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Openness of versality via coherent functors

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Abstract. We give a proof of openness of versality using coherent functors. As an application, we streamline Artin's criterion for algebraicity of a stack. We also introduce multi-step obstruction theories, employing them to produce obstruction theories for the stack of coherent sheaves, the Quot functor, and spaces of maps in the presence of non-flatness.

Introduction

In M. Artin's classic paper on stacks, a criterion for algebraicity is expounded [7, Theorem 5.3]. In the present paper, we take a novel approach to algebraicity, proving an algebraicity criterion for stacks which is easier to apply, more widely applicable, and admitting a substantially simpler proof.

Theorem A. *Fix an excellent scheme S and a category X that is fibered in groupoids over the category of S-schemes, $\mathbf{Sch}/S$. Then, X is an algebraic stack that is locally of finite presentation over S, if and only if the following conditions are satisfied.*

(1) [Stack] *X is a stack over the site $(\mathbf{Sch}/S)_{\text{Ét}}$.*

(2) [Limit preservation] *For any inverse system of affine S-schemes $\{\operatorname{Spec} A_j\}_{j \in J}$ with limit $\operatorname{Spec} A$, the natural functor*

$$\varinjlim_j X(\operatorname{Spec} A_j) \to X(\operatorname{Spec} A)$$

is an equivalence of categories.

(3) [Homogeneity] *For any diagram of affine S-schemes $[\operatorname{Spec} B \leftarrow \operatorname{Spec} A \xrightarrow{i} \operatorname{Spec} A']$, where i is a nilpotent closed immersion, the natural functor*

$$X(\operatorname{Spec}(B \times_A A')) \to X(\operatorname{Spec} A') \times_{X(\operatorname{Spec} A)} X(\operatorname{Spec} B)$$

is an equivalence of categories.

(4) [Effectivity] *For any* $\mathfrak{m}$*-adically complete local noetherian ring* $(B, \mathfrak{m})$ *with an* S*-scheme structure* $\operatorname{Spec} B \to S$ *such that the induced morphism* $\operatorname{Spec}(B/\mathfrak{m}) \to S$ *is locally of finite type, the natural functor*

$$X(\operatorname{Spec} B) \to \varprojlim_n X(\operatorname{Spec}(B/\mathfrak{m}^n))$$

is an equivalence of categories.

(5) [Conditions on automorphisms and deformations] *For any affine* S*-scheme* T *that is locally of finite type over* S *and* $\xi \in X(T)$*, the functors*

$$\operatorname{Aut}_{X/S}(\xi, -),\ \operatorname{Def}_{X/S}(\xi, -)\colon \mathbf{QCoh}(T) \to \mathbf{Ab}$$

are coherent.

(6) [Conditions on obstructions] *For any affine* S*-scheme* T *that is locally of finite type over* S *and* $\xi \in X(T)$*, there exist an integer* n *and a coherent* n*-step obstruction theory for* X *at* ξ*.*

Except for conditions (5) and (6), Theorem A is similar to Artin's criterion [7, Theorem 5.3]. Note, however, that we have fewer conditions, and these conditions are cleaner (e.g. no deformation situations). The conditions of Theorem A are also stable under composition, in the sense of [45].

This paper began with the realization that the homogeneity condition (3), which is stronger than the analogous condition of [7, (S1′)], together with conditions (5) and (6), simplifies and broadens the applicability of existing results.

Our usage of the term "coherent" in conditions (5) and (6) of Theorem A is in a different sense than what many readers may be familiar with and is due to M. Auslander [9]. For an affine scheme S, a functor $F\colon \mathbf{QCoh}(S) \to \mathbf{Ab}$ is *coherent* if there exists a morphism of quasi-coherent $\mathcal{O}_S$-modules $\mathcal{K}_1 \to \mathcal{K}_2$ such that for all $\mathcal{I} \in \mathbf{QCoh}(S)$, there is a natural isomorphism of abelian groups

$$F(\mathcal{I}) \cong \operatorname{coker}\bigl(\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{K}_2, \mathcal{I}) \to \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{K}_1, \mathcal{I})\bigr).$$

It is proven in [20] that most functors arising in moduli are coherent.

Relation with other work. The idea of using the Exal functors to simplify M. Artin's results is due to H. Flenner [15]. Our results and techniques are quite different, however. In particular, H. Flenner [15] does not address the relationship between formal smoothness and formal versality.

Independently, work in the Stacks Project [44, 07T0] has provided a different perspective on Artin's results. This approach, however, requires that the deformation–obstruction theory is given by a bounded complex. If there are non-flat or non-tame objects in the moduli problem, the existence of such a complex is subtle. Note that while the problems with non-tame stacks can be dealt with by [20, Theorem B], the problems with non-flatness likely needed to be handled by derived algebraic geometry [44, blog:2572].

Using the ideas of B. Töen and G. Vezzosi [46, §1.4], J. Lurie has developed a criterion for algebraicity in the derived context [31, Theorem 3.2.1]. Conditions (5) and (6) of

Theorem A are related to Lurie's requirement of the existence of a cotangent complex. Lurie's criterion is not applicable to Artin stacks, though it is a future intention [31, Remark 2]. J. Pridham has proved a criterion for Artin stacks [39, Theorem 3.16] which is related to the results of Lurie's Ph.D. thesis [30, Theorems 7.1.6, 7.5.1] and also exploits a derived analogue of the homogeneity condition (3) in order to simplify Lurie's conditions.

To prove that the Quot functors for separated Deligne–Mumford stacks are algebraic spaces, M. Olsson and J. Starr [38, Theorem 1.1] did not apply [7, Corollary 5.4], which like [7, Theorem 5.3] is formulated in terms of a single-step obstruction theory. The reason for this is simple: in the presence of non-flatness, it is difficult to formulate a single-step obstruction theory with good properties.

They circumvented this predicament by the use of Artin's original algebraicity criterion [6, Theorem 5.3]. This earlier algebraicity criterion is not formulated in terms of the existence and properties of a single-step obstruction theory – but in terms of certain explicit lifting problems – making its application more complicated (note that J. Starr [45, Theorem 2.15] has subsequently generalized the criteria of [6, Theorem 5.3] to stacks). To solve these lifting problems, M. Olsson and J. Starr [38, Lemma 2.5] used a 2-step process. This 2-step process is insufficiently functorial to define a multi-step obstruction theory in the sense of this paper. It is, however, closely related, and inspired the multi-step obstruction theories we define.

M. Olsson and J. Starr [38, p. 4077] noted that M. Artin had incorrectly computed the obstruction theory of the Quot functor in the presence of non-flatness [6, (6.4)]. We have also located some other articles in the literature that have not observed the subtlety of deformation theory in the presence of non-flatness (see Sections 8–9). We would like to emphasize that the impact of this on the main ideas of these articles is small. Indeed, the relevant arguments in these articles are still perfectly valid in the flat case, which covers most cases of interest to geometers. In the non-flat case, the relevant statements in these articles can be shown to hold with the techniques and examples of this article.

By work of M. Olsson [36, Remark 1.7], the conditions of Theorem A are seen to be necessary. The sufficiency of the conditions of Theorem A is demonstrated by the following sequence of observations:

(i) formally versal deformations exist,

(ii) algebraizations of formally versal deformations exist, and

(iii) formal versality at a point implies smoothness in a neighborhood.

Using the generalizations of M. Artin's techniques [7] due to B. Conrad and J. de Jong [11, Theorem 1.5], conditions (1)–(4) of Theorem A prove (i) and (ii). The main contribution of this paper is the usage of conditions (3), (5), and (6) of Theorem A to prove (iii).

Note that in our proof of (iii), the techniques of Artin approximation [5] are not used. This is in contrast to M. Artin's treatments [6, 7], where this technique features prominently. In a paper joint with D. Rydh [21], we illustrate how refinements of the homogeneity condition (3) clarify and simplify M. Artin's results on versality.

Outline. In Section 1, we discuss the notion of homogeneity. Homogeneity is a generalization of the Schlessinger–Rim criteria [12, Exposé VI]. This section is quite categorical, but it is the only section of the paper that is such. Morally, homogeneity provides a stack X with a linear structure at every point, which we describe in Section 2. To be precise, for any scheme T, together with an object $\xi \in X(T)$, homogeneity produces an additive functor $\mathrm{Exal}_X(\xi,-)\colon \mathbf{QCoh}(T) \to \mathbf{Ab}$ sharply controlling the deformation theory of ξ. The author learnt these ideas from J. Wise (in person) and his paper [48], though they are likely well known to experts, and go back at least as far as the work of H. Flenner [15]. In Section 3, we recall and generalize – to the relative setting – the notion of limit preserving groupoid [7, §1]. The results in Section 3 are similar to those obtained by Lieblich–Osserman [27, §2.4].

In Section 4, we recall the notions of formal versality and formal smoothness. We next recast these notions in terms of vanishing criteria for the functors $\mathrm{Exal}_X(T,-)$. The central technical result of this paper is Theorem 4.4 – our new proof of (iii).

In Section 5, we briefly review coherent functors. In Section 6, we formalize multi-step obstruction theories. In Section 7, we prove Theorem A.

The remainder of the paper is devoted to applications. In Section 8, we compute a 2-step obstruction theory for the stack of coherent sheaves. Finally, in Section 9, we compute a 2-step obstruction theory for the stack of morphisms between two algebraic stacks.

In Appendix A, we prove that pushouts of algebraic stacks along nilimmersions and affine morphisms exist. This aids in the verification of the homogeneity condition (3) in practice. In Appendix B, we consider left-exact sequences of Picard categories and a resulting 7-term exact sequence. In Appendix C, we state two basic results on local Tor-functors for morphisms of algebraic stacks.

Assumptions, conventions, and notations. If $\mathcal{C}$ is a category, then denote the opposite category by $\mathcal{C}^\circ$. A *fibration* of categories $Q\colon \mathcal{C} \to \mathcal{D}$ has the property that every arrow in the category $\mathcal{D}$ admits a strongly cartesian lift. For an object d of the category $\mathcal{D}$, we denote the resulting fiber category by $Q(d)$. It will also be convenient to say that the category $\mathcal{C}$ is *fibered* over $\mathcal{D}$. If the category $\mathcal{C}$ is fibered over $\mathcal{D}$ and every arrow in the category $\mathcal{C}$ is strongly cartesian, then we say that the functor Q is *fibered in groupoids*. The assumptions guarantee that if the category $\mathcal{C}$ is fibered in groupoids over $\mathcal{D}$, then for every object d of the category $\mathcal{D}$ the fiber category $Q(d)$ is a groupoid.

Let T be a scheme. Denote by $|T|$ the underlying topological space (with the Zariski topology) and $\mathcal{O}_T$ the (Zariski) sheaf of rings on $|T|$. If $t \in |T|$, then let $\kappa(t)$ denote its residue field. Denote by $\mathbf{QCoh}(T)$ the abelian category of quasi-coherent sheaves on the scheme T. Let $\mathbf{Sch}/T$ denote the category of schemes over T. The big étale site over T will be denoted by $(\mathbf{Sch}/T)_{\text{Ét}}$. If T is locally noetherian, then let $\mathbf{Coh}(T)$ denote the abelian category coherent sheaves on T.

Let A be a ring and let M be an A-module. Denote the quasi-coherent $\mathcal{O}_{\operatorname{Spec} A}$-module associated to M by $\widetilde{M}$. Denote the abelian category of all (resp. coherent) A-modules by $\mathbf{Mod}(A)$ (resp. $\mathbf{Coh}(A)$).

As in [44], we make no separation assumptions on our algebraic stacks and spaces. As in [37], we use the lisse-étale site for sheaves on algebraic stacks.

Fix a 1-morphism of algebraic stacks $f\colon X \to Y$. Given another 1-morphism of algebraic stacks $W \to Y$ we denote the pullback along this 1-morphism by $f_W\colon X_W \to W$.

A morphism of algebraic S-stacks $U \to V$ is a *locally nilpotent closed immersion* if it is a closed immersion defined by a quasi-coherent sheaf of ideals $\mathcal{I}$, such that fppf-locally (equivalently, smooth-locally) on V there always exists an integer n such that $\mathcal{I}^n = (0)$.

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1. Homogeneity

Schlessinger's conditions [42], for a functor of artinian rings, are fundamental to infinitesimal deformation theory. Schlessinger's conditions were generalized to groupoids by R. S. Rim [12, Exposé VI], clarifying infinitesimal deformation theory in the presence of automorphisms. Schlessinger's and Rim's conditions are both instances of the notion of homogeneity, which can be traced back to A. Grothendieck [18, no. 195]. A generalization of Rim's conditions was recently considered by J. Wise [48, §2]. In this section, we will develop a relative formulation of homogeneity for use in this paper.

Throughout this section, we let S be a scheme. An S-*groupoid* is a pair (X, a_X), where X is a category and $a_X\colon X \to \mathbf{Sch}/S$ is a fibration in groupoids. A 1-morphism of S-groupoids $\Phi\colon (Y, a_Y) \to (Z, a_Z)$ is a functor $\Phi\colon Y \to Z$ that commutes strictly over $\mathbf{Sch}/S$. We will typically refer to an S-groupoid (X, a_X) just as "X".

Example 1.1. For any S-scheme T, there is a canonical functor

$$\mathbf{Sch}/T \to \mathbf{Sch}/S\colon (W \to T) \mapsto (W \to T \to S)$$

which is faithful. In particular, we may view an S-scheme T as an S-groupoid. Thus a morphism of S-schemes $g\colon U \to V$ induces a 1-morphism of S-groupoids

$$\mathbf{Sch}/g\colon \mathbf{Sch}/U \to \mathbf{Sch}/V.$$

The converse is also true: any 1-morphism of S-groupoids $G\colon \mathbf{Sch}/U \to \mathbf{Sch}/V$ is uniquely isomorphic to a 1-morphism of the form $\mathbf{Sch}/g$ for some morphism of S-schemes $g\colon U \to V$.

Definition 1.2. Fix an S-groupoid X. An X-*scheme* is a pair (T, σ_T) consisting of an S-scheme T together with a 1-morphism of S-groupoids $\sigma_T\colon \mathbf{Sch}/T \to X$. A morphism of X-schemes $(f, \alpha_f)\colon (U, \sigma_U) \to (V, \sigma_V)$ is given by a morphism of S-schemes $f\colon U \to V$ together with a 2-morphism $\alpha_f\colon \sigma_U \Rightarrow \sigma_V \circ \mathbf{Sch}/f$. The collection of all X-schemes forms a 1-category, which we denote as $\mathbf{Sch}/X$.

For a 1-morphism of S-groupoids $\Phi\colon Y \to Z$ there is an induced functor

$$\mathbf{Sch}/\Phi\colon \mathbf{Sch}/Y \to \mathbf{Sch}/Z.$$

It is readily seen that for an S-groupoid X, the category $\mathbf{Sch}/X$ is also an S-groupoid. The content of the 2-Yoneda Lemma is essentially that the natural 1-morphism of S-groupoids $\mathbf{Sch}/X \to X$ is an equivalence. An inverse to this equivalence is given by picking a clivage for X.

The principal advantage of working with the fibered category $\mathbf{Sch}/X$ is that it admits a *canonical* clivage. In practice, this means that given an X-scheme V and a morphism of S-schemes $p\colon U \to V$, the way to make U an X-scheme is already chosen for us: it is the composition

$$\mathbf{Sch}/U \xrightarrow{\mathbf{Sch}/p} \mathbf{Sch}/V \to X.$$

It is for this reason that working with $\mathbf{Sch}/X$ greatly simplifies proofs and definitions. Calculations, however, are typically easier to perform in X.

Fix a class P of morphisms of S-schemes and an S-groupoid X. Then, a morphism of X-schemes $p\colon U \to V$ is P if the underlying morphism of S-schemes is P. The following squares will feature frequently and prominently throughout the article.

Definition 1.3. Fix a scheme S, a class P of morphisms of S-schemes, and an S-groupoid X. A *P-nil pair over X* is a pair

$$(V \xrightarrow{p} T, V \xrightarrow{j} V'),$$

where p and j are morphisms of X-schemes, p is P, and j is a locally nilpotent closed immersion. A *P-nil square over X* is a commutative diagram of X-schemes:

(1.1)
$$\begin{tikzcd} V \arrow[r, "p"] \arrow[d, hook, "j"'] & T \arrow[d, "i"] \\ V' \arrow[r, "p'"] & T', \end{tikzcd}$$

where the pair $(V \xrightarrow{p} T, V \xrightarrow{j} V')$ is P-nil over X. A P-nil square over X is *cocartesian* if it is cocartesian in the category of X-schemes. A P-nil square over X is *geometric* if p' is affine, i is a locally nilpotent closed immersion, and there is a natural isomorphism

$$\mathcal{O}_{T'} \to i_*\mathcal{O}_T \times_{p'_* j_* \mathcal{O}_V} p'_*\mathcal{O}_{V'}.$$

The following definition is a trivial generalization of the ideas of M. Olsson [34, Appendix A], J. Starr [45, §2], and J. Wise [48, §2].

Definition 1.4 (P-homogeneity). Fix a scheme S and a class P of morphisms of S-schemes. A 1-morphism of S-groupoids $\Phi\colon Y \to Z$ is *P-homogeneous* if the following two conditions are satisfied.

(H_1^P) A P-nil square over Y is cocartesian if and only if the induced P-nil square over Z is cocartesian.

(H_2^P) If a P-nil pair over Y can be completed to a cocartesian P-nil square over Z, then it can be completed to a P-nil square over Y.

An S-groupoid X is P-homogeneous if its structure 1-morphism is P-homogeneous.

For homogeneity, we will be interested in the following classes of morphisms:

Nil – locally nilpotent closed immersions,

Cl – closed immersions,

rNil – morphisms $V \to T$ such that there exists $(V_0 \to V) \in \mathbf{Nil}$ with the composition $(V_0 \to V \to T) \in \mathbf{Nil}$,

rCl – morphisms $V \to T$ such that there exists $(V_0 \to V) \in \mathbf{Nil}$ with the composition $(V_0 \to V \to T) \in \mathbf{Cl}$,

Aff – affine morphisms.

By [17, IV.18.12.11], universal homeomorphisms of schemes are integral, thus affine. Hence, there is a containment of classes of morphisms of S-schemes:

$$\begin{tikzcd}[column sep=small, row sep=small] & \mathbf{Cl} \arrow[dr, phantom, "\subseteq" rotate=-45] & \\ \mathbf{Nil} \arrow[ur, phantom, "\subseteq" rotate=45] \arrow[dr, phantom, "\subseteq" rotate=-45] & & \mathbf{rCl} \subseteq \mathbf{Aff}. \\ & \mathbf{rNil} \arrow[ur, phantom, "\subseteq" rotate=45] & \end{tikzcd}$$

In [21, Appendix A], it is shown that if X is limit preserving, in the sense of [7, §1], and a stack for the Zariski topology, then **rCl**-homogeneity is equivalent to the condition (S1′) of [7, (2.3)]. To assuage any fears of circularity, we would like to emphasize that this result will not be used in this paper.

