.*

(1) *The open and closed subgroup $G^0 \hookrightarrow G$ is linearly reductive (and in particular affine).*
(2) *$G/G^0 \to S$ is étale and separated and $G \to S$ is quasi-affine.*

Proof. This follows immediately from Proposition 18.4 since in the henselian case G^0 is the unique open and closed subscheme containing $(G_s)^0$. □

Theorem 18.9 (Identity component)**.** *Let S be an algebraic space and let $G \to S$ be a flat and separated group algebraic space of finite presentation with affine fibers. Suppose that $(G_s)^0$ is linearly reductive for every $s \in S$.*

(1) *There exist a unique linearly reductive and characteristic closed subgroup $G^0_{\mathrm{sm}} \hookrightarrow G$ smooth over S that restricts to $(G_s)^0_{\mathrm{red}}$ on fibers, and $G/G^0_{\mathrm{sm}} \to S$ is quasi-finite and separated.*
(2) *If S is of equal characteristic, then there exists a unique linearly reductive characteristic open and closed subgroup $G^0 \hookrightarrow G$ that restricts to $(G_s)^0$ on fibers, and $G/G^0 \to S$ is étale and separated.*
(3) *$G \to S$ is quasi-affine.*

The following are equivalent:

(4) *$G \to S$ is linearly reductive (in particular affine).*
(5) *$G/G^0_{\mathrm{sm}} \to S$ is finite and tame.*
(6) *(if S of equal characteristic) $G/G^0 \to S$ is finite and tame.*

In particular, if $G \to S$ is linearly reductive and S is of equal characteristic $p > 0$, then $G \to S$ is nice.

Proof. Let $S_1 \subseteq S$ be the open locus where $(G_s)^0$ is nice and let $S_2 \subseteq S$ be the open locus where G_s is smooth. Then $S = S_1 \cup S_2$. Over S_1, we define G^0_{sm} as in Lemma 18.7. Over S_2, we define $G^0_{\mathrm{sm}} = G^0$ as in Lemma 18.8. The first two statements follow. Since $G^0_{\mathrm{sm}} \to S$ is linearly reductive, it follows that $BG \to S$ is cohomologically affine if and only if $B(G/G^0_{\mathrm{sm}}) \to S$ is cohomologically affine [Alp13, Prop. 12.17]. If $B(G/G^0_{\mathrm{sm}}) \to S$ is cohomologically affine, then G/G^0_{sm} is finite [Alp14, Thm. 8.3.2]. Conversely, if G/G^0_{sm} is finite and tame then $BG \to S$ is cohomologically affine and $G \to S$ is affine. □

Corollary 18.10. *If S is a normal noetherian scheme with the resolution property (e.g., S is regular and separated, or S is quasi-projective) and $G \to S$ is linearly reductive, then G is embeddable.*

Proof. The stack BG^0_{sm} has the resolution property [Tho87, Cor. 3.2]. Since $BG^0_{\mathrm{sm}} \to BG$ is finite and faithfully flat, it follows that BG has the resolution property [Gro17], hence that G is embeddable. □

Remark 18.11. Let $G \to S$ be as in Theorem 18.9. When G/G^0_{sm} is merely finite, then $G \to S$ is geometrically reductive. This happens precisely when $G \to S$ is *pure* in the sense of Raynaud–Gruson [RG71, Défn. 3.3.3]. In particular, $G \to S$ is geometrically reductive if and only if $\pi\colon G \to S$ is affine and $\pi_*\mathcal{O}_G$ is a locally projective $\mathcal{O}_S$-module [RG71, Thm. 3.3.5].

19. Applications to equivariant geometry

19.1. **Generalization of Sumihiro's theorem on torus actions.** Sumihiro's theorem on torus actions in the relative case is the following. Let S be a noetherian scheme and $X \to S$ a morphism of scheme satisfying Sumihiro's condition (N), that is, $X \to S$ is flat and of finite type, X_s is geometrically normal for all generic points $s \in S$ and X_s is geometrically integral for all codimension 1 points $s \in S$ (which by a result of Raynaud implies that X is normal); see [Sum75, Defn. 3.4 and Rem. 3.5]. If S is normal and $T \to S$ is a smooth and Zariski-locally diagonalizable group scheme acting on X over S, then there exists a T-equivariant affine open neighborhood of any point of X [Sum75, Cor. 3.11]. We provide the following generalization of this result which simultaneously generalizes [AHR19, Thm. 4.4] to the relative case.

Theorem 19.1. *Let S be a quasi-separated algebraic space. Let G be an affine and flat group scheme over S of finite presentation. Let X be a quasi-separated algebraic space locally of finite presentation over S with an action of G. Let $x \in X$ be a point with image $s \in S$ such that $\kappa(x)/\kappa(s)$ is finite. Assume that x has linearly reductive stabilizer. Then there exists a G-equivariant étale neighborhood $(\operatorname{Spec} A, w) \to (X, x)$ that induces an isomorphism of residue fields and stabilizer groups at w.*

Proof. By applying Theorem 1.1 to $\mathcal{X} = [X/G]$ with $\mathcal{W}_0 = \mathcal{G}_x$ (the residual gerbe of x), we obtain an étale morphism $h\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ with $\mathcal{W}$ fundamental and $h|_{\mathcal{G}_x}$ an isomorphism. By applying Proposition 12.5(1) to the composition $\mathcal{W} \to \mathcal{X} \to BG$, we may shrink $\mathcal{W}$ around w so that $\mathcal{W} \to BG$ is affine. It follows that $W := \mathcal{W} \times_{\mathcal{X}} X$ is affine and $W \to X$ is G-equivariant. If we let $w' \in W$ be the unique preimage of x, then $(W, w') \to (X, x)$ is the desired étale neighborhood. □

Corollary 19.2. *Let S be a quasi-separated algebraic space, $T \to S$ be a group scheme of multiplicative type over S (e.g., a torus), and X be a quasi-separated algebraic space locally of finite presentation over S with an action of T. Let $x \in X$*

be a point with image $s \in S$ such that $\kappa(x)/\kappa(s)$ is finite. Then there exists a T-equivariant étale morphism $(\operatorname{Spec} A, w) \to (X, x)$ that induces an isomorphism of residue fields and stabilizer groups at w.