J. Wise [48, Proposition 2.1] has shown that every algebraic stack is **Aff**-homogeneous. In Appendix A, we generalize results of D. Ferrand [14] and obtain techniques to prove that many "geometric" moduli problems are **Aff**-homogeneous.

The following definition is a convenient computational tool. A 1-morphism of S-groupoids $\Phi\colon Y \to Z$ is *formally étale* if for every Z-scheme V' and every locally nilpotent closed immersion of Z-schemes $V \hookrightarrow V'$, then every Y-scheme structure on V that is compatible with its Z-scheme structure under Φ lifts uniquely to a compatible Y-scheme structure on V'. That is, there is always a unique solution to the following lifting problem:

$$\begin{tikzcd} V \arrow[r] \arrow[d, hook] & Y \arrow[d, "\Phi"] \\ V' \arrow[r] \arrow[ur, dashed, "\exists !"] & Z. \end{tikzcd}$$

Note that if Y is a stack for the étale topology, then it suffices to verify the above lifting property étale-locally on V'. Indeed, the uniqueness in the definition of formally étale guarantees the cocycle condition necessary to perform the descent. Also, if Y and Z are schemes, then the induced 1-morphism of S-groupoids $\mathbf{Sch}/Y \to \mathbf{Sch}/Z$ is formally étale if and only if the morphism of schemes $Y \to Z$ is formally étale [17, IV_4.17.1.1].

The following lemma provides several methods to prove that a 1-morphism of S-groupoids is P-homogeneous, at least in the situation where $P \subseteq \mathbf{Aff}$.

Lemma 1.5. *Fix a scheme S, a 1-morphism of S-groupoids $\Phi\colon Y \to Z$, and a class $P \subseteq$ **Aff** of morphisms of S-schemes.*

(1) *Every cocartesian P-nil square over Y is geometric. In particular, if Z satisfies (H_1^P), then every cocartesian P-nil square over Y is cocartesian over Z.*

(2) *Let $(V \xrightarrow{p} T, V \xrightarrow{j} V')$ be a P-nil pair over Y that may be completed to a cocartesian P-nil square over Z as in* (1.1). *If Φ is P-homogeneous, then this cocartesian P-nil square over Z lifts uniquely to a P-nil square over Y that is simultaneously cocartesian and geometric.*

(3) *Let W be a P-homogeneous S-groupoid. Then every P-nil pair $(V \xrightarrow{p} T, T \xrightarrow{j} V')$ over W can be completed to a P-nil square over W that is simultaneously cocartesian and geometric. In particular, P-nil squares over W are cocartesian if and only if they are geometric.*

(4) *If W is an S-groupoid that is a stack for the Zariski topology, then W is P-homogeneous if and only if for every P-nil pair $(\operatorname{Spec} A \to \operatorname{Spec} B, \operatorname{Spec} A \hookrightarrow \operatorname{Spec} A')$ over S, the naturally induced functor*

$$W(\operatorname{Spec}(B \times_A A')) \to W(\operatorname{Spec} B) \times_{W(\operatorname{Spec} A)} W(\operatorname{Spec} A')$$

is an equivalence of categories.

(5) *Let $\Psi\colon W \to Y$ be a 1-morphism of S-groupoids. If Φ is P-homogeneous, then Ψ is P-homogeneous if and only if $\Phi \circ \Psi$ is P-homogeneous.*

(6) *If Z is P-homogeneous, then the 1-morphism Φ is P-homogeneous if and only if for every Z-scheme W, the W-groupoid $Y \times_Z (\mathbf{Sch}/W)$ is P-homogeneous.*

(7) *If Z and Φ are P-homogeneous, then for every P-homogeneous 1-morphism of S-groupoids $\Psi\colon W \to Z$, the 1-morphism $Y \times_Z W \to W$ is P-homogeneous.*

(8) *If Z and Φ are P-homogeneous, then the diagonal 1-morphism $\Delta_\Phi\colon Y \to Y \times_Z Y$ is P-homogeneous.*

(9) *If Z is P-homogeneous and Φ is formally étale, then Φ and Y are P-homogeneous.*

Proof. For (1), fix a cocartesian P-nil square over Y as in (1.1). By [14, Théorème 7.1], the induced P-nil pair $(V \xrightarrow{p} T, V \xrightarrow{j} V')$ over Y may be completed to the following cocartesian P-nil square over S which is geometric:

$$\begin{tikzcd} V \arrow[r, "p"] \arrow[d, hook, "j"'] & T \arrow[d] \\ V' \arrow[r] & \tilde{T}. \end{tikzcd}$$

The universal properties produce a unique S-morphism $t\colon \tilde{T} \to T'$. The morphism t promotes $\tilde{T}$ to a Y-scheme and it follows from the universal property defining T' that there is a uniquely induced Y-morphism $u\colon T' \to \tilde{T}$ such that $tu = \mathrm{Id}_{T'}$. The universal property defining $\tilde{T}$ in the category of S-schemes shows that $ut = \mathrm{Id}_{\tilde{T}}$. Thus u is an isomorphism over Y and the result follows.

For (2), by (H_2^P), it follows that there is a P-nil square over Y,

$$\begin{tikzcd} V \arrow[r, "p"] \arrow[d, hook, "j"'] & T \arrow[d] \\ V' \arrow[r] & T''. \end{tikzcd}$$

The P-nil square over Y above induces a P-nil square over Z. Since the P-nil square over Z as in (1.1) is cocartesian, it follows that there is a uniquely induced Z-morphism $T' \to T''$ that is compatible with the data. Since T'' is a Y-scheme, T' inherits the structure of a Y-scheme. It follows that the cocartesian P-nil square over Z as in (1.1) lifts to a P-nil square over Y and, by (H_1^P), it is cocartesian and thus the lifting is unique. That the resulting square is geometric follows from (1).

The claims (3) and (5) both follow from (1) and (2).

The claims (4) and (6) both follow from (1), (2), and (3).

The claim (7) follows from (6).

For (8), by (7) and (5), we know that $Y \times_Z Y$ is P-homogeneous. The result now follows from (5) applied to $Y \to Y \times_Z Y \to Y$.

For (9), by (5), it is sufficient to prove that Φ is P-homogeneous. Since Φ is formally étale, we may use (1) to deduce that Φ satisfies (H_1^P). By (3), every cocartesian P-nil square over Z is geometric. Since Φ is formally étale, it follows that Φ satisfies (H_2^P). □

2. Extensions

The results of this section are well known to experts, being similar to those obtained by H. Flenner [15] and J. Wise [48, §2.3].

Fix a scheme S and an S-groupoid X. An X-*extension* is a square zero closed immersion of X-schemes $i\colon T \hookrightarrow T'$. An obligatory observation is that the $i^{-1}\mathcal{O}_{T'}$-module $\ker(i^{-1}\mathcal{O}_{T'} \to \mathcal{O}_T)$ is canonically a quasi-coherent $\mathcal{O}_T$-module. If the X-scheme T is affine, so is the scheme T'; see [17, I.5.1.9]. A morphism of X-extensions

$$(i_1\colon T_1 \hookrightarrow T_1') \to (i_2\colon T_2 \hookrightarrow T_2')$$

is a commutative diagram of X-schemes

$$\begin{tikzcd} T_1 \arrow[r, hook, "i_1"] \arrow[d] & T_1' \arrow[d] \\ T_2 \arrow[r, hook, "i_2"] & T_2'. \end{tikzcd}$$

In a natural way, the collection of X-extensions forms a category, which we denote as $\mathbf{Exal}_X$. There is a natural functor $\mathbf{Exal}_X \to \mathbf{Sch}/X\colon (i\colon T \hookrightarrow T') \to T$.

We denote by $\mathbf{Exal}_X(T)$ the fiber of the category $\mathbf{Exal}_X$ over the X-scheme T. An X-*extension of* T is an object of $\mathbf{Exal}_X(T)$. There is a natural functor

$$\mathbf{Exal}_X(T)^\circ \to \mathbf{QCoh}(T), \quad (i\colon T \hookrightarrow T') \mapsto \ker(i^{-1}\mathcal{O}_{T'} \to \mathcal{O}_T).$$

We denote by $\mathbf{Exal}_X(T, I)$ the fiber category of $\mathbf{Exal}_X(T)$ over the quasi-coherent $\mathcal{O}_T$-module I. An X-*extension of* T *by* I is an object of $\mathbf{Exal}_X(T, I)$.

A morphism $(T \hookrightarrow T_1') \to (T \hookrightarrow T_2')$ in $\mathbf{Exal}_X(T, I)$ induces a commutative diagram of sheaves of rings on the topological space $|T|$:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, equal] & \mathcal{O}_{T_2'} \arrow[r] \arrow[d] & \mathcal{O}_T \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \arrow[r] & \mathcal{O}_{T_1'} \arrow[r] & \mathcal{O}_T \arrow[r] & 0. \end{tikzcd}$$

The Snake Lemma implies that the morphism of S-schemes $T_1' \to T_2'$ is an isomorphism. Thus the category $\mathbf{Exal}_X(T, I)$ is a groupoid.

Example 2.1. If X is an algebraic stack, T is an X-scheme, and I is a quasi-coherent $\mathcal{O}_T$-module, then the groupoid $\mathbf{Exal}_X(T, I)$ is equivalent to the Picard category associated to the complex $\tau^{\leq 0}(\mathsf{RHom}_{\mathcal{O}_T}(\tau^{\geq -1} L_{T/X}, I)[1])$, where $\tau^{\geq -1} L_{T/X}$ is the truncated cotangent complex of [25, Chapter 17]. For a proof of this, see [36, §2.22 and Theorem A.7]. For background material on Picard categories see [8, XVIII.1.4].

Example 2.1 motivates many of the results in this section. The following is a triviality that we record here for future reference.

Lemma 2.2. *Fix a scheme S, a formally étale 1-morphism of S-groupoids $X \to Y$, an X-scheme T, and a quasi-coherent $\mathcal{O}_T$-module I. Then, the natural functor*

$$\mathbf{Exal}_X(T, I) \to \mathbf{Exal}_Y(T, I)$$

is an equivalence of categories.

Fix a scheme W and a quasi-coherent $\mathcal{O}_W$-module J. Then, the quasi-coherent $\mathcal{O}_W$-module $\mathcal{O}_W \oplus J$ is readily seen to be an $\mathcal{O}_W$-algebra. Indeed, for an open subset $U \subseteq |W|$ and $(w, j), (w', j') \in \Gamma(U, \mathcal{O}_W)$, let

$$(w, j) \cdot (w', j') = (ww', wj' + w'j),$$

which makes $\mathcal{O}_W \oplus J$ a sheaf of rings. The natural map $\mathcal{O}_W \to \mathcal{O}_W \oplus J$, $w \mapsto (w, 0)$ canonically defines an $\mathcal{O}_W$-algebra, which we denote as $\mathcal{O}_W[J]$. Let $W[J]$ be the W-scheme $\underline{\mathrm{Spec}}_W(\mathcal{O}_W[J])$. Corresponding to the natural surjection of $\mathcal{O}_W$-algebras $\mathcal{O}_W[J] \to \mathcal{O}_W$, there is a canonical W-extension of W by J, which we call the *trivial* W-extension of W by J and denote as $(i_{W,J} \colon W \hookrightarrow W[J])$. In particular, the structure morphism $r_{W,J} \colon W[J] \to W$ is a retraction of the morphism $i_{W,J} \colon W \to W[J]$.

For a morphism of X-schemes $q \colon U \to V$, let $\mathrm{Ret}_X(U/V)$ be the set of X-retractions to the morphism $q \colon U \to V$. That is,

$$\mathrm{Ret}_X(U/V) = \{r \in \mathrm{Hom}_{\mathbf{Sch}/X}(V, U) : rq = \mathrm{Id}_U\}.$$

Lemma 2.3. *Fix a scheme S, an S-groupoid X, an X-scheme T, a quasi-coherent T-module I, and an X-extension $(i \colon T \hookrightarrow T')$ of T by I. Then, there is a natural bijection:*

$$\mathrm{Hom}_{\mathbf{Exal}_X(T,I)}((i \colon T \hookrightarrow T'), (i_{T,I} \colon T \hookrightarrow T[I])) \to \mathrm{Ret}_X(T/T').$$

Proof. For a morphism of X-extensions $(T \hookrightarrow T') \to (T \hookrightarrow T[I])$, the composition $T' \to T[I] \xrightarrow{r_{T,I}} T$ defines an X-retraction to i. This assignment is bijective. □

With some homogeneity assumptions, we are able to prove something meaningful.

Proposition 2.4. *Fix a scheme S, an S-groupoid X, and an X-scheme T. Then the functor $\mathbf{Exal}_X(T) \to \mathbf{QCoh}(T)^\circ$ is a fibration in groupoids. If the S-groupoid X is* **Nil***-homogeneous, then $\mathbf{Exal}_X(T, I)$ is a Picard category for all $I \in \mathbf{QCoh}(T)$.*

Proof. Fix a morphism $\alpha^\circ\colon J \to I$ in $\mathbf{QCoh}(T)^\circ$. This corresponds to a morphism of quasi-coherent $\mathcal{O}_T$-modules $\alpha\colon I \to J$. Also, fix an X-extension $(i\colon T \hookrightarrow T'_I)$ of T by I. On the topological space $|T|$ we obtain a commutative diagram of sheaves of abelian groups with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "\alpha"'] & \mathcal{O}_{T'_I} \arrow[r] \arrow[d, "\widetilde{\alpha}"] & \mathcal{O}_T \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & J \arrow[r] & \mathcal{O}_{T'_I} \oplus_I J \arrow[r] & \mathcal{O}_T \arrow[r] & 0, \end{tikzcd}$$

where

$$\mathcal{O}_{T'_I} \oplus_I J = \operatorname{coker}\bigl(I \xrightarrow{i \mapsto (i, -\alpha(i))} \mathcal{O}_{T'_I} \oplus J\bigr).$$

It is easily verified that the sheaf of abelian groups $\mathcal{O}_{T'_J} = \mathcal{O}_{T'_I} \oplus_I J$ is a sheaf of rings and that the homomorphism $\widetilde{\alpha}$ is a ring homomorphism. The subsheaf $J \subseteq \mathcal{O}_{T'_J}$ defines a square zero sheaf of ideals and as J is quasi-coherent, one immediately concludes that the ringed space $(|T|, \mathcal{O}_{T'_J})$ is an S-scheme, T'_J, and that we have defined an S-extension $(i_\alpha\colon T \hookrightarrow T'_J)$ of T by J. The morphism of S-schemes $T'_J \to T'_I$ promotes the S-extension i_α to an X-extension of T by J. It is immediate that the resulting arrow $i_\alpha \to i$ in $\mathbf{Exal}_X(T)$ is strongly cartesian over the arrow $\alpha^\circ\colon J \to I$ in $\mathbf{QCoh}(T)^\circ$, and we deduce the first claim.

For the second claim, the fibration $\mathbf{Exal}_X(T) \to \mathbf{QCoh}(T)^\circ$ induces for each M and $N \in \mathbf{QCoh}(T)$ a functor

$$\pi_{M,N}\colon \mathbf{Exal}_X(T, M \times N) \to \mathbf{Exal}_X(T, M) \times \mathbf{Exal}_X(T, N).$$

Note that this functor is not unique, but for any other choice of such a functor $\pi'_{M,N}$, there is a unique natural isomorphism of functors $\pi_{M,N} \Rightarrow \pi'_{M,N}$. This renders the Picard category structure on $\mathbf{Exal}_X(T, I)$ as essentially unique, and on the level of isomorphism classes of objects, the abelian group structure is unique.

By [19, §1.2], it is sufficient to show that the functor $\pi_{M,N}$ is an equivalence, which we show using the arguments of [17, 0_{IV}.18.3]. For the essential surjectivity, we fix X-extensions $(i_M\colon T \hookrightarrow T'_M)$ and $(i_N\colon T \hookrightarrow T'_N)$ of T by M and N, respectively. By Lemma 1.5 (3), there is a geometric **Nil**-nil square over X

$$\begin{tikzcd} T \arrow[r, hook, "i_M"] \arrow[d, hook, "i_N"'] & T'_M \arrow[d] \\ T'_N \arrow[r] & T', \end{tikzcd}$$

In particular, the resulting closed immersion $i\colon T \hookrightarrow T'$ defines an X-extension of T by $M \times N$. Moreover, it is plain to see that $\pi_{M,N}(i) \cong (i_M, i_N)$. The full faithfulness of the functor $\pi_{M,N}$ follows from a similar argument. □

Denote the set of isomorphism classes of the category $\mathbf{Exal}_X(T, I)$ by $\mathrm{Exal}_X(T, I)$. By Proposition 2.4, if X is **Nil**-homogeneous, then there are additive functors

$$\mathrm{Der}_X(T, -)\colon \mathbf{QCoh}(T) \to \mathbf{Ab}, \quad I \mapsto \mathrm{Aut}_{\mathbf{Exal}_X(T,I)}(i_{T,I})$$

and

$$\mathrm{Exal}_X(T, -)\colon \mathbf{QCoh}(T) \to \mathbf{Ab}, \quad I \mapsto \mathrm{Exal}_X(T, I).$$

We note that the 0-object of the abelian group $\mathrm{Der}_X(T, I)$ corresponds to the identity automorphism and the 0-object of the group $\mathrm{Exal}_X(T, I)$ corresponds to the isomorphism class containing the trivial X-extension of T by I, $(i_{T,I}: T \hookrightarrow T[I])$. With a stronger homogeneity assumption, there is an important exact sequence.

Corollary 2.5. *Fix a scheme S, an* **rNil**-*homogeneous S-groupoid X, and an X-scheme T. Then for every short exact sequence of quasi-coherent $\mathcal{O}_T$-modules,*

$$0 \to K \xrightarrow{k} M \xrightarrow{c} C \to 0,$$

there is a natural 6*-term exact sequence of abelian groups:*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Der}_X(T,K) \arrow[r] & \mathrm{Der}_X(T,M) \arrow[r] & \mathrm{Der}_X(T,C) \arrow[dll, "\partial"] \\ & \mathrm{Exal}_X(T,K) \arrow[r] & \mathrm{Exal}_X(T,M) \arrow[r] & \mathrm{Exal}_X(T,C). \end{tikzcd}$$

Proof. If X is algebraic, then the exact sequence is a trivial consequence of [36, Theorem 1.1]. In general, the result can be recovered from [48, Proposition 2.3 (iv)], where it was shown that the fibered category $\mathbf{Exal}_X(T) \to \mathbf{QCoh}(T)^\circ$ is additive and left-exact, in the sense of [19]. We will follow a similar route, but instead employ the results of Appendix B.

By Proposition 2.4, the morphisms k and c induce functors

$$k_*: \mathbf{Exal}_X(T,K) \to \mathbf{Exal}_X(T,M) \quad \text{and} \quad c_*: \mathbf{Exal}_X(T,M) \to \mathbf{Exal}_X(T,C).$$

By Lemma B.1, it remains to prove that the following sequence of Picard categories is exact:

$$\mathbf{0} \xrightarrow{i_{T,K}} \mathbf{Exal}_X(T,K) \xrightarrow{k_*} \mathbf{Exal}_X(T,M) \xrightarrow{c_*} \mathbf{Exal}_X(T,C).$$

Since $c \circ k = 0$, it follows that there is a naturally induced 2-morphism $\delta: c_* \circ k_* \Rightarrow 0_{i_{T,C}} \circ 0$. Hence, there is a naturally induced morphism of Picard categories

$$\mathbf{Exal}_X(T,K) \to \mathbf{Exal}_X(T,M) \times_{c_*, \mathbf{Exal}_X(T,C), 0_{i_{T,C}}} \mathbf{0}.$$

It now remains to exhibit a quasi-inverse to the above functor. By Lemma 2.3, we may view an object of the right-hand side as being given by a pair $(i: T \hookrightarrow T'_M, r)$, where r is a retraction of the X-extension of T by C, $c_* i: T \hookrightarrow T'_C$. Note that since c is surjective with kernel K, the X-morphism $T'_C \to T'_M$ defines an X-extension of T'_C by K. In particular, we have an **rNil**-nil pair

$$(T'_C \to T'_M, T'_C \xrightarrow{r} T)$$

over X. Since X is **rNil**-homogeneous, Lemma 1.5 (3) implies that the **rNil**-nil pair over X in question can be completed to a cocartesian **rNil**-nil square over X which is geometric. In particular, the resulting morphism $j: T \hookrightarrow T'$ is an X-extension of T by K. Since j is defined by a universal property, we have defined a functor from the right-hand side above to $\mathbf{Exal}_X(T,K)$. The claim follows. □

Further strengthening our homogeneity assumption, we obtain more structure.

Corollary 2.6. *Fix a scheme S, an* **Aff**-*homogeneous S-groupoid X, and an X-scheme T. For all* affine *and étale morphisms $p\colon V \to T$ and quasi-coherent $\mathcal{O}_V$-modules M, there is an equivalence of Picard categories:*

$$\mathbf{Exal}_X(V, M) \to \mathbf{Exal}_X(T, p_*M).$$

Proof. Let $e\colon W \to T$ be an étale morphism. If $T \hookrightarrow T'$ is an X-extension of T by K, then there exists a unique X-extension $W \hookrightarrow W'$ of W by e^*K together with an étale morphism $W' \to T'$ such that $W' \times_{T'} T \cong W$ and the second projection coincides with $e\colon W \to T$; see [17, IV.18.1.2]. This describes a functor $e^*\colon \mathbf{Exal}_X(T, K) \to \mathbf{Exal}_X(W, e^*K)$. Taking $K = p_*M$ and $e = p$, we obtain a functor $\mathbf{Exal}_X(T, p_*M) \to \mathbf{Exal}_X(V, p^*p_*M)$. By Proposition 2.4, corresponding to the $\mathcal{O}_V$-module homomorphism $p^*p_*M \to M$, there is an induced functor $\mathbf{Exal}_X(V, p^*p_*M) \to \mathbf{Exal}_X(V, M)$. Composing these two functors produces a functor $\mathbf{Exal}_X(T, p_*M) \to \mathbf{Exal}_X(V, M)$.

Also, since p is affine, **Aff**-homogeneity implies that there is a functor

$$p_*\colon \mathbf{Exal}_X(V, M) \to \mathbf{Exal}_X(T, p_*M).$$

Indeed, for any X-extension $(j\colon V \hookrightarrow V')$ of V by M, the **Aff**-homogeneity of X and Lemma 1.5 (3) provide a cocartesian **Aff**-nil square over X as in (1.1) which is geometric. In particular, the X-morphism $(i\colon T \hookrightarrow T')$ defines an X-extension of T by p_*M. The functors $\mathbf{Exal}_X(T, p_*M) \rightleftarrows \mathbf{Exal}_X(V, M)$ are clearly quasi-inverse. □

3. Limit preservation

In this section we prove that the functors defined in Section 2, $M \mapsto \mathrm{Der}_X(T, M)$ and $M \mapsto \mathrm{Exal}_X(T, M)$, frequently preserve direct limits. We also relativize the notion of limit preserving S-groupoid [7, §1].

Definition 3.1. Let S be a scheme and let $\Phi\colon Y \to Z$ be a 1-morphism of S-groupoids. The 1-morphism Φ is *limit preserving* if for every inverse system of quasi-compact and quasi-separated Z-schemes with affine transition maps $\{T_j\}_{j \in J}$ and every Y-scheme T, such that as a Z-scheme T is an inverse limit of $\{T_j\}_{j \in J}$, then the following two conditions hold.

(LP$_1$) There exist $j_0 \in J$ and a Y-scheme structure on T_{j_0} such that the induced diagram of Y-schemes $\{T_j\}_{j \geq j_0}$ has inverse limit T.

(LP$_2$) Let $j_1 \in J$ and let T_{j_1} have two Y-scheme structures such that both of the induced diagrams of Y-schemes $\{T_j\}_{j \geq j_1}$ have inverse limit T. Then for all $j \gg j_1$, the two Y-scheme structures on T_j are isomorphic.

An S-groupoid X is *limit preserving* if its structure morphism to $\mathbf{Sch}/S$ is so. Similarly, an X-scheme T is *limit preserving* if its structure 1-morphism $\mathbf{Sch}/T \to X$ is so.

Analogous to Lemma 1.5, we have the following easily verified lemma.

Lemma 3.2. *Fix a scheme S and a 1-morphism of S-groupoids $\Phi: Y \to Z$.*

(1) *If Z is a Zariski stack, then it is limit preserving if and only if for every inverse system of affine S-schemes $\{\operatorname{Spec} A_j\}_{j \in J}$ with limit $\operatorname{Spec} A$, the natural functor*

$$\varinjlim_j Z(\operatorname{Spec} A_j) \to Z(\operatorname{Spec} A)$$

is an equivalence.

(2) *If Z is an algebraic stack, then it is limit preserving if and only if it is locally of finite presentation over S.*

(3) *If Φ is limit preserving, then for every other limit preserving 1-morphism $W \to Y$ the composition $W \to Z$ is limit preserving.*

(4) *The 1-morphism Φ is limit preserving if and only if for every Z-scheme T the T-groupoid $Y \times_Z \mathbf{Sch}/T$ is limit preserving.*

(5) *If Φ is limit preserving, then for every 1-morphism of S-groupoids $W \to Z$, the 1-morphism $Y \times_Z W \to W$ is limit preserving.*

(6) *If Φ is limit preserving, then the diagonal 1-morphism $\Delta_\Phi: Y \to Y \times_Z Y$ is limit preserving.*

Proof. The claims (1), (3), (4), and (5) are all obvious, thus their proofs are omitted. Claim (2) follows from (1) and [25, Propositions 4.15, 4.18]. For claim (6), combine (4) and (LP_2) (note that for a morphism of schemes, this just says that if a morphism is locally of finite presentation, then so too is its diagonal [17, IV.1.4.3.1]). □

Example 3.3. Fix a scheme S and a limit preserving S-groupoid X. Then, an X-scheme is limit preserving if and only if it is locally of finite presentation over S.

We now have the main result of this section.

Proposition 3.4. *Fix a scheme S, a* **Nil***-homogeneous S-groupoid X, and a quasi-compact, quasi-separated, limit preserving X-scheme T.*

(1) *The functor $M \mapsto \operatorname{Der}_X(T, M)$ preserves direct limits.*

(2) *If, in addition, X is limit preserving, then the functor $M \mapsto \operatorname{Exal}_X(T, M)$ preserves direct limits.*

Proof. Throughout we fix a directed system of quasi-coherent $\mathcal{O}_T$-modules $\{M_j\}_{j \in J}$ with direct limit M. In particular, the natural map

$$T[M] \to \varprojlim_j T[M_j]$$

is an isomorphism of X-schemes. For (1), by Lemma 2.3, there are natural isomorphisms

$$\operatorname{Der}_X(T, M) \cong \operatorname{Ret}_X(T/T[M]) \cong \varinjlim_j \operatorname{Ret}_X(T/T[M_j]) \cong \varinjlim_j \operatorname{Der}_X(T, M_j).$$

For (2), we first show that the map

$$\varinjlim_j \operatorname{Exal}_X(T, M_j) \to \operatorname{Exal}_X(T, M) \tag{3.1}$$

is injective. Lemma 2.3 shows that an X-extension $(T \hookrightarrow T'')$ of T by a quasi-coherent $\mathcal{O}_T$-module N represents 0 in $\operatorname{Exal}_X(T, N)$ if and only if $\operatorname{Ret}_X(T/T'') \neq \emptyset$. So, consider a compatible collection of X-extensions $(T \hookrightarrow T'_j)$ of T by M_i with limit $(T \hookrightarrow T')$. Since

$$\operatorname{Ret}_X(T/T') = \varinjlim_j \operatorname{Ret}_X(T/T'_j),$$

it follows that the map (3.1) is injective.

We now show that the natural map (3.1) is surjective. First, we prove the result in the case where $X = S$ and S and T are affine. Since T is affine and of finite presentation over S, there exist an integer n and a closed immersion $k\colon T \hookrightarrow \mathbb{A}^n_S$. By [17, 0_{IV}.20.2.3], there is a functorial surjection $\operatorname{Hom}_{\mathcal{O}_T}(k^*\Omega_{\mathbb{A}^n_S/S}, K) \twoheadrightarrow \operatorname{Exal}_S(T, K)$ for every quasi-coherent $\mathcal{O}_T$-module K. Since the $\mathcal{O}_T$-module $k^*\Omega_{\mathbb{A}^n_S/S}$ is finite free, it follows that the functor $K \mapsto \operatorname{Hom}_{\mathcal{O}_T}(k^*\Omega_{\mathbb{A}^n_S/S}, K)$ preserves direct limits. Direct limits are exact, so the map

$$\varinjlim_j \operatorname{Exal}_S(T, M_j) \twoheadrightarrow \operatorname{Exal}_S(T, M) \tag{3.2}$$

is surjective.

If S and T are no longer assumed to be affine, then a straightforward Zariski descent argument combined with the affine case already considered shows that the map (3.2) is bijective. For the general case, let $(T \hookrightarrow T') \in \operatorname{Exal}_X(T, M)$. By the above considerations, there exist a j_0 and an S-extension of T by M_{j_0}, $(T \hookrightarrow T'_{j_0})$, such that its pushforward along $M_{j_0} \to M$ is isomorphic to $(T \hookrightarrow T')$ as an S-extension. If $j \geq j_0$, then denote the pushforward of $(T \hookrightarrow T'_{j_0})$ along the morphism $M_{j_0} \to M_j$ by $(T \hookrightarrow T'_j)$. There is a natural morphism of S-schemes $T'_j \to T'_{j_0}$ and the resulting inverse system $\{T'_j\}_{j \geq j_0}$ has limit T'. Since X is a limit preserving S-groupoid, there exist $j_1 \geq j_0$ and an X-scheme structure on T'_{j_1} such that the resulting inverse system of X-schemes $\{T'_j\}_{j \geq j_1}$ has limit T'. The result follows. □

4. Formal smoothness and formal versality

In this section we prove the main result of the paper.

Definition 4.1. Fix a scheme S, an S-groupoid X, and an X-scheme V. Consider the following lifting problem in the category of X-schemes: given a pair of morphisms of X-schemes $(V \xrightarrow{p} T, V \xrightarrow{j} V')$, where j is a locally nilpotent closed immersion, complete the following diagram so that it commutes:

$$\begin{tikzcd} V \arrow[r, "p"] \arrow[d, hook, "j"'] & T \\ V' \arrow[ur, dashed] & \end{tikzcd} \tag{4.1}$$

We say that the X-scheme T is

formally smooth if the lifting problem above can always be solved Zariski-locally on V';

formally versal at $t \in |T|$ if the lifting problem can be solved whenever V is local artinian with closed point v such that $p(v) = t$, the induced map $\kappa(t) \to \kappa(v)$ is an isomorphism, and j is an X-extension of V by $\kappa(v)$.

We certainly have the following implication:

$$\text{formally smooth} \Rightarrow \text{formally versal at every } t \in |T|.$$

In general, there is no reverse implication. We will see, however, that this subtlety vanishes once the S-groupoid is **Aff**-homogeneous. The following lemma is hopefully clarifying. Note that we cannot immediately apply [25, Proposition 4.15 (ii)], because there G. Laumon and L. Moret-Bailly assume that solutions to the lifting problem exist étale-locally on V', whereas we only assume that they exist Zariski-locally.

Lemma 4.2. *Fix an S-groupoid X and an X-scheme T. If the 1-morphism $T \to X$ is representable by algebraic spaces that are locally of finite presentation, then the X-scheme T is formally smooth if and only if the 1-morphism $T \to X$ is representable by smooth morphisms of algebraic spaces.*

Proof. Suppose that T is a formally smooth X-scheme. To prove that $T \to X$ is representable by smooth morphisms, it is sufficient to prove that if W is an X-scheme, then the induced morphism of algebraic spaces $T_W \to W$, obtained by pulling back $T \to X$ along W, is smooth. Since T_W is an algebraic space, there exists an étale and surjective morphism $\tilde{T}_W \to T_W$, where $\tilde{T}_W$ is a scheme. It remains to prove that the morphism of schemes $\tilde{T}_W \to W$ is smooth. Since the morphism in question is locally of finite presentation, it remains to show that it satisfies the infinitesimal lifting criterion for smooth morphisms. We will use [17, IV.17.14.1], thus we must show that we can complete every 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[dd, hook] & \tilde{T}_W \arrow[d] & \\ & T_W \arrow[r] \arrow[d] & T \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[uur, dashed] & W \arrow[r] & X, \end{tikzcd}$$

where $A \to A_0$ is a surjection of local rings with square 0 kernel. Since $T \to X$ is formally smooth and A is a local ring, there exists a morphism $\operatorname{Spec} A \to T$ that makes the diagram 2-commute. The universal property of the 2-fiber product further implies that there is an induced morphism $\operatorname{Spec} A \to T_W$ that makes the diagram commute. But $\tilde{T}_W \to T_W$ is étale, surjective, and representable by schemes. It now follows from étale descent and again from [17, IV.17.14.1] that there is a unique morphism $\operatorname{Spec} A \to \tilde{T}_W$ completing the diagram. The result follows. The other direction is similar, thus is omitted. □

There is a tight connection between formal smoothness (resp. formal versality) and X-extensions in the *affine* setting. The next result has arguments similar to those of [15, Satz 3.2], but the definitions are slightly different.

Lemma 4.3. *Fix a scheme S, an S-groupoid X, and an* affine *X-scheme T.*

(1) *If X is* **Aff***-homogeneous and the abelian group* $\operatorname{Exal}_X(T, I)$ *is trivial for every quasi-coherent $\mathcal{O}_T$-module I, then the X-scheme T is formally smooth.*

(2) *If X is* **rCl***-homogeneous and* $\operatorname{Exal}_X(T, \kappa(t)) = 0$ *at a closed point $t \in |T|$, then the X-scheme T is formally versal at t.*

(3) *If X is* **Cl***-homogeneous and T is noetherian and formally versal at a closed point $t \in |T|$, then* $\operatorname{Exal}_X(T, \kappa(t)) = 0$.

Proof. For (1), fix a locally nilpotent closed immersion of X-schemes $j: V \hookrightarrow V'$. It suffices to construct an X-morphism $V' \to T$ Zariski-locally on V' that makes the diagram (4.1) commute. Thus we may assume V and V' are affine and the locally nilpotent closed immersion $j: V \to V'$ is defined by a quasi-coherent $\mathcal{O}_{V'}$-ideal J such that $J^n = 0$ for some integer n. By induction on the integer n, we may further reduce to the situation where $J^2 = 0$. In particular, j is a square zero extension of V by J and $(V \xrightarrow{p} T, V \xrightarrow{j} V')$ is an **Aff**-nil pair over X. Since X is **Aff**-homogeneous, Lemma 1.5 (3) implies that there is a cocartesian **Aff**-nil square over X as in (1.1) that is geometric. In particular, the resulting X-morphism $i: T \hookrightarrow T'$ defines an X-extension of T by $p_* J$. By hypothesis, $\operatorname{Exal}_X(T, p_* J) = 0$. Lemma 2.3 now provides an X-retraction $T' \to T$. The composition

$$V' \xrightarrow{p'} T' \to T$$

gives the required lifting.

The claim (2) follows from an identical argument just given for (1).

For (3), given an X-extension $T \hookrightarrow \tilde{T}$ of T by $\kappa(t)$, write $T = \operatorname{Spec} R$, $\tilde{T} = \operatorname{Spec} \tilde{R}$, $\mathfrak{m} = t \in |T|$, and $I = \ker(\tilde{R} \to R) \cong R/\mathfrak{m}$. Let the ideal $\tilde{\mathfrak{m}} \lhd \tilde{R}$ denote the (unique) maximal ideal induced by $\mathfrak{m}$. For $n \geq 0$ define $R_n = R/\mathfrak{m}^{n+1}$, $\tilde{R}_n = \tilde{R}/\tilde{\mathfrak{m}}^{n+1}$, and $I_n = \ker(\tilde{R}_n \to R_n)$. There is an induced surjective morphism $l_n: I \to I_n$ and since I is an R-module of length 1, there is an $n_0 \gg 0$ such that l_{n_0} is an isomorphism. Let $V = \operatorname{Spec} R_{n_0}$ and $V' = \operatorname{Spec} \tilde{R}_{n_0}$ and let $j: V \hookrightarrow V'$ be the resulting X-extension of V by $\kappa(t)$.

Formal versality at $t \in |T|$ gives an X-morphism $V' \to T$ as in (4.1). If $p: V \to T$ is the induced closed immersion, then $(V \xrightarrow{p} T, V \xrightarrow{j} V')$ is a **Cl**-nil pair. By Lemma 1.5 (3), there exists a cocartesian **Cl**-nil square over X as in (1.1) which is geometric. In particular, the resulting X-morphism $i: T \hookrightarrow T'$ defines an X-extension of T by $\kappa(t)$. The compatible X-morphism $V' \to T$ and the cocartesian **Cl**-nil square (1.1) prove that the X-extension $i: T \hookrightarrow T'$ admits a retraction $r: T' \to T$, thus defines a trivial extension of T by $\kappa(t)$ over X (Lemma 2.3). The cocartesian **Cl**-nil square (1.1) also produces a morphism of X-extensions of T by $\kappa(t)$ from $T \hookrightarrow \tilde{T}$ to $T \hookrightarrow T'$, which is automatically an isomorphism. The result follows. □

Fix an affine scheme T and an additive functor $F\colon \mathbf{QCoh}(T) \to \mathbf{Ab}$. The functor F is *finitely generated* if there exist a quasi-coherent $\mathcal{O}_T$-module I and an object $\eta \in F(I)$ such that for all $M \in \mathbf{QCoh}(T)$ the induced morphism of abelian groups $\operatorname{Hom}_{\mathcal{O}_T}(I, M) \to F(M)$ given by $f \mapsto f_*\eta$ is surjective. The notion of finite generation of a functor is due to M. Auslander [9].