Proof. This follows immediately from Theorem 19.1 as any subgroup of a fiber of $T \to S$ is linearly reductive. □

Remark 19.3. In [Bri15], Brion establishes several powerful structure results for actions of connected algebraic groups on varieties. In particular, [Bri15, Thm. 4.8] recovers the result above when S is the spectrum of a field, T is a torus and X is quasi-projective without the final conclusion regarding residue fields and stabilizer groups.

19.2. **Relative version of Luna's étale slice theorem.** We provide the following generalization of Luna's étale slice theorem [Lun73] (see also [AHR19, Thm. 4.5]) to the relative case.

Theorem 19.4. *Let S be a quasi-separated algebraic space. Let $G \to S$ be a smooth, affine group scheme. Let X be a quasi-separated algebraic space locally of finite presentation over S with an action of G. Let $x \in X$ be a point with image $s \in S$ such that $k(x)/k(s)$ is a finite separable extension. Assume that x has linearly reductive stabilizer G_x. Then there exists*

(1) *an étale morphism $(S', s') \to (S, s)$ and a $\kappa(s)$-isomorphism $\kappa(s') \cong \kappa(x)$;*
(2) *a geometrically reductive (linearly reductive if* $\operatorname{char} \kappa(s) > 0$ *or s has an open neighborhood of characteristic zero) closed subgroup $H \subseteq G' := G \times_S S'$ over S' such that $H_{s'} \cong G_x$; and*
(3) *an unramified H-equivariant S'-morphism $(W, w) \to (X', x')$ of finite presentation with W affine and $\kappa(w) \cong \kappa(x')$ such that $W \times^H G' \to X'$ is étale. Here $x' \in X' := X \times_S S'$ is the unique $\kappa(x)$-point over $x \in X$ and $s' \in S'$.*

Moreover, it can be arranged that

(4) *if $X \to S$ is smooth at x, then $W \to S'$ is smooth and there exists an H-equivariant section $\sigma \colon S' \to W$ such that $\sigma(s') = w$, and there exists a strongly étale H-equivariant morphism $W \to \mathbb{V}(\mathcal{N}_\sigma)$;*
(5) *if X admits an adequate GIT quotient by G (e.g., X is affine over S and G is geometrically reductive over S), and Gx is closed in X_s, then $W \times^H G' \to X'$ is strongly étale; and*
(6) *if $G \to S$ is embeddable, $H \to S$ is linearly reductive, and either*
    (a) *$X \to S$ is affine;*
    (b) *$G \to S$ has connected fibers, S is normal noetherian scheme, and $X \to S$ is flat of finite type with geometrically normal fibers, or*
    (c) *there exists a G-equivariant locally closed immersion $X \hookrightarrow \mathbb{P}(V)$ where V is a locally free $\mathcal{O}_S$-module with a G-action,*

    then $W \to X'$ is a locally closed immersion.

In the statement above, $W \times^H G'$ denotes the quotient $(W \times G')/H$ which inherits a natural action of G', and $\mathcal{N}_\sigma$ is the conormal bundle $\mathcal{I}/\mathcal{I}^2$ (where $\mathcal{I}$ is the sheaf of ideals in W defining σ) which inherits an action of H. If $H \to S$ is a flat and affine group scheme of finite presentation over an algebraic space S, and X and Y are algebraic spaces over S with an action of H which admit adequate GIT quotients (i.e. $[X/H]$ and $[Y/H]$ admit adequate moduli spaces), then an H-equivariant morphism $f \colon X \to Y$ is called *strongly étale* if $[X/H] \to [Y/H]$ is.

The section $\sigma \colon S' \to W$ of (4) induces an H-equivariant section $\widetilde{\sigma} \colon S' \to X'$. This factors as $S' \to G'/H \to W \times^H G' \to X'$. Since the last map is étale, we have that $L_{(G'/H)/X'} = \mathcal{N}_\sigma[1]$. The map $G'/H \to X'$ is unramified and its image is the orbit of $\widetilde{\sigma}$. We can thus think of $\mathcal{N}_\sigma$ as the *conormal bundle for the orbit of* $\widetilde{\sigma}$. We also have an exact sequence:

$$0 \to \mathcal{N}_\sigma \to \mathcal{N}_{\widetilde{\sigma}} \to \mathcal{N}_e \to 0$$

where $e\colon S' \to G'/H$ is the unit section.

Remark 19.5. A considerably weaker variant of this theorem had been established in [Alp10, Thm. 2], which assumed the existence of a section $\sigma\colon S \to X$ such that $X \to S$ is smooth along σ, the stabilizer group scheme G_σ of σ is smooth, and the induced map $G/G_\sigma \to X$ is a closed immersion.

Proof of Theorem 19.4. We start by picking an étale morphism $(S', s') \to (S, s)$ realizing (1) with S' affine. After replacing S' with an étale neighborhood, Proposition 18.4 yields a geometrically reductive closed subgroup scheme $H \subseteq G'$ such that $H_{s'} \cong G_x$. This can be made linearly reductive if $\operatorname{char} \kappa(s) > 0$ or s has an open neighborhood of characteristic zero (Proposition 16.14).

We apply the main theorem (Theorem 1.1) to $([X'/G'], x')$ and $h_0\colon \mathcal{W}_0 = BG_x \cong \mathcal{G}_{x'}$ where x' also denotes the image of x' in $[X'/G']$. This gives us a fundamental stack $\mathcal{W}$ and an étale morphism $h\colon (\mathcal{W}, w) \to ([X'/G'], x')$ such that $\mathcal{G}_w = BG_x$.

Since $G \to S$ is smooth, so is $G'/H \to S'$ and $[X'/H] \to [X'/G']$. The point $x' \in X'$ gives a canonical lift of $\mathcal{G}_w = BG_x \to [X'/G']$ to $\mathcal{G}_w = BG_x \to [X'/H]$. After replacing S' with and étale neighborhood, we can thus lift h to a map $q\colon (\mathcal{W}, w) \to ([X'/H], x')$ (Proposition 16.12). This map is unramified since h is étale and $[X'/H] \to [X'/G']$ is representable. After replacing $\mathcal{W}$ with an open neighborhood, we can also assume that $\mathcal{W} \to BH$ is affine by Proposition 12.5(1). Thus $\mathcal{W} = [W/H]$ where W is affine and q corresponds to an H-equivariant unramified map $W \to X'$. Note that since $w \in |\mathcal{W}|$ has stabilizer $H_{s'}$, there is a unique point $w \in |W|$ above $w \in |\mathcal{W}|$. This establishes (1)–(3).

If X is smooth, then so is $\mathcal{W} \to S'$ and (4) follows from Proposition 17.3 applied to $\mathcal{W} \to BH \to S'$. Note that unless H is smooth it is a priori not clear that $W \to S'$ is smooth. But the section $\sigma\colon S' \to W$ is a regular closed immersion since it is a pull-back of the regular closed immersion $BH \hookrightarrow \mathcal{W}$ given by Proposition 17.3. It follows that W is smooth in a neighborhood of σ.

If $[X/G]$ has an adequate moduli space, then $\mathcal{W} \to [X/G]$ becomes strongly étale after replacing $\mathcal{W}$ with a saturated open neighborhood by Luna's fundamental lemma (Theorem 3.14). This establishes (5).

Finally, for (6) we may assume that G is embeddable. If (b) holds, then there exists a G-quasi-projective G-invariant neighborhood $U \subseteq X$ of x [Sum75, Thm. 3.9]. Thus, cases (6a) and (6b) both reduce to case (6c). For (6c), we may assume that V is a free $\mathcal{O}_S$-module. As H is linearly reductive, there exists an H-semi-invariant function $f \in \Gamma(\mathbb{P}(V), \mathcal{O}(1))$ not vanishing at x. Then $\mathbb{P}(V)_f$ is an H-invariant affine open neighborhood. Applying Proposition 17.3 to $[\mathbb{P}(V)_f/H] \to BH \to S$ gives, after replacing S with an étale neighborhood, an affine open H-invariant neighborhood $U \subseteq \mathbb{P}(V)_f$, a section $\widetilde{\sigma}\colon BH \to [U/H]$ and a strongly étale morphism $U \to \mathbb{V}(\mathcal{N}_{\widetilde{\sigma}})$. We now consider the composition $\sigma\colon BH \to [U/H] \to [\mathbb{P}(V)/G]$. Since σ_s is a closed immersion, it becomes unramified after replacing S with an open neighborhood. This gives the exact sequence

$$0 \to \mathcal{N}_\sigma \to \mathcal{N}_{\widetilde{\sigma}} \to \Omega_{BH/BG} \to 0.$$

Since H is linearly reductive, this sequence splits. After choosing a splitting, we obtain an H-equivariant closed subscheme $\mathbb{V}(\mathcal{N}_\sigma) \hookrightarrow \mathbb{V}(\mathcal{N}_{\widetilde{\sigma}})$ and by pull back, an H-equivariant closed subscheme $W \hookrightarrow U$. By construction $[W/H] \to [U/H] \to [\mathbb{P}(V)/G]$ is étale at x. Finally, we replace W with an affine open H-saturated neighborhood of x in the quasi-affine scheme $W \cap X$. $\square$

20. Applications to henselizations

20.1. **Existence of henselizations.** Let $\mathcal{X}$ be an algebraic stack with affine stabilizers and let $x \in |\mathcal{X}|$ be a point with linearly reductive stabilizer. We have already seen that the completion $\widehat{\mathcal{X}}_x$ exists if $\mathcal{X}$ is noetherian (Theorem 11.2). In this section we will prove that there also is a henselization $\mathcal{X}_x^h$ when $\mathcal{X}$ is of finite presentation over an algebraic space S and $\kappa(x)/\kappa(s)$ is finite.

We say that an algebraic stack $\mathcal{G}$ is a *one-point gerbe* if $\mathcal{G}$ is noetherian and an fppf-gerbe over the spectrum of a field k, or, equivalently, if $\mathcal{G}$ is reduced, noetherian and $|\mathcal{G}|$ is a one-point space. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is called *pro-étale* if $\mathcal{X}$ is the inverse limit of a system of étale morphisms $\mathcal{X}_\lambda \to \mathcal{Y}$ such that $\mathcal{X}_\mu \to \mathcal{X}_\lambda$ is affine for all sufficiently large λ and all $\mu \geq \lambda$.