The functor F is *half-exact* if for every short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathbf{QCoh}(T)$, the sequence $F(M') \to F(M) \to F(M'')$ is exact.

If, in addition, T is noetherian and sends coherent $\mathcal{O}_T$-modules to coherent $\mathcal{O}_T$-modules, then A. Ogus and G. Bergman have shown [33, Theorem 2.1] that if for all closed points $t \in |T|$ we have $F(\kappa(t)) = 0$, then F is the zero functor. If F is finitely generated, then this result can be refined. Indeed, it is shown in [20, Corollary 6.7] that if $F(\kappa(t)) = 0$, then there exists an affine open subscheme $p\colon U \hookrightarrow T$ such that the composition $F \circ p_*(-)\colon \mathbf{QCoh}(U) \to \mathbf{Ab}$ is identically zero. We now use this to prove the main technical result of the paper.

Theorem 4.4. *Fix a locally noetherian scheme S, an* **Aff***-homogeneous and limit preserving S-groupoid X, and an affine X-scheme T, locally of finite type over S. If the functor $M \mapsto \operatorname{Exal}_X(T, M)$ is finitely generated and T is formally versal at a closed point $t \in |T|$, then it is formally smooth in an open neighborhood of t.*

Proof. By Lemma 4.3 (3), we have $\operatorname{Exal}_X(T, \kappa(t)) = 0$. By Corollary 2.5, the functor $M \mapsto \operatorname{Exal}_X(T, M)$ is half-exact, and by Proposition 3.4 it commutes with direct limits. As $\operatorname{Exal}_X(T, -)$ is finitely generated, Corollary 6.7 of [20] now applies. Thus, there exists an affine open neighborhood $p\colon U \hookrightarrow T$ of t such that the functor $\operatorname{Exal}_X(T, p_*(-))\colon \mathbf{QCoh}(U) \to \mathbf{Ab}$ is the zero functor. By Corollary 2.6, $\operatorname{Exal}_X(U, -)$ is also the zero functor. By Lemma 4.3 (1), we conclude that U is a formally smooth X-scheme. □

5. Coherent functors

Fix a ring A. An additive functor $F : \mathbf{Mod}(A) \to \mathbf{Ab}$ is *coherent*, if there exist an A-module homomorphism $f\colon I \to J$ and an element $\eta \in F(I)$, inducing an exact sequence for every A-module M,

$$\operatorname{Hom}_A(J, M) \to \operatorname{Hom}_A(I, M) \to F(M) \to 0.$$

We refer to the data $(f\colon I \to J, \eta)$ as a *presentation* for F. For a comprehensive account of coherent functors, we refer the interested reader to [9]. Some stronger results that are available in the noetherian situation are developed in [23]. Here we record some simple consequences of [9, p. 200].

Lemma 5.1. *Fix a ring A. For each $i = 1, \dots, 5$, let $H_i\colon \mathbf{Mod}(A) \to \mathbf{Ab}$ be an additive functor fitting into an exact sequence*

$$H_1 \to H_2 \to H_3 \to H_4 \to H_5.$$

(1) *If H_2 and H_4 are finitely generated and H_5 is coherent, then H_3 is finitely generated.*

(2) *If H_1 is finitely generated and H_2, H_4, and H_5 are coherent, then H_3 is coherent.*

We now have the following important example of a coherent functor, which is a special case of [9, p. 200].

Example 5.2. Let A be a ring. If $Q^\bullet$ is a bounded above complex of A-modules, then the functor $M \mapsto \operatorname{Ext}^i_A(Q^\bullet, M)$ is coherent for every integer i. Indeed, there is a quasi-isomorphism $P^\bullet \simeq Q^\bullet$, where $P^\bullet$ is a complex of A-modules that is term-by-term projective. By definition, for every A-module M and integer i there is a natural isomorphism:

$$\operatorname{Ext}^i_A(Q^\bullet, M) = \frac{\ker(\operatorname{Hom}_A(P^i, M) \to \operatorname{Hom}_A(P^{i-1}, M))}{\operatorname{im}(\operatorname{Hom}_A(P^{i+1}, M) \to \operatorname{Hom}_A(P^i, M))}.$$

Since the functor $M \mapsto \operatorname{Hom}_A(P^j, M)$ is coherent for every integer j, Lemma 5.1 (2) implies that the functor $M \mapsto \operatorname{Ext}^i_A(Q^\bullet, M)$ is coherent for every integer i. Using Spaltenstein resolutions [43], this example extends to where $Q^\bullet$ is unbounded.

Example 5.3. Let R be a noetherian ring. If $Q^\bullet$ is a bounded above complex of R-modules with coherent cohomology, then the functor $M \mapsto \operatorname{Tor}^R_i(Q^\bullet, M)$ is coherent for every integer i. Indeed, there is a quasi-isomorphism $F^\bullet \to Q^\bullet$, where $F^\bullet$ is a bounded above complex of finitely generated free R-modules. Arguing as in Example 5.2, it remains to show that if F is a finitely generated and free R-module, then the functor $M \mapsto F \otimes_R M$ is coherent. But this is clear: there is a natural isomorphism $F \otimes_R M = \operatorname{Hom}_R(F^\vee, M)$ for every R-module M. Thus the functor in question is isomorphic to $\operatorname{Hom}_R(F^\vee, -)$, which is coherent.

The following lemma is crucial for the proof of Theorem A.

Lemma 5.4. *Fix a scheme S and an algebraic S-stack X. If T is an affine X-scheme, then the functors $M \mapsto \operatorname{Der}_X(T, M)$ and $M \mapsto \operatorname{Exal}_X(T, M)$ are coherent.*

Proof. By [36, Theorem 1.1], there is a bounded above complex of $\mathcal{O}_T$-modules $L_{T/X}$, with quasi-coherent cohomology sheaves, as well as functorial isomorphisms

$$\operatorname{Der}_X(T, M) \cong \operatorname{Ext}^0_{\mathcal{O}_T}(L_{T/X}, M) \quad \text{and} \quad \operatorname{Exal}_X(T, M) \cong \operatorname{Ext}^1_{\mathcal{O}_T}(L_{T/X}, M)$$

for all quasi-coherent $\mathcal{O}_T$-modules M. By Example 5.2, the result follows. □

The next example is [20, Theorem C] and is crucial for the applications in Sections 8–9.

Example 5.5. Fix an affine scheme S and a morphism of algebraic stacks $f : X \to S$ that is locally of finite presentation. If $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$ and $\mathcal{N} \in \mathbf{QCoh}(X)$, where $\mathcal{N}$ is of finite presentation, flat over S, with support proper over S, then the functor

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes_{\mathcal{O}_X} f^*(-)) : \mathbf{QCoh}(S) \to \mathbf{Ab}$$

is coherent. Stated in this generality, the coherence of the above functor is non-trivial. If S is noetherian, f is projective, and $\mathcal{M} \in \mathbf{Coh}(X)$, then a direct proof of the coherence of the above functor can be found in [23, Example 2.7]. If S is noetherian and admits a dualizing complex (e.g., when S is of finite type over a field or $\mathbb{Z}$; see [22, V.2]), f is a proper morphism of algebraic stacks, and $\mathcal{M} \in \mathsf{D}^-_{\mathrm{Coh}}(X)$, then the coherence is simpler (see [20, Proposition 2.1], which extends Flenner's arguments in the analytic case [16, Satz 2.1] to algebraic stacks).

If S is noetherian, $\mathcal{M} \in \mathsf{D}^-_{\mathrm{Coh}}(X)$, and f is a proper morphism of schemes or algebraic spaces, then the coherence is also a consequence of some (now) standard facts. Indeed, by [28, Theorem 4.1] (if X is a scheme) or [44, Tag 08HP] (if X is an algebraic space), there exist a perfect complex $\mathcal{P}$ on X and a morphism $p\colon \mathcal{P} \to \mathcal{M}$ that induces a quasi-isomorphism $\tau^{\geq 0}\mathcal{P} \to \tau^{\geq 0}\mathcal{M}$. There is now a natural sequence of isomorphisms for every $\mathcal{I} \in \mathbf{QCoh}(S)$:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes_{\mathcal{O}_X} f^*\mathcal{I}) &\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{N} \otimes_{\mathcal{O}_X} f^*\mathcal{I}) \\
&\cong \mathcal{H}^0\big(\mathsf{R}\Gamma(X, \mathcal{P}^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} [\mathcal{N} \otimes_{\mathcal{O}_X} f^*\mathcal{I}])\big) \quad (\mathcal{P} \text{is perfect}) \\
&\cong \mathcal{H}^0\big(\mathsf{R}\Gamma(X, \mathcal{P}^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*\mathcal{I})\big) \quad (\mathcal{N} \text{ is flat over } S) \\
&\cong \mathcal{H}^0\big(\mathsf{R}\Gamma(X, \mathcal{P}^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N}) \otimes^{\mathsf{L}}_{\mathcal{O}_S} \mathcal{I}\big).
\end{aligned}$$

The last isomorphism is the projection formula, see [32, Proposition 5.3] (if X is a scheme) or [44, Tag 08IN] (if X is an algebraic space). Since f is proper, $\mathsf{R}\Gamma(X, \mathcal{P}^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N})$ is a bounded above complex of $R = \Gamma(S, \mathcal{O}_S)$-modules with coherent cohomology, see [10, III.2.2.1] (if X is a scheme) or [44, Tag 08GK] (if X is an algebraic space). The result now follows from Example 5.3.

6. Automorphisms, deformations, obstructions, and composition

A hypothesis in Theorem 4.4 is that the functor $M \mapsto \mathrm{Exal}_X(T, M)$ is finitely generated. We have found the direct verification of this hypothesis to be difficult. In this section, we provide some exact sequences to remedy this situation. We also take the opportunity to formalize and relativize obstruction theories.

Fix a scheme S and a 1-morphism of S-groupoids $\Phi\colon Y \to Z$. We write $\mathbf{Def}_\Phi$ for the category with objects the pairs $(i\colon T \hookrightarrow T', r\colon T' \to T)$, where i is a Y-extension and r is a Z-retraction of i. A morphism $(i_1\colon T_1 \hookrightarrow T_1', r_1\colon T_1' \to T_1) \to (i_2\colon T_2 \hookrightarrow T_2', r_2\colon T_2' \to T_2)$ in $\mathbf{Def}_\Phi$ is a morphism of Y-extensions $i_1 \to i_2$ such that the resulting diagram of Z-schemes commutes,

$$\begin{tikzcd} T_1' \arrow[r, "r_1"] \arrow[d] & T_1 \arrow[d] \\ T_2' \arrow[r, "r_2"] & T_2. \end{tikzcd}$$

By Lemma 2.3, $\mathbf{Def}_\Phi$ can be viewed as the category of completions of the following diagram:

$$\begin{tikzcd} T \arrow[r] \arrow[d, hook] & Y \arrow[d, "\Phi"] \\ T[I] \arrow[r] \arrow[ur] & Z, \end{tikzcd}$$

where I is a quasi-coherent $\mathcal{O}_T$-module and $T[I]$ is the trivial Z-extension of T by I. There is a natural functor $\mathbf{Def}_\Phi \to \mathbf{Exal}_Y$, which sends $(i\colon T \hookrightarrow T', r\colon T' \to T)$ to $(i\colon T \hookrightarrow T')$. If T

is a Y-scheme, then we denote the fiber of this functor over $\mathbf{Exal}_Y(T)$ by $\mathbf{Def}_\Phi(T)$. It follows that there is an induced functor $\mathbf{Def}_\Phi(T) \to \mathbf{QCoh}(T)^\circ$. We denote the fiber of this functor over a quasi-coherent $\mathcal{O}_T$-module I as $\mathbf{Def}_\Phi(T, I)$. This category is naturally pointed by the trivial Y-extension of T by I. The following example is related to Example 2.1.

Example 6.1. Let S be a scheme and let $\Phi\colon Y \to Z$ be a 1-morphism of algebraic stacks. If T is a Y-scheme, which we regard as being given by a 1-morphism $t\colon T \to Y$, and I is a quasi-coherent $\mathcal{O}_T$-module, then the category $\mathbf{Def}_\Phi(T, I)$ is naturally equivalent to the Picard category represented by the complex $\tau^{\leq 0}\mathsf{R}\mathrm{Hom}_{\mathcal{O}_T}(\tau^{\geq 0}\mathsf{L}t^*\tau^{\geq 0}L_{Y/Z}, I)$, where $\tau^{\geq 0}L_{Y/Z}$ is the truncated cotangent complex defined in [25, Chapter 17]. In particular, many of the results of this section, when Φ is a 1-morphism of algebraic stacks, can be viewed as mild generalizations or simple consequences of the results appearing in the latter parts of [25, Chapter 17].

The following lemma, which is related to [25, Lemme 17.15.1], will be important and explains why the groupoids $\mathbf{Def}_\Phi(T, I)$ are more amenable to calculation than $\mathbf{Exal}_Y(T, I)$ and $\mathbf{Exal}_Z(T, I)$. Indeed, the groupoid $\mathbf{Def}_\Phi$ has base change properties, while $\mathbf{Exal}_Y(T, I)$ typically does not. This will be revisited in Lemma 6.9 and Corollary 6.14.

Lemma 6.2. *Fix a scheme S and a 2-cartesian diagram of S-groupoids:*

$$\begin{array}{ccc} Y_W & \xrightarrow{p_Y} & Y \\ {\scriptstyle \Phi_W}\downarrow & & \downarrow{\scriptstyle \Phi} \\ W & \xrightarrow{p} & Z. \end{array}$$

Let T be a Y_W-scheme and let $I \in \mathbf{QCoh}(T)$. Then the natural functor

$$\mathbf{Def}_{\Phi_W}(T, I) \to \mathbf{Def}_\Phi(T, I)$$

induces an equivalence of categories.

Proof. We prove that the functor in question induces an equivalence of categories by constructing a quasi-inverse. If $(i\colon T \hookrightarrow T', r\colon T' \to T)$ belongs to $\mathbf{Def}_\Phi(T, I)$, then the retraction r endows T' with a structure of a W-scheme, which as a Z-scheme is isomorphic to its other Z-scheme structure obtained from its Y-scheme structure. The universal property of the 2-fiber product implies that T' becomes a Y_W-scheme, the Y-morphism i is a Y_W-morphism, and the Z-morphism r is a W-morphism. It follows that we have functorially defined an object of $\mathbf{Def}_{\Phi_W}(T, I)$, thus there is an induced functor $\mathbf{Def}_\Phi(T, I) \to \mathbf{Def}_{\Phi_W}(T, I)$. That this functor is quasi-inverse to $\mathbf{Def}_{\Phi_W}(T, I) \to \mathbf{Def}_\Phi(T, I)$ is clear. □

We record for future reference the following trivial observations.

Lemma 6.3. *Fix a scheme S, 1-morphisms of S-groupoids $X \xrightarrow{\Psi} Y \xrightarrow{\Phi} Z$, an X-scheme T, and a quasi-coherent $\mathcal{O}_T$-module I. If the 1-morphism $\Psi\colon X \to Y$ is formally étale, then the natural functor*

$$\mathbf{Def}_{\Phi\circ\Psi}(T, I) \to \mathbf{Def}_\Phi(T, I)$$

is an equivalence of categories.

Lemma 6.4. *Fix a scheme S, a class of morphisms $P \subseteq$* **Aff***, a 1-morphism of P-homogeneous S-groupoids $\Phi\colon Y \to Z$, a morphism of Y-schemes $p\colon V \to T$ where $p \in P$, and $K \in \mathbf{QCoh}(V)$. Then the natural functor*

$$\mathbf{Def}_{\Phi}(T, p_* K) \to \mathbf{Def}_{\Phi}(V, K)$$

is an equivalence of categories.

The proof of the next result is similar to Proposition 2.4, thus is omitted.

Proposition 6.5. *Fix a scheme S, a 1-morphism of* **Nil***-homogeneous S-groupoids $\Phi\colon Y \to Z$, a Y-scheme T, and a quasi-coherent $\mathcal{O}_T$-module I. Then the category $\mathbf{Def}_{\Phi}(T, I)$ admits a natural structure as a Picard category.*

Denote the set of isomorphism classes of the Picard category $\mathbf{Def}_{\Phi}(T, I)$ by $\mathrm{Def}_{\Phi}(T, I)$. Thus, by Proposition 6.5, we obtain functors

$$\mathrm{Def}_{\Phi}(T, -)\colon \mathbf{QCoh}(T) \to \mathbf{Ab}, \quad I \mapsto \mathrm{Def}_{\Phi}(T, I)$$

and

$$\mathrm{Aut}_{\Phi}(T, -)\colon \mathbf{QCoh}(T) \to \mathbf{Ab}, \quad I \mapsto \mathrm{Aut}_{\mathbf{Def}_{\Phi}(T,I)}(i_{T,I}).$$

We include the following corollary for its intended reference in [21]. Its proof is almost identical to that of Corollary 2.5 and [48, Proposition 2.2 (iv)], thus is omitted.

Corollary 6.6. *Fix a scheme S, a 1-morphism of* **rNil***-homogeneous S-groupoids $\Phi\colon Y \to Z$, and a Y-scheme T. Then for every short exact sequence in $\mathbf{QCoh}(T)$,*

$$0 \to K \xrightarrow{k} M \xrightarrow{c} C \to 0.$$

there is a natural exact sequence of abelian groups

$$\begin{aligned} 0 \longrightarrow \mathrm{Aut}_{\Phi}(T, K) \longrightarrow \mathrm{Aut}_{\Phi}(T, M) \longrightarrow \mathrm{Aut}_{\Phi}(T, C) \longrightarrow \\ \longrightarrow \mathrm{Def}_{\Phi}(T, K) \longrightarrow \mathrm{Def}_{\Phi}(T, M) \longrightarrow \mathrm{Def}_{\Phi}(T, C). \end{aligned}$$

We now have an exact sequence that greatly aids computations.

Proposition 6.7. *Fix a scheme S, a 1-morphism of* **Nil***-homogeneous S-groupoids $\Phi\colon Y \to Z$, a Y-scheme T, and a quasi-coherent $\mathcal{O}_T$-module I. Then there is a natural exact sequence of abelian groups*

$$\begin{aligned} 0 \longrightarrow \mathrm{Aut}_{\Phi}(T, I) \longrightarrow \mathrm{Der}_Y(T, I) \longrightarrow \mathrm{Der}_Z(T, I) \longrightarrow \\ \longrightarrow \mathrm{Def}_{\Phi}(T, I) \longrightarrow \mathrm{Exal}_Y(T, I) \longrightarrow \mathrm{Exal}_Z(T, I). \end{aligned}$$

Proof. By Lemma B.1, it is sufficient to show that the following sequence of Picard categories is left-exact:

$$\mathbf{0} \xrightarrow{(i_{T,I}, r_{T,I})} \mathbf{Def}_{\Phi}(T, I) \to \mathbf{Exal}_{Y}(T, I) \to \mathbf{Exal}_{Z}(T, I).$$

By Lemma 2.3 and the explicit description of the 2-fiber product of Picard categories given in Appendix B, this is clear, and the result follows. □

We now introduce multi-step relative obstruction theories. For single-step obstruction theories, this definition is similar to [7, (2.6)] and [34, Definition A.10].