Let $\mathcal{X}$ be an algebraic stack and let $x \in |\mathcal{X}|$ be a point. Consider the inclusion $i \colon \mathcal{G}_x \hookrightarrow \mathcal{X}$ of the residual gerbe of x. Let $\nu \colon \mathcal{G} \to \mathcal{G}_x$ be a pro-étale morphism of one-point gerbes. The *henselization of* $\mathcal{X}$ *at* ν is by definition an initial object in the 2-category of 2-commutative diagrams

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[dr, "\nu"'] & \mathcal{X}' \arrow[d, "f"] \\ & \mathcal{X} \end{tikzcd} \tag{20.1}$$

where f is pro-étale (but not necessarily representable even if ν is representable). If $\nu \colon \mathcal{G}_x \to \mathcal{G}_x$ is the identity, we say that $\mathcal{X}_x^h := \mathcal{X}_\nu^h$ is the *henselization at* x.

Proposition 20.1 (Henselizations for stacks with good moduli spaces)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal and good moduli space $\pi \colon \mathcal{X} \to X$. If $x \in |\mathcal{X}|$ is a point such that $x \in |\mathcal{X}_{\pi(x)}|$ is closed, then the henselization $\mathcal{X}_x^h$ of $\mathcal{X}$ at x exists. Moreover*

1. *$\mathcal{X}_x^h = \mathcal{X} \times_X \operatorname{Spec} \mathcal{O}_{X,\pi(x)}^h$,*
2. *$\mathcal{X}_x^h$ is linearly fundamental, and*
3. *$(\mathcal{X}_x^h, \mathcal{G}_x)$ is a henselian pair.*

Proof. Let $\mathcal{X}_x^h := \mathcal{X} \times_X \operatorname{Spec} \mathcal{O}_{X,\pi(x)}^h$ which has good moduli space $\operatorname{Spec} \mathcal{O}_{X,\pi(x)}^h$. The pair $(\mathcal{X}_x^h, \mathcal{G}_x)$ is henselian (Theorem 3.6) and linearly fundamental (Theorem 13.1). It thus satisfies the hypotheses of Setup 16.1(c). To see that it is the henselization, we note that Proposition 16.4 trivially extends to pro-étale morphisms $\mathcal{X}' \to \mathcal{X}$ and implies that a section $\mathcal{G}_x \to \mathcal{X}' \times_{\mathcal{X}} \mathcal{G}_x$ extends to a morphism $\mathcal{X}_x^h \to \mathcal{X}'$. ◻

Remark 20.2. In the previous proposition, it is enough that $\mathcal{X}$ has separated diagonal and π is of finite presentation. If $\mathcal{X}$ does not have separated diagonal, it is still true that $(\mathcal{X} \times_X \operatorname{Spec} \mathcal{O}_{X,\pi(x)}^h, \mathcal{G}_x)$ is a henselian pair but it need not be the henselization. In Example 3.16 the pair $(\mathcal{Y}, B\mathbb{Z}/2\mathbb{Z})$ is henselian with non-separated diagonal and the henselization map $\mathcal{X} \to \mathcal{Y}$ is non-representable.

Theorem 20.3 (Existence of henselizations)**.** *Let S be a quasi-separated algebraic space. Let $\mathcal{X}$ be an algebraic stack, locally of finite presentation and quasi-separated over S, with affine stabilizers. Let $x \in |\mathcal{X}|$ be a point such that the residue field extension $\kappa(x)/\kappa(s)$ is finite and let $\nu \colon \mathcal{G} \to \mathcal{G}_x$ be a pro-étale morphism such that $\mathcal{G}$ is a one-point gerbe with linearly reductive stabilizer. Then the henselization $\mathcal{X}_\nu^h$ of $\mathcal{X}$ at ν exists. Moreover, $\mathcal{X}_\nu^h$ is a linearly fundamental algebraic stack and $(\mathcal{X}_\nu^h, \mathcal{G})$ is a henselian pair.*

Remark 20.4. If $x \in |\mathcal{X}|$ has linearly reductive stabilizer, the theorem above shows that the henselization $\mathcal{X}_x^h$ of $\mathcal{X}$ at x exists and moreover that $\mathcal{X}_x^h$ is linearly fundamental and $(\mathcal{X}_x^h, \mathcal{G}_x)$ is a henselian pair.

Proof of Theorem 20.3. By definition, we can factor ν as $\mathcal{G} \to \mathcal{G}_1 \to \mathcal{G}_x$ where $\mathcal{G} \to \mathcal{G}_1$ is pro-étale and representable and $\mathcal{G}_1 \to \mathcal{G}_x$ is étale. We can also arrange so that $\mathcal{G} \to \mathcal{G}_1$ is stabilizer-preserving. Then $\mathcal{G} = \mathcal{G}_1 \times_{k_1} \operatorname{Spec} k$ where k/k_1 is a separable field extension and $\mathcal{G}_1$ has linearly reductive stabilizer.

By Theorem 1.1 we can find a fundamental stack $\mathcal{W}$, a closed point $w \in \mathcal{W}$ and an étale morphism $(\mathcal{W}, w) \to (\mathcal{X}, x)$ such that $\mathcal{G}_w = \mathcal{G}_1$. Then $\mathcal{W}^h_{\nu_1} = \mathcal{W} \times_W \operatorname{Spec} \mathcal{O}^h_{W,\pi(w)}$, where $\pi\colon \mathcal{W} \to W$ is the adequate moduli space. Indeed, $\mathcal{W} \times_W \operatorname{Spec} \mathcal{O}^h_{W,\pi(w)}$ is linearly fundamental (Corollary 13.7) so Proposition 20.1 applies. Finally, we obtain $\mathcal{W}^h_\nu$ by base changing along a pro-étale morphism $W' \to W$ extending k/k_1. $\square$

20.2. Étale-local equivalences.

Theorem 20.5. *Let S be a quasi-separated algebraic space. Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks, locally of finite presentation and quasi-separated over S, with affine stabilizers. Suppose $x \in |\mathcal{X}|$ and $y \in |\mathcal{Y}|$ are points with linearly reductive stabilizers above a point $s \in |S|$ such that $\kappa(x)/\kappa(s)$ and $\kappa(y)/\kappa(s)$ are finite. Then the following are equivalent:*

(1) *There exists an isomorphism $\mathcal{X}^h_x \to \mathcal{Y}^h_y$ of henselizations.*
(2) *There exists a diagram of étale pointed morphisms*

$$\begin{tikzcd} & ([\operatorname{Spec} A/\mathrm{GL}_n], w) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ (\mathcal{X}, x) & & (\mathcal{Y}, y) \end{tikzcd}$$

such that both f and g induce isomorphisms of residual gerbes at w.

If S is locally noetherian, then the conditions above are also equivalent to:

(1′) *There exists an isomorphism $\widehat{\mathcal{X}}_x \to \widehat{\mathcal{Y}}_y$ of completions.*

Proof. The implications (2) $\Longrightarrow$ (1) and (2) $\Longrightarrow$ (1′) are clear. For the converses, we may reduce to the case when S is excellent in which case the argument of [AHR19, Thm. 4.19] is valid if one applies Theorem 1.1 instead of [AHR19, Thm. 1.1]. $\square$

Appendix A. Counterexamples in mixed characteristic

We first recall the following conditions on an algebraic stack $\mathcal{W}$ introduced in Section 15.

(FC) There is only a finite number of different characteristics in $\mathcal{W}$.
(PC) Every closed point of $\mathcal{W}$ has positive characteristic.
(N) Every closed point of $\mathcal{W}$ has nice stabilizer.

We also introduce the following condition which is implied by (FC) or (PC).

($\mathbb{Q}_{\mathrm{open}}$) Every closed point of $\mathcal{W}$ that is of characteristic zero has a neighborhood of characteristic zero.

In this appendix we will give examples of schemes and linearly fundamental stacks in mixed characteristic with various bad behavior.

(1) A noetherian linearly fundamental stack $\mathcal{X}$ with good moduli space $\mathcal{X} \to X$ such that X does not satisfy condition ($\mathbb{Q}_{\mathrm{open}}$) and we cannot write $\mathcal{X} = [\operatorname{Spec}(B)/G]$ with G linearly reductive étale-locally on X or étale-locally on $\mathcal{X}$ (Appendix A.1). In particular, condition ($\mathbb{Q}_{\mathrm{open}}$) is necessary in Theorem 13.1 and the similar condition is necessary in Corollary 17.4(7).
(2) A non-noetherian linearly fundamental stack $\mathcal{X}$ that cannot be written as an inverse limit of noetherian linearly fundamental stacks (Appendices A.2 and A.3).

(3) A noetherian scheme satisfying ($\mathbb{Q}_{\mathrm{open}}$) but neither (FC) nor (PC) (Appendix A.4).

Such counterexamples must have infinitely many different characteristics and closed points of characteristic zero.

Throughout this appendix, we work over the base scheme $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$. Let SL_2 act on $\mathfrak{sl}_2$ by conjugation. Then $\mathcal{Y} = [\mathfrak{sl}_2/\mathrm{SL}_2]$ is a fundamental stack with adequate moduli space $\mathcal{Y} \to Y := \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 = \operatorname{Spec}\mathbb{Z}[\frac{1}{2}, t]$ given by the determinant. Indeed, this follows from Zariski's main theorem and the following description of the orbits over algebraically closed fields. For $t \neq 0$, there is a unique orbit with Jordan normal form

$$\begin{bmatrix} \sqrt{-t} & 0 \\ 0 & -\sqrt{-t} \end{bmatrix}$$

and stabilizer $\mathbb{G}_m$. For $t = 0$, there are two orbits, one closed and one open, with Jordan normal forms

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and stabilizers SL_2 and $\mu_2 \times \mathbb{G}_a$ respectively. The nice locus is $Y_{\mathrm{nice}} = \{t \neq 0\}$. The linearly reductive locus is $\{t \neq 0\} \cup \mathbb{A}^1_{\mathbb{Q}}$.

A.1. **A noetherian example.** Let $A = \mathbb{Z}[\frac{1}{2}, t, \frac{1}{t+p}] \subseteq \mathbb{Q}[t]$ where p ranges over the set of all odd primes P.

- A is a noetherian integral domain: the localization of $\mathbb{Z}[\frac{1}{2}, t]$ in the multiplicative set generated by $(t+p)$.
- $A/(t) = \mathbb{Q}$.

We let $X = \operatorname{Spec} A$, let $X \to Y$ be the natural map (a flat monomorphism) and let $\mathcal{X} = \mathcal{Y} \times_Y X$. Then $\mathcal{X}$ is linearly fundamental with good moduli space X.

The nice locus of X is $\{t \neq 0\}$ and the complement consists of a single closed point x of characteristic zero. Any neighborhood of this point contains points of positive characteristic. It is thus impossible to write $\mathcal{X} = [\operatorname{Spec} B/G]$, with a linearly reductive group G, after restricting to any étale neighborhood of $x \in X$, or more generally, after restricting to any étale neighborhood in $\mathcal{X}$ of the unique closed point above x.