Definition 6.8. Fix a scheme S, a 1-morphism of **Nil**-homogeneous S-groupoids $\Phi\colon Y \to Z$, and an integer $n \geq 1$. For a Y-scheme T, an *n-step relative obstruction theory for Φ at T* is a sequence of additive functors (the obstruction spaces)

$$\mathrm{O}^i(T, -)\colon \mathbf{QCoh}(T) \to \mathbf{Ab}, \quad I \mapsto \mathrm{O}^i(T, I) \quad \text{for } i = 1, \dots, n$$

as well as natural transformations of functors (the obstruction maps)

$$\begin{aligned} &\mathrm{o}^1(T, -)\colon \mathrm{Exal}_Z(T, -) \Rightarrow \mathrm{O}^1(T, -), \\ &\mathrm{o}^i(T, -)\colon \ker \mathrm{o}^{i-1}(T, -) \Rightarrow \mathrm{O}^i(T, -) \quad \text{for } i = 2, \dots, n, \end{aligned}$$

such that the natural transformation of functors

$$\mathrm{Exal}_Y(T, -) \Rightarrow \mathrm{Exal}_Z(T, -)$$

has image $\ker \mathrm{o}^n(T, -)$. For an *affine* Y-scheme T, an n-step relative obstruction theory at T is *coherent* if the functors $\{\mathrm{O}^i(T, -)\}_{i=1}^n$ are all coherent.

We feel that it is important to point out that simply taking the cokernel of the last morphism in the exact sequence of Proposition 6.7 produces a 1-step relative obstruction theory, which we denote as $(\mathrm{obs}_\Phi, \mathrm{Obs}_\Phi)$ and call the *minimal* relative obstruction theory. This obstruction theory generalizes to the relative setting the minimal obstruction theory described in [15]. In practice, the minimal obstruction theory is a difficult object to explicitly describe. The following base change result is useful, however.

Lemma 6.9. *Fix a scheme S and a 2-cartesian diagram of* **Nil**-*homogeneous S-groupoids:*

$$\begin{tikzcd} Y_W \arrow[r] \arrow[d, "\Phi_W"'] & Y \arrow[d, "\Phi"] \\ W \arrow[r, "p"] & Z. \end{tikzcd}$$

*If T is a Y_W-scheme and $I \in$ **QCoh**(T), then*

$$\mathrm{Obs}_{\Phi_W}(T, I) \subseteq \mathrm{Obs}_{\Phi}(T, I).$$

Proof. By Proposition 6.7, there is a commutative diagram with exact rows:

$$\begin{tikzcd} \mathrm{Exal}_{Y_W}(T,I) \arrow[r] \arrow[d] & \mathrm{Exal}_W(T,I) \arrow[r] \arrow[d] & \mathrm{Obs}_{\Phi_W}(T,I) \arrow[r] & 0 \\ \mathrm{Exal}_Y(T,I) \arrow[r] & \mathrm{Exal}_Z(T,I) \arrow[r] & \mathrm{Obs}_\Phi(T,I) \arrow[r] & 0. \end{tikzcd}$$

It follows that there is a naturally induced morphism $\mathrm{Obs}_{\Phi_W}(T,I) \to \mathrm{Obs}_\Phi(T,I)$, which we will now prove to be injective. Fix a W-extension $(T \hookrightarrow T')$ of T by I. If this W-extension lifts, as a Z-extension, to a Y-extension, then the universal property of the 2-fiber product implies that it lifts to a Y_W-extension. A standard diagram chase now shows that this proves the injectivity of the map in question. □

Example 6.10. Note that the injection of Lemma 6.9 is rarely a bijection. Indeed, if $\Phi\colon Y \to Z$ admits a section s, then for any Z-scheme T and quasi-coherent $\mathcal{O}_T$-module I it follows that $\mathrm{Exal}_Y(T,I) \to \mathrm{Exal}_Z(T,I)$ also admits a section and is thus surjective. In particular, $\mathrm{Obs}_\Phi(T,I) = 0$. Note that this implies that $\mathrm{Obs}_{\Phi_Y}(T,I) = 0$ for every Y-scheme T and quasi-coherent $\mathcal{O}_T$-module I. To obtain an explicit counterexample, it suffices to find a Φ, a T, and an I such that $\mathrm{Obs}_\Phi(T,I) \neq 0$. For this, let $\Phi\colon Y \to Z$ be the 1-morphism of S-groupoids given by a non-smooth morphism of affine schemes. Let $T = Y$, which we view as a Y-scheme in the obvious way. Then $\mathrm{Exal}_Y(T,I) = 0$ for every quasi-coherent $\mathcal{O}_T$-module I. Since $T \to Z$ is not smooth, Lemmas 4.2 and 4.3 imply that $\mathrm{Exal}_Z(T,I_0) \neq 0$ for some quasi-coherent $\mathcal{O}_T$-module I_0. In particular, $\mathrm{Obs}_\Phi(T,I_0) = \mathrm{Exal}_Z(T,I_0) \neq 0$.

Combining Lemmas 6.3 and 2.2, we obtain the following.

Lemma 6.11. *Fix a scheme S, 1-morphisms of* **Nil**-*homogeneous S-groupoids*

$$X \xrightarrow{\Psi} Y \xrightarrow{\Phi} Z,$$

an X-scheme T, and a quasi-coherent $\mathcal{O}_T$-module I. If Ψ is formally étale, then every n-step relative obstruction theory for Φ at T lifts to an n-step relative obstruction theory for $\Phi \circ \Psi$ with the same obstruction spaces.

What follows is an immediate consequence of Proposition 6.7 and Lemma 5.1.

Corollary 6.12. *Fix a scheme S, a 1-morphism of* **Nil**-*homogeneous S-groupoids $\Phi\colon Y \to Z$, an affine Y-scheme T, and an integer $n \geq 1$. Suppose there exists a coherent n-step relative obstruction theory at T.*

(1) *If the functor $M \mapsto \mathrm{Exal}_Z(T,M)$ is coherent, then the minimal obstruction theory* $(\mathrm{obs}_\Phi, \mathrm{Obs}_\Phi)$ *is coherent at T.*

(2) *If the functors $M \mapsto \mathrm{Def}_\Phi(T,M)$, $\mathrm{Exal}_Z(T,M)$ are finitely generated, then the functor $M \mapsto \mathrm{Exal}_Y(T,M)$ is finitely generated.*

Proof. For (1), we note that for every quasi-coherent $\mathcal{O}_T$-module M and $i = 2, \dots, n$ there are natural exact sequences

$$
\begin{aligned}
&0 \to \ker \mathrm{o}^1(T, M) \to \mathrm{Exal}_Z(T, M) \xrightarrow{\mathrm{o}^1(T,M)} \mathrm{O}^1(T, M),\\
&0 \to \ker \mathrm{o}^i(T, M) \to \ker \mathrm{o}^{i-1}(T, M) \xrightarrow{\mathrm{o}^i(T,M)} \mathrm{O}^i(T, M),\\
&0 \to \ker \mathrm{o}^n(T, M) \to \mathrm{Exal}_Z(T, M) \to \mathrm{Obs}_\Phi(T, M) \to 0.
\end{aligned}
$$

Combining the first exact sequence with Lemma 5.1 (2), we see that the functor $\ker \mathrm{o}^1(T, -)$ is coherent. Working by induction on i, the second exact sequence combined with Lemma 5.1 (2) proves that the functor $\ker \mathrm{o}^n(T, -)$ is coherent. The third exact sequence and Lemma 5.1 (2) now prove that $\mathrm{Obs}_\Phi(T, -)$ is coherent.

The claim (2) is an immediate consequence of the exact sequence of Proposition 6.7 and Lemma 5.1 (1). □

The result that follows shows the stability of the conditions of Theorem A under composition, in the sense of J. Starr [45]. The following result also extends – by four terms to the right – the exact sequence [34, §A.15].

Proposition 6.13. *Fix a scheme S and 1-morphisms of* **Nil***-homogeneous S-groupoids*

$$X \xrightarrow{\Psi} Y \xrightarrow{\Phi} Z,$$

an X-scheme T, and a quasi-coherent $\mathcal{O}_T$-module I. Then there is a natural 9-term exact sequence of abelian groups

$$
\begin{aligned}
&0 \longrightarrow \mathrm{Aut}_\Psi(T, I) \longrightarrow \mathrm{Aut}_{\Phi\circ\Psi}(T, I) \longrightarrow \mathrm{Aut}_\Phi(T, I) \longrightarrow\\
&\quad\longrightarrow \mathrm{Def}_\Psi(T, I) \longrightarrow \mathrm{Def}_{\Phi\circ\Psi}(T, I) \longrightarrow \mathrm{Def}_\Phi(T, I) \longrightarrow\\
&\quad\longrightarrow \mathrm{Obs}_\Psi(T, I) \longrightarrow \mathrm{Obs}_{\Phi\circ\Psi}(T, I) \longrightarrow \mathrm{Obs}_\Phi(T, I) \longrightarrow 0.
\end{aligned}
$$

In particular, there are natural isomorphisms

$$\mathrm{Aut}_\Psi(T, I) \cong \mathrm{Def}_{\Delta_\Psi}(T, I) \quad \textit{and} \quad \mathrm{Def}_\Psi(T, I) \cong \mathrm{Obs}_{\Delta_\Psi}(T, I).$$

Proof. The latter claims follow by combining the result with the triple

$$X \xrightarrow{\Delta_\Psi} X \times_Y X \to X$$

and Lemma 6.2.

By Lemma B.1, we will obtain the first seven terms of the exact sequence if the following sequence of Picard categories is left-exact:

$$\mathbf{0} \xrightarrow{(i_{T,I}, r_{T,I})} \mathbf{Def}_\Psi(T, I) \to \mathbf{Def}_{\Phi\circ\Psi}(T, I) \to \mathbf{Def}_\Phi(T, I).$$

By Lemma 2.3 and the explicit description of the 2-fiber product of Picard categories given in Appendix B, this is clear. For the remaining four terms of the exact sequence: we first apply the Snake Lemma to the commutative diagram with exact rows

$$\begin{tikzcd} & \mathrm{Exal}_X(T,I) \arrow[r] \arrow[d] & \mathrm{Exal}_Y(T,I) \arrow[r] \arrow[d] & \mathrm{Obs}_\Psi(T,I) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Exal}_Z(T,I) \arrow[r] & \mathrm{Exal}_Z(T,I) \arrow[r] & 0, & \end{tikzcd}$$

which produces an exact sequence

$$K_{\Phi\circ\Psi}(T,I) \to K_\Phi(T,I) \to \mathrm{Obs}_\Psi(T,I) \to \mathrm{Obs}_{\Phi\circ\Psi}(T,I) \to \mathrm{Obs}_\Phi(T,I) \to 0, \tag{6.1}$$

where

$$\begin{aligned} K_{\Phi\circ\Psi}(T,I) &= \ker\big(\mathrm{Exal}_X(T,I) \to \mathrm{Exal}_Z(T,I)\big), \\ K_\Psi(T,I) &= \ker\big(\mathrm{Exal}_Y(T,I) \to \mathrm{Exal}_Z(T,I)\big). \end{aligned}$$

By Proposition 6.7, we obtain a commutative diagram with exact rows

$$\begin{tikzcd} & \mathrm{Der}_Z(T,I) \arrow[r] \arrow[d] & \mathrm{Def}_{\Phi\circ\Psi}(T,I) \arrow[r] \arrow[d] & K_{\Phi\circ\Psi}(T,I) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Der}_Z(T,I)/\mathrm{Der}_Y(T,I) \arrow[r] & \mathrm{Def}_\Phi(T,I) \arrow[r] & K_\Phi(T,I) \arrow[r] & 0. \end{tikzcd}$$

By the Snake Lemma, we thus obtain an isomorphism

$$\mathrm{coker}\big(\mathrm{Def}_{\Phi\circ\Psi}(T,I) \to \mathrm{Def}_\Phi(T,I)\big) \cong \mathrm{coker}\big(K_{\Phi\circ\Psi}(T,I) \to K_\Phi(T,I)\big).$$

Combining this isomorphism with the exact sequence (6.1), we deduce that the sequence

$$\mathrm{Def}_{\Phi\circ\Psi}(T,I) \to \mathrm{Def}_\Phi(T,I) \to \mathrm{Obs}_\Psi(T,I) \to \mathrm{Obs}_{\Phi\circ\Psi}(T,I) \to \mathrm{Obs}_\Phi(T,I) \to 0$$

is exact. Splicing the 7-term exact sequence which we earlier obtained from the left-exact sequence of Picard categories to the 6-term exact sequence above gives the result. □

We now arrive at the final result of this section, which is instrumental to the bootstrapping argument employed to prove Theorem A.

Corollary 6.14. *Let $\Psi\colon X \to Y$ be a* **Nil**-*homogeneous* 1-*morphism of S-groupoids. Let W be an $(X \times_{\Psi,Y,\Psi} X)$-scheme and let $(\Delta_\Psi)_W\colon D_{\Psi,W} \to W$ be the pullback of Δ_Ψ along W. If T is a $D_{\Psi,W}$-scheme and M is a quasi-coherent $\mathcal{O}_T$-module, then*

$$\begin{aligned} \mathrm{Aut}_{(\Delta_\Psi)_W}(T,M) &= 0, \\ \mathrm{Def}_{(\Delta_\Psi)_W}(T,M) &\cong \mathrm{Aut}_\Psi(T,M), \\ \mathrm{Obs}_{(\Delta_\Psi)_W}(T,M) &\subseteq \mathrm{Def}_\Psi(T,M). \end{aligned}$$

Proof. The third diagonal of Ψ is an isomorphism, so $\mathrm{Obs}_{\Delta_{\Delta_{\Delta_\Psi}}}(T,M) = 0$. By Lemmas 6.2 and 6.9 and Proposition 6.13, there are natural isomorphisms:

$$\begin{aligned} &\mathrm{Aut}_{(\Delta_\Psi)_W}(T,M) \cong \mathrm{Aut}_{\Delta_\Psi}(T,M) \cong \mathrm{Def}_{\Delta_{\Delta_\Psi}}(T,M) \cong \mathrm{Obs}_{\Delta_{\Delta_{\Delta_\Psi}}}(T,M) \cong 0, \\ &\mathrm{Def}_{(\Delta_\Psi)_W}(T,M) \cong \mathrm{Def}_{\Delta_\Psi}(T,M) \cong \mathrm{Aut}_\Psi(T,M), \\ &\mathrm{Obs}_{(\Delta_\Psi)_W}(T,M) \subseteq \mathrm{Obs}_{\Delta_\Psi}(T,M) \cong \mathrm{Def}_\Psi(T,M). \end{aligned}$$

□

7. Proof of Theorem A

In this section we prove Theorem A. Before we do this, however, we will prove the following theorem.

Theorem 7.1. *Fix an excellent scheme S. An S-groupoid X is an algebraic S-stack that is locally of finite presentation over S if and only if the following conditions are satisfied.*

(1) *X is a stack over the site $(\mathbf{Sch}/S)_{\text{Ét}}$.*

(2) *X is limit preserving.*

(3) *X is **Aff**-homogeneous.*

(4) *The diagonal $\Delta_{X/S}\colon X \to X \times_S X$ is representable by algebraic spaces.*

(5) *For any $\mathfrak{m}$-adically complete local noetherian ring $(B, \mathfrak{m})$ with an S-scheme structure* $\operatorname{Spec} B \to S$ *such that the induced morphism* $\operatorname{Spec}(B/\mathfrak{m}) \to S$ *is locally of finite type, the natural functor*
$$X(\operatorname{Spec} B) \to \varprojlim_n X(\operatorname{Spec}(B/\mathfrak{m}^n))$$
induces an equivalence of categories.

(6) *For any affine X-scheme T that is locally of finite type over S, the functor*
$$M \mapsto \operatorname{Exal}_X(T, M)$$
is finitely generated.

Proof. Fix a morphism $x\colon \operatorname{Spec} \Bbbk \to S$, where $\Bbbk$ is a field. Denote by $\mathcal{A}_S(x)$ the category whose objects are pairs (A, a), where A is a local artinian ring with residue field $\Bbbk$, and $a\colon \operatorname{Spec} A \to S$ is a morphism of schemes, such that the composition $\operatorname{Spec} A_{\mathrm{red}} \to \operatorname{Spec} A \to S$ agrees with x. Morphisms $(A, a) \to (B, b)$ in $\mathcal{A}_S(x)$ are ring homomorphisms $A \to B$ that preserve the data. For $\xi \in X(x)$, there is an induced category fibered in groupoids $X_\xi\colon \mathcal{C}_\xi \to \mathcal{A}_S(x)^\circ$. The **Aff**-homogeneity of the S-groupoid X implies the homogeneity (in the sense of [12, Exposé VI, Definition 2.5]) of the cofibered category $X_\xi^\circ\colon \mathcal{C}_\xi^\circ \to \mathcal{A}_S(x)$.

If the morphism x is locally of finite type, then, by (6) and [20, Lemma 6.6], the $\Bbbk$-vector space $\operatorname{Exal}_X(\xi, \Bbbk)$ is finite dimensional. By Lemma 5.4 and again by [20, Lemma 6.6], the $\Bbbk$-vector space $\operatorname{Der}_S(x, \Bbbk)$ is also finite dimensional. Thus, by Proposition 6.7, the $\Bbbk$-vector space $\operatorname{Def}_{X/S}(\xi, \Bbbk)$ is finite dimensional. By definition, $\operatorname{Def}_{X/S}(\xi, \Bbbk)$ is the set of isomorphism classes of the category $X_\xi(\xi[\epsilon])$.

Thus, by (5), Theorem 1.5 of [11] applies, and so for every such ξ, there is a pointed and affine X-scheme (Q_ξ, q), locally of finite type over S, such that the X-scheme $\operatorname{Spec} \kappa(q)$ is isomorphic to ξ, and Q_ξ is formally versal at q, and q is a closed point of Q_ξ. We now apply Theorem 4.4 to conclude that we may (by passing to an open subscheme) assume that Q_ξ is a formally smooth X-scheme containing q. Condition (4) and Lemma 4.2 now imply that the X-scheme Q_ξ is representable by smooth morphisms.