A.2. **A non-noetherian example.** Let $A = \mathbb{Z}[\frac{1}{2}, t, \frac{t-1}{p}] \subseteq \mathbb{Q}[t]$ where p ranges over the set of all odd primes P. Note that

- A is a non-noetherian integral domain,
- $A = \mathbb{Z}[\frac{1}{2}, t, (x_p)_{p\in P}]/(px_p - t + 1)_{p \in P}$,
- $A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[x_p]$ is regular, and thus noetherian, for every $p \in P$,
- $A \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[t]$,
- $A/(t) = \mathbb{Q}$,
- $A/(t-1) = \mathbb{Z}[\frac{1}{2}, (x_p)_{p\in P}]/(px_p)_{p\in P}$ has infinitely many irreducible components: the spectrum is the union of $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{A}^1_{\mathbb{F}_p}$ for every $p \in P$, and
- $\operatorname{Spec} A \to \operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ admits a section: $t = 1$, $x_p = 0$ for all $p \in P$.

We let $X = \operatorname{Spec} A$, let $X \to Y$ be the natural map and let $\mathcal{X} = \mathcal{Y} \times_Y X$. Then $\mathcal{X}$ is linearly fundamental with good moduli space X. Note that $\mathcal{X} \to X$ is of finite presentation as it is a pull-back of $\mathcal{Y} \to Y$.

Proposition A.1. *There does not exist a noetherian linearly fundamental stack $\mathcal{X}_\alpha$ and an affine morphism $\mathcal{X} \to \mathcal{X}_\alpha$.*

Proof. Suppose that such an $\mathcal{X}_\alpha$ exists. Then we may write $\mathcal{X} = \varprojlim \mathcal{X}_\lambda$ where the $\mathcal{X}_\lambda$ are affine and of finite presentation over $\mathcal{X}_\alpha$. Let $\mathcal{X}_\lambda \to X_\lambda$ denote the good moduli space which is of finite type [AHR19, Thm. A.1]. Thus, $\mathcal{X} \to \mathcal{X}_\lambda \times_{X_\lambda} X$ is affine and of finite presentation. For all sufficiently large λ we can thus find an affine finitely presented morphism $\mathcal{X}'_\lambda \to \mathcal{X}_\lambda$ such that $\mathcal{X} \to \mathcal{X}'_\lambda \times_{X_\lambda} X$ is an isomorphism. Since also $\mathcal{X} \to \mathcal{Y} \times_Y X$ is an isomorphism, it follows that there is an isomorphism $\mathcal{X}'_\lambda \to \mathcal{Y} \times_Y X_\lambda$ for all sufficiently large λ.

To prove the proposition, it is thus enough to show that there does not exist a factorization $X \to X_\lambda \to Y$ with X_λ noetherian and affine such that $\mathcal{Y} \times_Y X_\lambda$ is linearly fundamental. This follows from the following lemma. □

Lemma A.2. *Let Z be an integral affine scheme together with a morphism $Z \to Y = \operatorname{Spec} \mathbb{Z}[\frac{1}{2}, t]$ such that*

(1) *$f_\mathbb{Q} \colon Z_\mathbb{Q} \to \operatorname{Spec} \mathbb{Q}[t]$ is an isomorphism,*
(2) *$f^{-1}(0)$ is of pure characteristic zero, and*
(3) *$f^{-1}(1)$ admits a section s.*

Then Z is not noetherian.

Proof. For $a \in \mathbb{Z}$ and $p \in P$, let a_p (resp. $a_\mathbb{Q}$) denote the point in Y corresponding to the prime ideal $(p, t-a)$ (resp. $(t-a)$). Similarly, let η_p (resp. η) denote the points corresponding to the prime ideals (p) (resp. 0). Let $W = \operatorname{Spec} \mathbb{Z}[\frac{1}{2}] \hookrightarrow Z$ be the image of the section s and let $1_p \in Z$ also denote the unique point of characteristic p on W.

Suppose that the local rings of Z are noetherian. We will prove that $f^{-1}(1)$ then has infinitely many irreducible components. Since $f^{-1}(1)$ is the union of the closed subschemes W and $W_p := f^{-1}(1_p)$, $p \in P$, it is enough to prove that W_p has (at least) dimension 1 for every p.

Note that $\mathcal{O}_{Z,1_p}$ is (at least) 2-dimensional since there is a chain $1_p \leq 1_\mathbb{Q} \leq \eta$ of length 2 (here we use (1)). By Krull's Hauptidealsatz, $\mathcal{O}_{Z,1_p}/(p)$ has (at least) dimension 1 (here we use that the local ring is noetherian). The complement of $\operatorname{Spec} \mathcal{O}_{W_p,1_p} \hookrightarrow \operatorname{Spec} \mathcal{O}_{Z,1_p}/(p)$ maps to η_p. It is thus enough to prove that $f^{-1}(\eta_p) = \emptyset$.