Define K to be the set of all morphisms $x\colon \operatorname{Spec} \Bbbk \to S$ that are locally of finite type, where $\Bbbk$ is a field. Set $Q = \amalg_{\kappa \in K, \xi \in X(\kappa)} Q_\xi$. Then, we have seen that the X-scheme Q is representable by smooth morphisms, and it remains to show that it is representable by surjective morphisms. Since the stack X is limit preserving, it suffices to show that if V is an affine X-scheme that is locally of finite type over S, then the morphism of algebraic S-spaces $Q \times_X V \to V$ is surjective. But $Q \times_X V \to V$ smooth and its image contains all the points $v \in |V|$ such that the morphism $\operatorname{Spec} \kappa(v) \to S$ is locally of finite type. Since V is locally of finite type over S, the result follows. □

To deduce Theorem A from Theorem 7.1 we will use a bootstrapping process. This process begins with the following corollary.

Corollary 7.2. *Fix an excellent scheme S and an S-groupoid X. If X satisfies the conditions of Theorem* A *and $\Delta_{X/S}\colon X \to X \times_S X$ is representable, then X is an algebraic stack that is locally of finite presentation over S.*

Proof. Note that conditions (1) and (2), combined with Lemma 3.2 (1), imply that the S-groupoid X is limit preserving. Also, conditions (1) and (3) combined with Lemma 1.5 (4) imply that X is **Aff**-homogeneous. Conditions (5) and (6), together with Corollary 6.12, imply that for every affine X-scheme V that is locally of finite type over S, the functor $M \mapsto \operatorname{Exal}_X(V, M)$ is finitely generated. Thus, Theorem 7.1 implies that X is an algebraic stack that is locally of finite presentation over S. □

We now come to the proof of Theorem A.

Proof of Theorem A. By Corollary 7.2, it remains to prove that $\Delta_{X/S}$ is representable. To show this, it remains to prove that for any $(X \times_S X)$-scheme T, the T-groupoid $D_{X/S,T}$, which is obtained by pulling back $\Delta_{X/S}$ along T, is an algebraic stack.

Arguing as in the proof of Corollary 7.2, X is limit preserving and **Aff**-homogeneous. By Lemmas 1.5 (8) and 3.2 (6), the diagonal 1-morphism $\Delta_{X/S}\colon X \to X \times_S X$ is **Aff**-homogeneous and limit preserving. By Lemmas 1.5 (7), (5) and 3.2 (5), (3), the S-groupoid $X \times_S X$ is **Aff**-homogeneous and limit preserving. Thus, by Lemmas 1.5 (7) and 3.2 (5) the T-groupoid $D_{X/S,T}$ is limit preserving and **Aff**-homogeneous. Representability of $D_{X/S,T}$ is local on T for the Zariski topology, thus we may assume that T is an affine scheme. Since $X \times_S X$ is limit preserving, every affine $(X \times_S X)$-scheme T factors through an affine $(X \times_S X)$-scheme T_0 that is locally of finite type over S. Thus, we may assume henceforth that T is locally of finite type over S, and is consequently excellent. By Corollary 6.14, the T-groupoid $D_{X/S,T}$ satisfies all the conditions of Theorem A. Thus, by Corollary 7.2, it remains to prove that $\Delta_{D_{X/S,T}/T}$ is representable. Replacing $X \to S$ by $D_{X/S,T} \to T$ and repeating the above analysis we are further reduced to proving that the diagonal of $D_{D_{X/S,T},V} \to V$ is representable for every affine $(D_{X/S,T} \times_T D_{X/S,T})$-scheme V that is locally of finite type over T. Since $\Delta_{\Delta_{X/S}}$ is a monomorphism, it follows that the diagonal of $D_{X/S,T}$ is a monomorphism. In particular, $D_{D_{X/S,T},V} \to V$ is a monomorphism. Thus, $\Delta_{D_{D_{X/S,T}/T,V}/V}$ is an isomorphism, which is representable, and the result follows. □

8. Application I: The stack of quasi-coherent sheaves

Fix a scheme S. For an algebraic S-stack Y and a property P of quasi-coherent $\mathcal{O}_Y$-modules, denote by $\mathbf{QCoh}^P(Y)$ the full subcategory of $\mathbf{QCoh}(Y)$ consisting of objects which are P. We will be interested in the following properties P of quasi-coherent $\mathcal{O}_Y$-modules and their combinations:

fp – finitely presented,

fl – Y-flat,

flb – S-flat,

prb – S-proper support.

Fix a morphism of algebraic stacks $f: X \to S$. For any S-scheme T, consider a property P of quasi-coherent $\mathcal{O}_{X_T}$-modules. Define

$$\underline{\mathrm{QCoh}}^P_{X/S}$$

to be the category with objects a pair $(T, \mathcal{M})$, where T is an S-scheme and $\mathcal{M} \in \mathbf{QCoh}^P(X_T)$. A morphism $(a, \alpha): (V, \mathcal{N}) \to (T, \mathcal{M})$ in the category $\underline{\mathrm{QCoh}}^P_{X/S}$ consists of an S-scheme morphism $a: V \to T$ together with an $\mathcal{O}_{X_V}$-isomorphism $\alpha: a^*_{X_T}\mathcal{M} \to \mathcal{N}$. Set

$$\underline{\mathrm{Coh}}_{X/S} = \underline{\mathrm{QCoh}}^{\mathrm{flb,fp,prb}}_{X/S}.$$

In this section, we will prove the following theorem.

Theorem 8.1. *Fix a scheme S and a morphism of algebraic stacks $f: X \to S$. If f is separated and locally of finite presentation, then $\underline{\mathrm{Coh}}_{X/S}$ is an algebraic stack that is locally of finite presentation over S with affine diagonal over S.*

A proof of Theorem 8.1, without the statement about the diagonal, appeared in [26, Theorem 2.1], though was light on details. In particular, no explicit obstruction theory was given and, as we will see, the obstruction theory is subtle when f is not flat (and is not a standard fact). There was also a minor error in the statement – that the morphism f is separated is essential [29]. The statement about the diagonal of $\underline{\mathrm{Coh}}_{X/S}$ was addressed by M. Roth and J. Starr [40, §2]. Their approach, however, is completely different, and relies on [26, Proposition 2.3]. In the setting of analytic spaces, the properties of the diagonal were addressed by H. Flenner [16, Korollar 3.2].

Just as in [26, Proposition 2.7], an immediate consequence of Theorems 8.1 and [20, Theorem D] is the existence of Quot spaces. Recall that for a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the presheaf $\underline{\mathrm{Quot}}_{X/S}(\mathcal{F}): (\mathbf{Sch}/S)^\circ \to \mathsf{Sets}$ is defined as follows:

$$\underline{\mathrm{Quot}}_{X/S}(\mathcal{F})[T \xrightarrow{\tau} S] = \left\{\tau_X^*\mathcal{F} \twoheadrightarrow \mathcal{Q} : \mathcal{Q} \in \mathbf{QCoh}^{\mathrm{flb,fp,prb}}(X_T)\right\}/\cong.$$

Corollary 8.2. *Fix a scheme S, a morphism of algebraic stacks $f: X \to S$, and $\mathcal{F} \in \mathbf{QCoh}(X)$. If f is separated and locally of finite presentation over S, then $\underline{\mathrm{Quot}}_{X/S}(\mathcal{F})$ is an algebraic space that is separated over S. If, in addition, $\mathcal{F}$ is of finite presentation, then $\underline{\mathrm{Quot}}_{X/S}(\mathcal{F})$ is locally of finite presentation over S.*

When $\mathcal{F}$ is of finite presentation, Corollary 8.2 was proved by M. Olsson and J. Starr [38, Theorem 1.1] and M. Olsson [35, Theorem 1.5]. When $\mathcal{F}$ is quasi-coherent and $X \to S$ is locally projective, Corollary 8.2 was recently addressed by G. Di Brino [13, Theorem 0.0.1] using different methods.

To prove Theorem 8.1 we use Theorem A. Note that there are inclusions

$$\underline{\mathrm{QCoh}}^{\mathrm{flb,fp,prb}}_{X/S} \subseteq \underline{\mathrm{QCoh}}^{\mathrm{flb,fp}}_{X/S} \subseteq \underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}.$$

The first inclusion is trivially formally étale. By Lemma A.5 (1) the second inclusion is also formally étale. Thus, by Lemma 1.5 (9), if $\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}$ is **Aff**-homogeneous, then $\underline{\mathrm{Coh}}_{X/S}$ is **Aff**-homogeneous. Also, by Lemmas 6.3 and 6.11, it is sufficient to determine the automorphisms, deformations, and obstructions for $\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}$.

Throughout, we fix a clivage for $\underline{\mathrm{QCoh}}_{X/S}$. This gives an equivalence of S-groupoids $\underline{\mathrm{QCoh}}_{X/S} \to \mathbf{Sch}/\underline{\mathrm{QCoh}}_{X/S}$, which we will use without further comment.

Lemma 8.3. *Fix a scheme S and a morphism of algebraic stacks $f\colon X \to S$. Then the S-groupoid $\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}$ is* **Aff**-*homogeneous.*

Proof. First we check $(\mathrm{H}_1^{\mathbf{Aff}})$. Fix a commutative diagram of $\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}$-schemes:

$$\begin{array}{ccc} (T_0, \mathcal{M}_0) & \xhookrightarrow{(i,\phi)} & (T_1, \mathcal{M}_1) \\ {\scriptstyle (p,\pi)}\downarrow & & \downarrow{\scriptstyle (p',\pi')} \\ (T_2, \mathcal{M}_2) & \xrightarrow{(i',\phi')} & (T_3, \mathcal{M}_3), \end{array} \tag{8.1}$$

where p is affine and i is a locally nilpotent closed immersion. Set

$$(g, \gamma) = (i', \phi') \circ (p, \pi)\colon (T_0, \mathcal{M}_0) \to (T_3, \mathcal{M}_3).$$

Lemma 1.5 (1) implies that if the diagram (8.1) is cocartesian in the category of $\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}$-schemes, then it remains cocartesian in the category of S-schemes. Conversely, suppose that the diagram (8.1) is cocartesian in the category of S-schemes. By Lemma 1.5 (1) (applied to $Y = Z = S$), i' is a locally nilpotent closed immersion and p' is affine. Let $(W, \mathcal{N})$ be a $\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}$-scheme and for $k \neq 3$ fix $\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}$-scheme maps $(y_k, \psi_k)\colon (T_k, \mathcal{M}_k) \to (W, \mathcal{N})$. Since the diagram (8.1) is cocartesian in the category of S-schemes, there exists a unique S-morphism $y_3\colon T_3 \to W$ that is compatible with this data. By adjunction, we obtain unique maps of $\mathcal{O}_{X_W}$-modules

$$\mathcal{N} \to (y_1)_*\mathcal{M}_1 \times_{(y_0)_*\mathcal{M}_0} (y_2)_*\mathcal{M}_2 \cong \{(y_3)_* p'_*\mathcal{M}_1\} \times_{\{(y_3)_* g_*\mathcal{M}_0\}} \{(y_3)_* i'_*\mathcal{M}_2\}.$$

The functor $(y_3)_*$ is left-exact, so there is a functorial isomorphism of $\mathcal{O}_{X_W}$-modules

$$\{(y_3)_* p'_*\mathcal{M}_1\} \times_{\{(y_3)_* g_*\mathcal{M}_0\}} \{(y_3)_* i'_*\mathcal{M}_2\} \cong (y_3)_*\{p'_*\mathcal{M}_1 \times_{g_*\mathcal{M}_0} i'_*\mathcal{M}_2\}.$$

The commutativity of the diagram (8.1) posits a uniquely induced morphism

$$\delta\colon \mathcal{M}_3 \to p'_*\mathcal{M}_1 \times_{g_*\mathcal{M}_0} i'_*\mathcal{M}_2 \cong p'_* p'^*\mathcal{M}_3 \times_{g_* g^*\mathcal{M}_3} i'_* i'^*\mathcal{M}_3.$$

It suffices to prove that δ is an isomorphism, which is local for the smooth topology. So, we immediately reduce to the affine case, where $S = \operatorname{Spec} A$, $X = \operatorname{Spec} D$, and $f\colon X \to S$ is given by a ring homomorphism $A \to D$. For each l we may take $T_l = \operatorname{Spec} A_l$ and we set $D_l = D \otimes_A A_3$. Also, $\mathcal{M}_3 \cong \widetilde{M}_3$, where M_3 is a D_3-module which is A_3-flat. Now, there is an exact sequence of A_3-modules

$$0 \to A_3 \to A_1 \times A_2 \to A_0 \to 0.$$

Applying the exact functor $M_3 \otimes_{A_3} -$ to this sequence produces an exact sequence

$$0 \to M_3 \to (M_3 \otimes_{A_3} A_1) \times (M_3 \otimes_{A_3} A_2) \to M_3 \otimes_{A_3} A_0 \to 0.$$

Since $M_3 \otimes_{A_3} A_l \cong M_3 \otimes_{D_3} D_l$, we obtain the required isomorphism δ:

$$\begin{aligned} M_3 &\cong (M_3 \otimes_{A_3} A_1) \times_{(M_3 \otimes_{A_3} A_0)} (M_3 \otimes_{A_3} A_2) \\ &\cong (M_3 \otimes_{D_3} D_1) \times_{(M_3 \otimes_{D_3} D_0)} (M_3 \otimes_{D_3} D_2). \end{aligned}$$

Next we check condition ($\mathrm{H}_2^{\mathbf{Aff}}$). Fix a diagram of $\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}$-schemes,

$$\big[(T_1, \mathcal{M}_1) \xleftarrow{(i,\phi)} (T_0, \mathcal{M}_0) \xrightarrow{(p,\pi)} (T_2, \mathcal{M}_2)\big],$$

where i is a locally nilpotent closed immersion and p is affine. Given a cocartesian square of S-schemes:

(8.2)
$$\begin{tikzcd} T_0 \arrow[r, hook, "i"] \arrow[d, "p"'] & T_1 \arrow[d, "p'"] \\ T_2 \arrow[r, hook, "i'"] & T_3, \end{tikzcd}$$

write $g = i'p$ and set

$$\mathcal{M}_3 = \ker\big((p'_{X_{T_3}})_* \mathcal{M}_1 \times (i'_{X_{T_3}})_* \mathcal{M}_2 \xrightarrow{d} g_* \mathcal{M}_0\big) \in \mathbf{QCoh}(X_{T_3}),$$

where d is the map $(m_1, m_2) \mapsto (g_*\phi)(m_1) - (g_*\pi)(m_2)$. It remains to show that $\mathcal{M}_3$ is T_3-flat, that the induced morphisms of quasi-coherent $\mathcal{O}_{X_2}$-modules $\phi'\colon i'^*_{X_{T_3}} \mathcal{M}_3 \to \mathcal{M}_2$ and $\pi'\colon p'^*_{X_{T_3}} \mathcal{M}_3 \to \mathcal{M}_1$ are isomorphisms, and that the following diagram commutes:

$$\begin{tikzcd}[column sep=small] & i^*_{X_{T_1}} p'^*_{X_{T_3}} \mathcal{M}_3 \arrow[ld] \arrow[r, "i^*\pi'"] & i^*_{X_{T_1}} \mathcal{M}_1 \arrow[rd, "\phi"] & \\ g^* \mathcal{M}_3 & & & \mathcal{M}_0. \\ & p^*_{X_{T_2}} i'^*_{X_{T_3}} \mathcal{M}_3 \arrow[lu] \arrow[r, "p^*_{X_{T_2}}\phi'"] & p^*_{X_{T_2}} \mathcal{M}_2 \arrow[ru, "\pi"'] & \end{tikzcd}$$

Indeed, this shows that the pairs (i', ϕ') and (p', π') define $\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}$-morphisms, and that the resulting completion of the diagram (8.2) commutes.

Now, these claims may all be verified locally for the smooth topology. Thus, we reduce to the affine situation as before, with the modification that for $k \neq 3$ we have $\mathcal{M}_k \cong \widetilde{M}_k$, where M_k is a D_k-module which is flat over A_k, and $\mathcal{M}_3 \cong \widetilde{M}_3 \cong \widetilde{M}_1 \times_{\widetilde{M}_0} \widetilde{M}_2$. The result now follows from [14, Théorème 2.2]. □

We now determine the automorphisms, deformations, and obstructions. Let $(T, \mathcal{M})$ be a $\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}$-scheme, and fix a quasi-coherent $\mathcal{O}_T$-module I. For an S-extension $i \colon T \hookrightarrow T'$ of T by I, form the 2-cartesian diagram

$$\begin{tikzcd} X_T \arrow[r, hook, "j"] \arrow[d, "f_T"'] & X_{T'} \arrow[r] \arrow[d, "f_{T'}"] & X \arrow[d, "f"] \\ T \arrow[r, hook, "i"] \arrow[rr, bend right, "\tau"'] & T' \arrow[r, "\tau'"] & S. \end{tikzcd}$$

Set $J = j^* \ker(\mathcal{O}_{X_{T'}} \to j_*\mathcal{O}_{X_T})$. Fixing a $\underline{\mathrm{QCoh}}_{X/S}$-extension $(i, \phi) \colon (T, \mathcal{M}) \to (T', \mathcal{M}')$, we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I \arrow[r, two heads] \arrow[dr] & \mathcal{M} \otimes_{\mathcal{O}_{X_T}} J \arrow[d, two heads] \\ & j^* \ker(\mathcal{M}' \to j_*\mathcal{M}). \end{tikzcd}$$

By the local criterion for flatness, $\mathcal{M}'$ is T'-flat if and only if the diagonal arrow above is an isomorphism. Thus, if a $\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}$-extension $(i, \phi) \colon (T, \mathcal{M}) \to (T', \mathcal{M}')$ exists, the top map must be an isomorphism. This is how we will describe our first obstruction.

Example 8.4. This obstruction can be non-trivial when f is not flat and i is not split. Indeed, let $S = \operatorname{Spec} \mathbb{C}[x, y]$ and take $0 = (x, y) \in |S|$ to be the origin. Set

$$X = \mathrm{Bl}_0 S = \underline{\mathrm{Proj}}_S \mathcal{O}_S[U, V]/(xV - yU),$$

$f \colon X \to S$ the induced map, and let $E = f^{-1}(0)$ be the exceptional divisor. Now take $\mathcal{M} = \mathcal{O}_E$ and consider the S-extension $T = \operatorname{Spec} \kappa(0) \hookrightarrow T' = \operatorname{Spec} \mathbb{C}[x, y]/(x^2, y)$. A straightforward calculation shows that $\mathcal{M} \otimes_{\mathcal{O}_{X_T}} J$ is the skyscraper sheaf supported at the point of E corresponding to the $y = 0$ line in S. Also, $f_T^* I = \mathcal{O}_{X_T}$ and so $\mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I \cong \mathcal{O}_E$. The resulting map $\mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I \to \mathcal{M} \otimes_{\mathcal{O}_{X_T}} J$ is not injective.

Observe that there is a short exact sequence of $\mathcal{O}_{T'}$-modules

$$0 \to i_*I \to \mathcal{O}_{T'} \to i_*\mathcal{O}_T \to 0.$$

By Theorem C.1 we obtain an exact sequence of quasi-coherent $\mathcal{O}_{X_{T'}}$-modules

$$\mathcal{T}or_1^{S,\tau',f}(i_*\mathcal{O}_T, \mathcal{O}_X) \to f_{T'}^* i_* I \to j_* J \to 0.$$

Since there is a functorial isomorphism $f_{T'}^* i_* I \cong j_* f_T^* I$, Corollary C.3 provides a natural exact sequence of quasi-coherent $\mathcal{O}_{X_T}$-modules

$$\mathcal{T}or_1^{S,\tau,f}(\mathcal{O}_T, \mathcal{O}_X) \to f_T^* I \to J \to 0.$$

Applying the functor $\mathcal{M} \otimes_{\mathcal{O}_{X_T}} -$ to this sequence produces another exact sequence

$$\mathcal{M} \otimes_{\mathcal{O}_{X_T}} \mathcal{T}or_1^{S,\tau,f}(\mathcal{O}_T, \mathcal{O}_X) \xrightarrow{\mathrm{o}^1((T,\mathcal{M}),I)(i)} \mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I \to \mathcal{M} \otimes_{\mathcal{O}_{X_T}} J \to 0.$$

Thus, we have defined a natural class

$$\mathrm{o}^1((T,\mathcal{M}),I)(i) \in \mathrm{Hom}_{\mathcal{O}_{X_T}}\big(\mathcal{M} \otimes_{\mathcal{O}_{X_T}} \mathcal{T}or_1^{S,\tau,f}(\mathcal{O}_T,\mathcal{O}_X), \mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I\big),$$

whose vanishing is necessary and sufficient for the map $\mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I \to \mathcal{M} \otimes_{\mathcal{O}_{X_T}} J$ to be an isomorphism. By functoriality of the class $\mathrm{o}^1((T,\mathcal{M}),I)(i)$, we obtain a natural transformation of functors

$$\mathrm{o}^1((T,\mathcal{M}),-)\colon \mathrm{Exal}_S(T,-) \Rightarrow \mathrm{Hom}_{\mathcal{O}_{X_T}}\big(\mathcal{M}\otimes_{\mathcal{O}_{X_T}} \mathcal{T}or_1^{S,\tau,f}(\mathcal{O}_T,\mathcal{O}_X), \mathcal{M}\otimes_{\mathcal{O}_{X_T}} f_T^*(-)\big).$$

Suppose that the S-extension $i\colon T \hookrightarrow T'$ now has the property that the map

$$\mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I \to \mathcal{M} \otimes_{\mathcal{O}_{X_T}} J$$

is an isomorphism. Let $\gamma_{\mathcal{M},I}$ denote the inverse to this map. Then [24, IV.3.1.12] gives a naturally defined obstruction

$$\mathrm{o}^2((T,\mathcal{M}),I)(i) \in \mathrm{Ext}^2_{j_*\mathcal{O}_{X_T}}\big(j_*\mathcal{M}, j_*(\mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I)\big) \cong \mathrm{Ext}^2_{\mathcal{O}_{X_T}}(\mathcal{M}, \mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I)$$

whose vanishing is a necessary and sufficient condition for there to exist a lift of $\mathcal{M}$ over T'. Thus, there is a natural transformation

$$\mathrm{o}^2((T,\mathcal{M}),-)\colon \ker \mathrm{o}^1((T,\mathcal{M}),-) \Rightarrow \mathrm{Ext}^2_{\mathcal{O}_{X_T}}(\mathcal{M}, \mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^*(-))$$

such that the pair $\{\mathrm{o}^1((T,\mathcal{M}),-), \mathrm{o}^2((T,\mathcal{M}),-)\}$ defines a 2-step obstruction theory for the S-groupoid $\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}$ at $(T,\mathcal{M})$.

In the case where $i = i_{T,I}\colon T \hookrightarrow T[I]$, the trivial X-extension of T by I, then the map $\mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I \to \mathcal{M} \otimes_{\mathcal{O}_{X_T}} J$ is an isomorphism. By [24, IV.3.1.12], we obtain natural isomorphisms of abelian groups:

$$\begin{aligned}
\mathrm{Aut}_{\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}/S}((T,\mathcal{M}),I) &\cong \mathrm{Hom}_{j_*\mathcal{O}_{X_T}}\big(j_*\mathcal{M}, j_*(\mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I)\big),\\
&\cong \mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{M}, \mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I),\\
\mathrm{Def}_{\underline{\mathrm{QCoh}}^{\mathrm{flb}}_{X/S}/S}((T,\mathcal{M}),I) &\cong \mathrm{Ext}^1_{j_*\mathcal{O}_{X_T}}\big(j_*\mathcal{M}, j_*(\mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I)\big),\\
&\cong \mathrm{Ext}^1_{\mathcal{O}_{X_T}}(\mathcal{M}, \mathcal{M} \otimes_{\mathcal{O}_{X_T}} f_T^* I).
\end{aligned}$$

Proof of Theorem 8.1. Using standard reductions [41, Appendix B], we are free to assume that f is, in addition, finitely presented, and the scheme S is affine and of finite type over $\operatorname{Spec}\mathbb{Z}$ (in particular, it is noetherian and excellent). We now verify the conditions of Theorem A. Certainly, the S-groupoid $\underline{\mathrm{Coh}}_{X/S}$ is a limit preserving étale stack over S. By Lemma 8.3, it is also **Aff**-homogeneous. Also, for a noetherian local ring $(B,\mathfrak{m})$ that is $\mathfrak{m}$-adically complete and a map $\operatorname{Spec} B \to S$, the canonical functor

$$\mathbf{QCoh}^{\mathrm{flb,fp,prb}}(X_{\operatorname{Spec} B}) \to \varprojlim_n \mathbf{QCoh}^{\mathrm{flb,fp,prb}}(X_{\operatorname{Spec}(B/\mathfrak{m}^n)})$$

is an equivalence of categories [35, Theorem 1.4].

If $(T,\mathcal{M})$ is a $\underline{\mathrm{Coh}}_{X/S}$-scheme, then we have proved that

$$\begin{aligned}
\mathrm{Aut}_{\underline{\mathrm{Coh}}_{X/S}/S}((T,\mathcal{M}),-) &= \mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{M},\mathcal{M}\otimes_{\mathcal{O}_{X_T}} f_T^*(-)),\\
\mathrm{Def}_{\underline{\mathrm{Coh}}_{X/S}/S}((T,\mathcal{M}),-) &= \mathrm{Ext}^1_{\mathcal{O}_{X_T}}(\mathcal{M},\mathcal{M}\otimes_{\mathcal{O}_{X_T}} f_T^*(-)),\\
\mathrm{O}^1((T,\mathcal{M}),-) &= \mathrm{Hom}_{\mathcal{O}_{X_T}}\big(\mathcal{M}\otimes_{\mathcal{O}_{X_T}} \mathcal{T}or_1^{S,\tau,f}(\mathcal{O}_T,\mathcal{O}_X),\mathcal{M}\otimes_{\mathcal{O}_{X_T}} f_T^*(-)\big),\\
\mathrm{O}^2((T,\mathcal{M}),-) &= \mathrm{Ext}^2_{\mathcal{O}_{X_T}}(\mathcal{M},\mathcal{M}\otimes_{\mathcal{O}_{X_T}} f_T^*(-)),
\end{aligned}$$

where $\{\mathrm{O}^1((T,\mathcal{M}),-),\mathrm{O}^2((T,\mathcal{M}),-)\}$ are the obstruction spaces for a 2-step obstruction theory. If T is assumed to be locally noetherian, then by Theorem C.1, the $\mathcal{O}_{X_T}$-module $\mathcal{T}or_1^{S,\tau,f}(\mathcal{O}_T,\mathcal{O}_X)$ is coherent. In addition, if T is affine, Example 5.5 implies that the functors listed above are coherent. Having met the conditions of Theorem A, we see that the S-groupoid $\underline{\mathrm{QCoh}}^{\mathrm{flb,fp,prb}}_{X/S}$ is algebraic and locally of finite presentation over S.

It remains to show that the diagonal of $\underline{\mathrm{Coh}}_{X/S}$ is affine. If $(T,\mathcal{M})$ and $(T,\mathcal{N})$ are $\underline{\mathrm{Coh}}_{X/S}$-schemes, then the commutative diagram in the category of T-presheaves

$$\begin{tikzcd}
\underline{\mathrm{Isom}}_{\underline{\mathrm{QCoh}}_{X/S}}((T,\mathcal{M}),(T,\mathcal{N})) \arrow[r] \arrow[d, "\lambda\mapsto(\lambda,\lambda^{-1})"'] & \mathrm{Hom}_T(-,T) \arrow[d, "(\mathrm{Id}_{\mathcal{M}},\mathrm{Id}_{\mathcal{N}})"] \\
\underline{\mathrm{Hom}}_{\mathcal{O}_{X_T}/T}(\mathcal{M},\mathcal{N})\times\underline{\mathrm{Hom}}_{\mathcal{O}_{X_T}/T}(\mathcal{N},\mathcal{M}) \arrow[r] & \underline{\mathrm{Hom}}_{\mathcal{O}_{X_T}/T}(\mathcal{M},\mathcal{M})\times\underline{\mathrm{Hom}}_{\mathcal{O}_{X_T}/T}(\mathcal{N},\mathcal{N}),
\end{tikzcd}$$

where the morphism along the base is $(\mu,\nu)\mapsto(\nu\circ\mu,\mu\circ\nu)$, is cartesian. By [20, Theorem D], we deduce the result. □

We conclude this section with the following observations. Let $X\to S$ be a morphism of algebraic stacks and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let

$$(T\xrightarrow{\tau}S,\varphi\colon\tau_X^*\mathcal{F}\twoheadrightarrow\mathcal{G})=(T,\varphi)$$

be a $\underline{\mathrm{Quot}}_{X/S}(\mathcal{F})$-scheme. Then minor variations of the arguments given in the determination of the deformation and obstruction theory for $\underline{\mathrm{Coh}}_{X/S}$ show that there is a 2-step obstruction theory for $\underline{\mathrm{Quot}}_{X/S}(\mathcal{F})$:

$$\begin{aligned}
\mathrm{o}^1((T,\varphi),-)&\colon \mathrm{Exal}_S(T,-)\Rightarrow \mathrm{Hom}_{\mathcal{O}_{X_T}}\big(\mathcal{T}or_1^{S,\tau,f}(\mathcal{O}_T,\mathcal{F}),\mathcal{G}\otimes_{\mathcal{O}_{X_T}} f_T^*(-)\big),\\
\mathrm{o}^2((T,\varphi),-)&\colon \ker \mathrm{o}^1((T,\varphi),-)\Rightarrow \mathrm{Ext}^1_{\mathcal{O}_{X_T}}(\ker\varphi,\mathcal{G}\otimes_{\mathcal{O}_{X_T}} f_T^*(-)).
\end{aligned}$$

Moreover, we also have a functorial isomorphism

$$\mathrm{Def}_{\underline{\mathrm{Quot}}_{X/S}(\mathcal{F})}((T,\varphi),-)\cong \mathrm{Hom}_{\mathcal{O}_{X_T}}(\ker\varphi,\mathcal{G}\otimes_{\mathcal{O}_{X_T}} f_T^*(-)).$$

9. Application II: The Hilbert stack and spaces of morphisms

Fix a scheme S and a 1-morphism of algebraic stacks $f\colon X\to S$. For an S-scheme T, consider a property P of a morphism $Z\to X_T$. Such properties P could be (but not limited to):

qf – quasi-finite,

lfpb – the composition $Z \to X_T \to T$ is locally of finite presentation,

prb – the composition $Z \to X_T \to T$ is proper,

flb – the composition $Z \to X_T \to T$ is flat.

Define $\underline{\mathrm{Mor}}^P_{X/S}$ to be the category with objects pairs $(T, Z \xrightarrow{g} X_T)$, where T is an S-scheme and $g\colon Z \to X_T$ is a *representable* morphism of algebraic S-stacks that is P. Morphisms

$$(p,\pi)\colon (V, W \xrightarrow{h} X_V) \to (T, Z \xrightarrow{g} X_T)$$

in the category $\underline{\mathrm{Mor}}^P_{X/S}$ are 2-cartesian diagrams

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "\pi"'] & X_V \arrow[r, "f_V"] \arrow[d, "p_{X_T}"'] & V \arrow[d, "p"] \\ Z \arrow[r, "g"'] & X_T \arrow[r, "f_T"'] & T. \end{tikzcd}$$

If the property P is reasonably well-behaved, the natural functor $\underline{\mathrm{Mor}}^P_{X/S} \to \mathbf{Sch}/S$ defines an S-groupoid. We define the *Hilbert stack*, $\underline{\mathrm{HS}}_{X/S}$, to be the S-groupoid $\underline{\mathrm{Mor}}^{\mathrm{flb,lfpb,prb,qf}}_{X/S}$. This Hilbert stack contains Vistoli's Hilbert stack [47] as well as the stack of branchvarieties [3]. We will prove the following theorem.

Theorem 9.1. *Fix a scheme S and a morphism of algebraic stacks $f\colon X \to S$ that is separated and locally of finite presentation. Then $\underline{\mathrm{HS}}_{X/S}$ is an algebraic stack, which is locally of finite presentation over S with affine diagonal over S.*

Theorem 9.1 was the result alluded to in the title of M. Lieblich's paper [26], though a precise statement was not given. Theorem 9.1 was established in [26] using an auxiliary representability result [26, Proposition 2.3] combined with [26, Theorem 2.1] (Theorem 8.1 above). In the non-flat case, the obstruction theory used in the proof of [26, Proposition 2.3] is incorrect (a variant of Example 8.4 can be made into a counterexample in this setting also). The stated obstruction theory can be made into the second step of a 2-step obstruction theory, however. The properties of the diagonal of $\underline{\mathrm{HS}}_{X/S}$ have not been addressed previously.

Corollary 9.2. *Fix a scheme S and morphisms of algebraic stacks $f\colon X \to S$ and $g\colon Y \to S$. Assume that f is locally of finite presentation, proper, and flat and that g is locally of finite presentation with finite diagonal. Then $\underline{\mathrm{Hom}}_S(X,Y)$ is an algebraic stack, which is locally of finite presentation over S with affine diagonal over S.*

Corollary 9.2 can be used in the construction of the stack of twisted stable maps [1, Proposition 4.2]. The original construction of the stack of twisted stable maps utilized an incorrect obstruction theory in the non-flat case [2, Lemma 5.3.3]. The original proof of Corollary 9.2, due to M. Aoki [4, §3.5], also has an incorrect obstruction theory in the case of a non-flat target. The stated obstruction theories, as before, can be realized as the second step of a 2-step obstruction theory. A variant of Example 8.4 can be made into counterexamples in these settings too.

To prove Theorem 9.1, we will apply Theorem A directly (though as mentioned previously, this could be done as in [26] using Theorem 8.1). With Theorem 9.1 proven it is easy to deduce Corollary 9.2 via the standard method of associating to a morphism its graph, thus the proof is omitted. Now, just as in Section 8, there are inclusions

$$\underline{\mathrm{Mor}}^{\mathrm{flb,lfpb,prb,qf}}_{X/S} \subseteq \underline{\mathrm{Mor}}^{\mathrm{flb,lfpb,prb}}_{X/S} \subseteq \underline{\mathrm{Mor}}^{\mathrm{flb,lfpb}}_{X/S} \subseteq \underline{\mathrm{Mor}}^{\mathrm{flb}}_{X/S}.$$

The first two inclusions are trivially formally étale. By Lemma A.6, the third inclusion is formally étale. By Lemma 1.5 (9), they will all be **Aff**-homogeneous if $\underline{\mathrm{Mor}}^{\mathrm{flb}}_{X/S}$ is **Aff**-homogeneous. Also, by Lemmas 6.3 and 6.11, descriptions of the automorphisms, deformations, and obstructions for $\underline{\mathrm{Mor}}^{\mathrm{flb}}_{X/S}$ descend to the subcategories listed above.

Lemma 9.3. *Fix a scheme S and a morphism of algebraic stacks $f\colon X \to S$. Then the S-groupoid $\underline{\mathrm{Mor}}^{\mathrm{flb}}_{X/S}$ is* **Aff***-homogeneous.*

Proof. First we check ($\mathrm{H}_2^{\mathbf{Aff}}$). Fix a diagram of $\underline{\mathrm{Mor}}^{\mathrm{flb}}_{X/S}$-schemes

$$\big[(T_1, Z_1 \xrightarrow{g_1} X_{T_1}) \xleftarrow{(i,\phi)} (T_0, Z_0 \xrightarrow{g_0} X_{T_0}) \xrightarrow{(p,\pi)} (T_2, Z_2 \xrightarrow{g_2} X_{T_2})\big],$$

where i is a locally nilpotent closed immersion and p is affine, and a cocartesian square of S-schemes

$$\begin{tikzcd} T_0 \arrow[r, hook, "i"] \arrow[d, "p"'] & T_1 \arrow[d, "p'"] \\ T_2 \arrow[r, hook, "i'"] & T_3. \end{tikzcd}$$

By Proposition A.2, there exists a 2-commutative diagram of algebraic S-stacks

$$\begin{tikzcd} & Z_0 \arrow[rr, "\phi"] \arrow[dl, "\pi"'] \arrow[dd] & & Z_1 \arrow[dl, "\pi'"] \arrow[dd] \\ Z_2 \arrow[rr, "\phi'"] \arrow[dd] & & Z_3 \arrow[dd] & \\ & T_0 \arrow[rr, "i"] \arrow[dl, "p"'] & & T_1 \arrow[dl, "p'"] \\ T_2 \arrow[rr, "i'"] & & T_3, & \end{tikzcd}$$

where the left and rear faces of the cube are 2-cartesian, and the top and bottom faces are 2-cocartesian in the 2-category of algebraic S-stacks. Thus, the universal properties guarantee the existence of a unique T_3-morphism $Z_3 \xrightarrow{g_3} X_{T_3}$. By Lemma A.4, the morphism $Z_3 \to T_3$ is flat and all faces of the cube are 2-cartesian. In particular, the resulting $\underline{\mathrm{Mor}}^{\mathrm{flb}}_{X/S}$-scheme diagram

$$\begin{tikzcd} (T_0, Z_0 \xrightarrow{g_0} X_{T_0}) \arrow[r, hook, "(i,\phi)"] \arrow[d, "(p,\pi)"'] & (T_1, Z_1 \xrightarrow{g_1} X_{T_1}) \arrow[d, "(p',\pi')"] \\ (T_2, Z_2 \xrightarrow{g_2} X_{T_2}) \arrow[r, hook, "(i',\phi')"] & (T_3, Z_3 \xrightarrow{g_3} X_{T_3}) \end{tikzcd}$$

is cocartesian in the category of $\underline{\mathrm{Mor}}_{X/S}$-schemes. Condition $(\mathrm{H}_1^{\mathbf{Aff}})$ follows from a similar argument as that given in the proof Lemma 8.3. □