Consider the local ring $\mathcal{O}_{Y,0_p}$. This is a regular local ring of dimension 2. Since $f_\mathbb{Q}$ is an isomorphism, $Z \times_Y \operatorname{Spec} \mathcal{O}_{Y,\eta_p} \to \operatorname{Spec} \mathcal{O}_{Y,\eta_p}$ is a birational affine morphism to the spectrum of a DVR. Thus, either $f^{-1}(\eta_p) = \emptyset$ or $Z \times_Y \operatorname{Spec} \mathcal{O}_{Y,\eta_p} \to \operatorname{Spec} \mathcal{O}_{Y,\eta_p}$ is an isomorphism. In the latter case, $f^{-1}(\operatorname{Spec} \mathcal{O}_{Y,0_p}) = f^{-1}(\operatorname{Spec} \mathcal{O}_{Y,0_p} \smallsetminus 0_p) \cong \operatorname{Spec} \mathcal{O}_{Y,0_p} \smallsetminus 0_p$ which contradicts that f is affine. □

A.3. **A variant of the non-noetherian example.** Let $A = \mathbb{Z}[\frac{1}{2}, t, \frac{t-1}{p^a}] \subseteq \mathbb{Q}[t]$ where $a \geq 1$ and p ranges over the set of all odd primes P. Note that

- A is a non-noetherian integral domain,
- $A = \mathbb{Z}[\frac{1}{2}, t, (x_{p,a})_{p \in P, a \geq 1}]/(px_{p,1} - t + 1, px_{p,a+1} - x_{p,a})_{p \in P, a \geq 1}$,
- $A \otimes_\mathbb{Z} \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[(x_{p,a})_{a \geq 1}]/(px_{p,a+1} - x_{p,a})_{a \geq 1}$ is two-dimensional and integral but not noetherian, for every $p \in P$,
- $A \otimes_\mathbb{Z} \mathbb{Q} = \mathbb{Q}[t]$,
- $A/(t) = \mathbb{Q}$,
- $A/(t-1) = \mathbb{Z}[\frac{1}{2}, (x_{p,a})_{p \in P, a \geq 1}]/(px_{p,1}, px_{p,a+1} - x_{p,a})_{p \in P, a \geq 1}$ is non-reduced with one irreducible component: the nil-radical is $(x_{p,a})_{p \in P, a \geq 1}$.
- $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$ admits a section: $t = 1$, $x_{p,a} = 0$ for all $p \in P$, $a \geq 1$.

This also gives a counterexample, exactly as in the previous subsection.

A.4. **Condition ($\mathbb{Q}_{\text{open}}$).** We provide examples illustrating that condition ($\mathbb{Q}_{\text{open}}$) is slightly weaker than conditions (FC) and (PC) even in the noetherian case. A non-connected example is given by $S = \operatorname{Spec}(\mathbb{Z} \times \mathbb{Q})$ which has infinitely many different characteristics and a closed point of characteristic zero. A connected counterexample is given by the push-out $S = \operatorname{Spec}\bigl(\mathbb{Z} \times_{\mathbb{F}_p} (\mathbb{Z}_{(p)}[x])\bigr)$ for any choice of prime number p. Note that the irreducible component $\operatorname{Spec}(\mathbb{Z}_{(p)}[x])$ has closed points of characteristic zero, e.g., the prime ideal $(px-1)$. The push-out is noetherian by Eakin–Nagata's theorem.

For an irreducible noetherian scheme, condition $(\mathbb{Q}_{\mathrm{open}})$ implies (FC) or (PC). That is, an irreducible noetherian scheme with a dense open of equal characteristic zero, has only a finite number of characteristics. This follows from Krull's Hauptidealsatz. We also note that for a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$, there are no closed points of characteristic zero so $(\mathbb{Q}_{\mathrm{open}})$ and (PC) hold trivially.

References

[AHH18] J. Alper, D. Halpern-Leistner, and J. Heinloth, *Existence of moduli spaces for algebraic stacks*, preprint, Dec 2018, arXiv:1812.01128, p. 71 pp.

[AHHR] J. Alper, J. Hall, D. Halpern-Leistner, and D. Rydh, *Artin algebraization for pairs and applications to the local structure of stacks and Ferrand pushouts*, in preparation.

[AHR19] J. Alper, J. Hall, and D. Rydh, *A Luna étale slice theorem for algebraic stacks*, accepted in Ann. of Math., 2019, arXiv:1504.06467.

[Alp10] J. Alper, *On the local quotient structure of Artin stacks*, Journal of Pure and Applied Algebra **214** (2010), no. 9, 1576 – 1591.

[Alp12] J. Alper, *Local properties of good moduli spaces*, Tohoku Math. J. (2) **64** (2012), no. 1, 105–123.

[Alp13] J. Alper, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 6, 2349–2402.

[Alp14] J. Alper, *Adequate moduli spaces and geometrically reductive group schemes*, Algebr. Geom. **1** (2014), no. 4, 489–531.

[AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[Ana73] S. Anantharaman, *Schémas en groupes, espaces homogènes et espaces algébriques sur une base de dimension 1*, Sur les groupes algébriques, Soc. Math. France, Paris, 1973, pp. 5–79. Bull. Soc. Math. France, Mémoire 33.

[AOV11] D. Abramovich, M. Olsson, and A. Vistoli, *Twisted stable maps to tame Artin stacks*, J. Algebraic Geom. **20** (2011), no. 3, 399–477.

[Art69] M. Artin, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 23–58.

[Aus66] M. Auslander, *Coherent functors*, Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965), Springer, New York, 1966, pp. 189–231.

[BR85] P. Bardsley and R. W. Richardson, *Étale slices for algebraic transformation groups in characteristic p*, Proc. London Math. Soc. (3) **51** (1985), no. 2, 295–317.

[Bri15] M. Brion, *On linearization of line bundles*, J. Math. Sci. Univ. Tokyo **22** (2015), no. 1, 113–147.

[Con] B. Conrad, *Formal GAGA for Artin stacks*, available on homepage.

[Con10] B. Conrad, *Smooth linear algebraic groups over the dual numbers*, MathOverflow question at `http://mathoverflow.net/questions/22078/smooth-linear-algebraic-groups-over-the-dual-numbers`, 2010.

[EGA] A. Grothendieck, *Éléments de géométrie algébrique*, I.H.E.S. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32** (1960, 1961, 1961, 1963, 1964, 1965, 1966, 1967).

[Elk73] R. Elkik, *Solutions d'équations à coefficients dans un anneau hensélien*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 553–603 (1974).

[God56] R. Godement, *Topologies $\mathfrak{m}$-adiques*, Séminaire H. Cartan et C. Chevalley: Géométrie Algébrique, (E. N. S., 1955/56), 8ème année, Exp. No. 18, Secrétariat mathématique, Paris, 1956, p. 12.

[Gro17] P. Gross, *Tensor generators on schemes and stacks*, Algebr. Geom. **4** (2017), no. 4, 501–522.

[GZB15] A. Geraschenko and D. Zureick-Brown, *Formal GAGA for good moduli spaces*, Algebr. Geom. **2** (2015), no. 2, 214–230.

[Hal14] J. Hall, *Cohomology and base change for algebraic stacks*, Math. Z. **278** (2014), no. 1-2, 401–429.

[Hal17] J. Hall, *Openness of versality via coherent functors*, J. Reine Angew. Math. **722** (2017), 137–182.

[Hoy17] M. Hoyois, *The six operations in equivariant motivic homotopy theory*, Adv. Math. **305** (2017), 197–279.

[HP14] D. Halpern-Leistner and A. Preygel, *Mapping stacks and categorical notions of properness*, preprint, Feb 2014, arXiv:1402.3204.

[HR14] J. Hall and D. Rydh, *The Hilbert stack*, Adv. Math. **253** (2014), 194–233.

[HR15] J. Hall and D. Rydh, *Algebraic groups and compact generation of their derived categories of representations*, Indiana Univ. Math. J. **64** (2015), no. 6, 1903–1923.

[HR16] J. Hall and D. Rydh, *Mayer–Vietoris squares in algebraic geometry*, June 2016, arXiv:1606.08517.

[HR17] J. Hall and D. Rydh, *Perfect complexes on algebraic stacks*, Compositio Math. **153** (2017), no. 11, 2318–2367.

[HR19] J. Hall and D. Rydh, *Coherent Tannaka duality and algebraicity of* Hom*-stacks*, Algebra Number Theory **13** (2019), no. 7, 1633–1675.

[KS16] K. Kurano and K. Shimomoto, *Ideal-adic completion of quasi-excellent rings (after Gabber)*, preprint, September 2016, arXiv:1609.09246.

[LMB] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 39, Springer-Verlag, Berlin, 2000.

[Lun73] D. Luna, *Slices étales*, Sur les groupes algébriques, Soc. Math. France, Paris, 1973, pp. 81–105. Bull. Soc. Math. France, Mémoire 33.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994.

[Nag62] M. Nagata, *Complete reducibility of rational representations of a matric group.*, J. Math. Kyoto Univ. **1** (1961/1962), 87–99.

[Ols05] M. Olsson, *On proper coverings of Artin stacks*, Adv. Math. **198** (2005), no. 1, 93–106.

[Ols06] M. Olsson, *Deformation theory of representable morphisms of algebraic stacks*, Math. Z. **253** (2006), no. 1, 25–62.

[Pop86] D. Popescu, *General Néron desingularization and approximation*, Nagoya Math. J. **104** (1986), 85–115.

[Ray68] M. Raynaud, *Un critère d'effectivité de descente*, Séminaire d'Algèbre Commutative dirigé par Pierre Samuel: 1967–1968. Les épimorphismes d'anneaux, Exp. No. 5, Secrétariat mathématique, Paris, 1968, p. 22.

[RG71] M. Raynaud and L. Gruson, *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89.

[Ryd11a] D. Rydh, *The canonical embedding of an unramified morphism in an étale morphism*, Math. Z. **268** (2011), no. 3-4, 707–723.

[Ryd11b] D. Rydh, *Étale dévissage, descent and pushouts of stacks*, J. Algebra **331** (2011), 194–223.

[Ryd15] D. Rydh, *Noetherian approximation of algebraic spaces and stacks*, J. Algebra **422** (2015), 105–147.

[Ryd16] D. Rydh, *Approximation of sheaves on algebraic stacks*, Int. Math. Res. Not. **2016** (2016), no. 3, 717–737.

[Ses77] C. S. Seshadri, *Geometric reductivity over arbitrary base*, Advances in Math. **26** (1977), no. 3, 225–274.

[SGA3_{I}] *Schémas en groupes. I: Propriétés générales des schémas en groupes*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 151, Springer-Verlag, Berlin, 1970.

[SGA3_{II}] *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152, Springer-Verlag, Berlin, 1970.

[SGA3_{III}] *Schémas en groupes. III: Structure des schémas en groupes réductifs*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153, Springer-Verlag, Berlin, 1970.

[SGA4_3] *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Mathematics, Vol. 305, Springer-Verlag, Berlin, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.

[Stacks] The Stacks Project Authors, *Stacks Project*, `http://math.columbia.edu/algebraic_geometry/stacks-git`.

[Sum75] H. Sumihiro, *Equivariant completion. II*, J. Math. Kyoto Univ. **15** (1975), no. 3, 573–605.

[Tho87] R. W. Thomason, *Equivariant resolution, linearization, and Hilbert's fourteenth problem over arbitrary base schemes*, Adv. in Math. **65** (1987), no. 1, 16–34.

[TK10] A. Touzé and W. van der Kallen, *Bifunctor cohomology and cohomological finite generation for reductive groups*, Duke Math. J. **151** (2010), no. 2, 251–278.

[Tot04] B. Totaro, *The resolution property for schemes and stacks*, J. Reine Angew. Math. **577** (2004), 1–22.

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