Fix a $\underline{\mathrm{Mor}}^{\mathrm{flb}}_{X/S}$-scheme $(T, Z \xrightarrow{g} X_T)$ and a quasi-coherent $\mathcal{O}_T$-module I. Unravelling the definitions and applying the results of [36] demonstrates that there are natural isomorphisms of abelian groups

$$\begin{aligned}\mathrm{Aut}_{\underline{\mathrm{Mor}}^{\mathrm{flb}}_{X/S}/S}((T, Z \xrightarrow{g} X_T), I) &\cong \mathrm{Hom}_{\mathcal{O}_Z}(L_{Z/X_T}, g^* f_T^* I),\\ \mathrm{Def}_{\underline{\mathrm{Mor}}^{\mathrm{flb}}_{X/S}/S}((T, Z \xrightarrow{g} X_T), I) &\cong \mathrm{Ext}^1_{\mathcal{O}_Z}(L_{Z/X_T}, g^* f_T^* I).\end{aligned}$$

Using identical ideas to those developed in Section 8, together with [36], we obtain a 2-term obstruction theory for the S-groupoid $\underline{\mathrm{Mor}}^{\mathrm{flb}}_{X/S}$ at $(T, Z \xrightarrow{g} X_T)$:

$$\begin{aligned}&\mathrm{o}^1((T, Z \xrightarrow{g} X_T), -)\colon \mathrm{Exal}_S(T, -) \Rightarrow \mathrm{Hom}_{\mathcal{O}_Z}\big(g^* \mathcal{T}or_1^{S,\tau,f}(\mathcal{O}_T, \mathcal{O}_X), g^* f_T^*(-)\big),\\ &\mathrm{o}^2((T, Z \xrightarrow{g} X_T), -)\colon \ker \mathrm{o}^1((T, Z \xrightarrow{g} X_T), -) \Rightarrow \mathrm{Ext}^2_{\mathcal{O}_Z}(L_{Z/X_T}, g^* f_T^*(-)).\end{aligned}$$

Proof of Theorem 9.1. The proof that the S-groupoid $\underline{\mathrm{HS}}_{X/S}$ is algebraic and locally of finite presentation is essentially identical to the proof of Theorem 8.1, thus is omitted. It remains to show that the diagonal is affine. If

$$(T, Z_1 \xrightarrow{g_1} X_T) \quad \text{and} \quad (T, Z_2 \xrightarrow{g_2} X_T)$$

are $\underline{\mathrm{HS}}_{X/S}$-schemes, then the inclusion of T-presheaves

$$\begin{aligned}&\underline{\mathrm{Isom}}_{\underline{\mathrm{HS}}_{X/S}}\big((T, Z_1 \xrightarrow{g_1} X_T), (T, Z_2 \xrightarrow{g_2} X_T)\big)\\ &\quad \subseteq \underline{\mathrm{Isom}}_{\underline{\mathrm{QCoh}}_{X/S}}\big((T, (g_2)_*\mathcal{O}_{Z_2}), (T, (g_1)_*\mathcal{O}_{Z_1})\big)\end{aligned}$$

is representable by closed immersions. By Theorem 8.1, the result follows. □

A. Homogeneity of stacks

The results of this section are routine bootstrapping arguments. They are included so that **Aff**-homogeneity can be proved for moduli problems involving stacks.

Definition A.1. Fix a 2-commutative diagram of algebraic stacks

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\not\Downarrow_\alpha"] & X_1 \arrow[d, "f'"] \\ X_2 \arrow[r, hook, "i'"'] & X_3, \end{tikzcd}$$

where i and i' are closed immersions and f and f' are affine. If the induced map

$$\mathcal{O}_{X_3} \to i'_*\mathcal{O}_{X_2} \times_{(i'f)_*\mathcal{O}_{X_0}} f'_*\mathcal{O}_{X_1}$$

is an isomorphism of sheaves, then we say that the diagram is a *geometric pushout*, and that X_3 is a *geometric pushout* of the diagram

$$\big[X_2 \xleftarrow{f} X_0 \xrightarrow{i} X_1\big].$$

The main result of this section is the following proposition.

Proposition A.2. *Any diagram of algebraic stacks $[X_2 \xleftarrow{f} X_0 \xrightarrow{i} X_1]$, where i is a locally nilpotent closed immersion and f is affine, admits a geometric pushout X_3. The resulting geometric pushout diagram is 2-cartesian and 2-cocartesian in the 2-category of algebraic stacks.*

We now need to collect some results which aid with the bootstrapping process.

Lemma A.3. *Fix a 2-commutative diagram of algebraic stacks*

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow"] & X_1 \arrow[d, "f'"] \\ X_2 \arrow[r, hook, "i'"'] & X_3. \end{tikzcd}$$

(1) *If the diagram above is a geometric pushout diagram, then it is 2-cartesian.*

(2) *If the diagram above is a geometric pushout diagram, then it remains so after flat base change on X_3.*

(3) *If after fppf base change on X_3, the above diagram is a geometric pushout diagram, then it was a geometric pushout prior to base change.*

Proof. The claim (1) is local on X_3 for the smooth topology, thus we may assume that everything in sight is affine; whence the result follows from [14, Théorème 2.2]. Claims (2) and (3) are trivial applications of flat descent. □

Lemma A.4. *Consider a 2-commutative diagram of algebraic stacks*

$$\begin{tikzcd}[row sep=small, column sep=small] & U_0 \arrow[rr, hook] \arrow[dl] \arrow[dd] & & U_1 \arrow[dl] \arrow[dd] \\ U_2 \arrow[rr, hook, crossing over] \arrow[dd] & & U_3 & \\ & X_0 \arrow[rr, hook] \arrow[dl] & & X_1 \arrow[dl] \\ X_2 \arrow[rr, hook] & & X_3 \arrow[from=uu, crossing over] & \end{tikzcd}$$

where the back and left faces of the cube are 2-cartesian, the top and bottom faces are geometric pushout diagrams, and for $j = 0, 1, 2$, the morphisms $U_j \to X_j$ are flat. Then all faces of the cube are 2-cartesian and the morphism $U_3 \to X_3$ is flat.

Proof. By Lemma A.3 (2), this is all smooth local on X_3 and U_3, thus we immediately reduce to the case where everything in sight is affine. Fix a diagram of rings $[A_2 \to A_0 \xleftarrow{p} A_1]$ where $p\colon A_1 \to A_0$ is surjective. For $j = 0, 1, 2$ fix flat A_j-algebras B_j, and A_0-isomorphisms

$B_2 \otimes_{A_2} A_0 \cong B_0$ and $B_1 \otimes_{A_1} A_0 \cong B_0$. Set $A_3 = A_2 \times_{A_0} A_1$ and $B_3 = B_2 \times_{B_0} B_1$, then we have to prove that the A_3-algebra B_3 is flat, the natural maps $B_3 \otimes_{A_3} A_j \to B_j$ are isomorphisms, and that these isomorphisms are compatible with the given isomorphisms. This is an immediate consequence of [14, Théorème 2.2], since these are questions about modules. □

We omit the proof of the following easy result from commutative algebra.

Lemma A.5. *Fix a surjection of rings $A \to A_0$ and let $I = \ker(A \to A_0)$. Suppose that there is a k such that $I^k = 0$.*

(1) *Given a map of A-modules $u\colon M \to N$ such that $u \otimes_A A_0$ is surjective, then u is surjective.*

(2) *For an A-module M, if $M \otimes_A A_0$ is finitely generated, then M is finitely generated.*

(3) *Given an A-algebra B and a B-module M, let $M_0 = A_0 \otimes_A M$.*

   (1) *If M is A-flat and M_0 is B_0-finitely presented, then M is B-finitely presented.*

   (2) *If B_0 is a finite type A_0-algebra, then B is a finite type A-algebra.*

   (3) *If B is a flat A-algebra and B_0 is a finitely presented A_0-algebra, then B is a finitely presented A-algebra.*

Lemma A.6. *Fix a flat morphism $f\colon X \to Y$ of algebraic stacks and a locally nilpotent closed immersion $Y_0 \hookrightarrow Y$. Then f is locally of finite presentation, respectively smooth, if and only if the same is true of $X \times_Y Y_0 \to Y_0$.*

Proof. Observe that for flat morphisms which are locally of finite presentation, smoothness is a fibral condition, thus follows from the first claim. The first claim is smooth local on Y and X, thus follows from Lemma A.5 (3). □

Lemma A.7. *Let $X \hookrightarrow X'$ be a locally nilpotent closed immersion of algebraic stacks and let $U \to X$ be a smooth morphism, where U is an affine scheme. Then there exists a smooth morphism $U' \to X'$ which pulls back to $U \to X$.*

Proof. Since U is quasi-compact, it is sufficient to treat the case where the locally nilpotent closed immersion $X \hookrightarrow X'$ is square zero. By [36, Theorem 1.4], the only obstruction to the existence of a flat lift of $U \to X$ over X' lies in an abelian group of the form $\operatorname{Ext}^2_{\mathcal{O}_U}(L_{U/X}, M)$, where M is a quasi-coherent $\mathcal{O}_U$-module. The morphism $U \to X$ is smooth, U is affine, and the $\mathcal{O}_U$-module $\mathcal{H}om_{\mathcal{O}_U}(\Omega_{U/X}, M)$ is quasi-coherent, thus

$$\operatorname{Ext}^2_{\mathcal{O}_U}(L_{U/X}, M) = H^2\big(U, \mathcal{H}om_{\mathcal{O}_U}(\Omega_{U/X}, M)\big) = 0.$$

Finally, by Lemma A.6, any such lift that is flat, is also smooth. □

The following result is a variation of [48, Proposition 2.1].

Lemma A.8. *Fix a 2-commutative diagram of algebraic stacks*

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow\alpha"] & X_1 \arrow[d, "f'"] \\ X_2 \arrow[r, hook, "i'"'] & X_3. \end{tikzcd}$$

If the diagram is a geometric pushout diagram and i is a locally nilpotent closed immersion, then it is 2-cartesian and 2-cocartesian in the 2-category of algebraic stacks.

Proof. That the diagram is 2-cartesian is Lemma A.3 (1). It remains to show that we can uniquely complete all 2-commutative diagrams of algebraic stacks

$$\begin{tikzcd} & X_1 \arrow[dr, "f'"] \arrow[drrr, bend left, "\psi_1"] & & & \\ X_0 \arrow[ur, hook, "i"] \arrow[dr, "f"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & X_3 \arrow[rr, phantom, "\Downarrow\beta"] & & W. \\ & X_2 \arrow[ur, hook, "i'"'] \arrow[urrr, bend right, "\psi_2"'] & & & \end{tikzcd}$$

By smooth descent, this is smooth-local on X_3, so we may reduce to the situation where the $X_j = \operatorname{Spec} A_j$ are all affine schemes. Since X_3 is a geometric pushout of the diagram

$$\big[X_2 \xleftarrow{f} X_0 \xrightarrow{i} X_2\big],$$

it follows that $A_3 \cong A_2 \times_{A_0} A_1$.

Let $q\colon \operatorname{Spec} B \to W$ be a smooth morphism such that the pullback $v_j\colon U_j \to X_j$ of q along ψ_j is surjective for $j \in \{0, 1, 2\}$, which exists because the X_j are all quasi-compact. There are compatibly induced morphisms of algebraic spaces $\psi_{j,B}\colon U_j \to \operatorname{Spec} B$ for $j = 1$ and 2 and $f_B\colon U_0 \to U_2$ and $i_B\colon U_0 \hookrightarrow U_1$.

Let $c_2\colon \operatorname{Spec} C_2 \to U_j$ be an étale morphism such that $v_2 \circ c_2$ is smooth and surjective. The morphism c_2 pulls back along f_B to give an étale morphism $c_0\colon \operatorname{Spec} C_0 \to U_0$ such that $v_0 \circ c_0$ is smooth and surjective. Let $\tilde{f}\colon \operatorname{Spec} C_0 \to \operatorname{Spec} C_2$ and $\tilde{\psi}_2\colon \operatorname{Spec} C_2 \to \operatorname{Spec} B$ be the resulting morphisms.

Since c_0 is étale and i is a locally nilpotent closed immersion, there is an étale morphism $c_1\colon \operatorname{Spec} C_1 \to X_1$ whose pullback along i_B is isomorphic to c_0; see [17, IV.18.1.2]. Let $C_3 = C_2 \times_{C_0} C_1$. Then there is a uniquely induced ring homomorphism $A_3 \to C_3$. By Lemma A.4, the morphism $c_3\colon \operatorname{Spec} C_3 \to \operatorname{Spec} A_3$ is flat and surjective and by Lemma A.6 it follows that c_3 is smooth and surjective. Hence, we may replace $\operatorname{Spec} A_j$ by $\operatorname{Spec} C_j$ and further assume that the ψ_j for $j = 0$, 1, and 2 factor through some smooth morphism $q\colon \operatorname{Spec} B \to W$. In particular, there is an induced morphism $\psi_3\colon \operatorname{Spec} A_3 \to \operatorname{Spec} B \to W$. It remains to prove that the morphism ψ_3 is unique up to a unique choice of 2-morphism. Let ψ_3 and $\psi_3'\colon \operatorname{Spec} A_3 \to W$ be two compatible morphisms. That these morphisms are isomorphic can be checked smooth-locally on $\operatorname{Spec} A_3$. But smooth-locally, the morphisms ψ_3 and ψ_3' both factor through some $\operatorname{Spec} B \to W$ and the morphisms $\operatorname{Spec} A_j \to \operatorname{Spec} A_3 \to \operatorname{Spec} B$ coincide for $j = 0$, 1, and 2, thus ψ_3 and ψ_3' are isomorphic. To show that the isomorphism between ψ_3 and ψ_3' is unique, we just repeat the argument, and the result follows. □

We finally come to the proof of Proposition A.2.

Proof of Proposition A.2. By Lemma A.8, it suffices to prove the existence of geometric pushouts. Let $\mathcal{C}_0$ denote the category of affine schemes. For $d = 1, 2, 3$, let $\mathcal{C}_d$ denote the full 2-subcategory of the 2-category of algebraic stacks with affine dth diagonal. Note that $\mathcal{C}_3$ is the full 2-category of algebraic stacks. We will prove by induction on $d \geq 0$ that if

$$[X_2 \xleftarrow{f} X_0 \xrightarrow{i} X_1] \tag{A.1}$$

belongs to $\mathcal{C}_d$, then it admits a geometric pushout. For the base case, where $d = 0$, take $X_3 = \operatorname{Spec}(\mathcal{O}_{X_2}(X_2) \times_{\mathcal{O}_{X_0}(X_0)} \mathcal{O}_{X_1}(X_1))$ and the result is clear. Now let $d > 0$ and assume that (A.1) belongs to $\mathcal{C}_d$. Fix a smooth surjection $\amalg_{l \in \Lambda} X_2^l \to X_2$, where X_2^l is an affine scheme for all $l \in \Lambda$. Set $X_0^l = X_2^l \times_{X_2} X_0$. Then as f is affine, the scheme X_0^l is also affine. By Lemma A.7, the resulting smooth morphism $X_0^l \to X_0$ lifts to a smooth morphism $X_1^l \to X_1$, with X_1^l affine, and $X_0^l \cong X_1^l \times_{X_1} X_0$. For $m = 0, 1$, and 2 and $u, v, w \in \Lambda$ set

$$X_m^{uv} = X_m^u \times_{X_m} X_m^v \quad \text{and} \quad X_m^{uvw} = X_m^u \times_{X_m} X_m^v \times_{X_m} X_m^w.$$

Note that for $m = 0, 1$, and 2 and all $u, v, w \in \Lambda$ we have $X_m^{uv}, X_m^{uvw} \in \mathcal{C}_{d-1}$. By the inductive hypothesis, for $I = u$, uv or uvw, a geometric pushout X_3^I of the diagram $[X_2^I \leftarrow X_0^I \to X_1^I]$ exists. By Lemma A.8, there are uniquely induced morphisms $X_m^{uv} \to X_m^u$. For $m \neq 3$, these morphisms are clearly smooth, and by Lemmas A.4 and A.6 the morphisms $X_3^{uv} \to X_3^u$ are smooth. It is easily verified that the universal properties give rise to a smooth groupoid $[\amalg_{u,v \in \Lambda} X_3^{uv} \rightrightarrows \amalg_{w \in \Lambda} X_3^w]$. The quotient X_3 of this groupoid in the category of stacks is algebraic. By Lemma A.3 (3), it is also a geometric pushout of (A.1). The result follows. □

B. Fibre products of Picard categories

For background material and conventions on Picard categories, we refer the reader to [8, XVIII.1.4]. In this appendix, we describe a variant of the exact sequence appearing in [19, (2.5.2)].

Let $f: P' \to P$ and $g: P \to Q$ be 1-morphisms of Picard categories. Define $P' \times_{f,P,g} Q$ to be the groupoid with objects (p', q, ξ), where $p' \in P'$ and $q \in Q$ and $\xi: f(p') \to g(q)$, and morphisms $(\phi, \chi): (p_1', q_1, \xi_1) \to (p_2', q_2, \xi_2)$, where $\phi: p_1' \to p_2'$ and $\chi: q_1 \to q_2$ are morphisms such that the following diagram commutes:

$$\begin{tikzcd} f(p_1') \arrow[r, "f_*\phi"] \arrow[d, "\xi_1"'] & f(p_2') \arrow[d, "\xi_2"] \\ g(q_1) \arrow[r, "g_*\chi"] & g(q_2). \end{tikzcd}$$

It is easily shown that $P' \times_{f,P,g} Q$ admits a natural structure of a Picard category such that the induced projections $f': P' \times_{f,P,g} Q \to Q$ and $g': P' \times_{f,P,g} Q \to P$ are 1-morphisms of Picard categories. There is also a canonically induced 2-morphism $\alpha: f \circ g' \Rightarrow g \circ f'$. In particular, there is a 2-commutative diagram of Picard categories,

$$\begin{tikzcd} P' \times_{f,P,g} Q \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\swarrow\alpha"] & P' \arrow[d, "f"] \\ Q \arrow[r, "g"] & P. \end{tikzcd}$$

It is easily shown that the 2-commutative diagram above is 2-cartesian in the 2-category of Picard categories.

Let $\mathbf{0}$ denote the Picard category with one object, whose abelian group of automorphisms is 0. If P is a Picard category and 0_P is a zero object of P, then there is an induced 1-morphism of Picard categories $0_P : \mathbf{0} \to P$. Also, there is a unique 1-morphism of Picard categories $0 \colon P \to \mathbf{0}$. Finally, let $\overline{P}$ be the abelian group of isomorphism classes of P.

Let $f_1 \colon P_1 \to P_2$ and $f_2 \colon P_2 \to P_3$ be 1-morphisms of Picard categories and let 0_{P_1} be a 0-object of P_1. Let $0_{P_3} = f_2 \circ f_1(0_{P_1})$. We say that the sequence of Picard categories

$$\mathbf{0} \xrightarrow{0_{P_1}} P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3$$

is *left-exact* if there exists a 2-morphism $\delta \colon f_2 \circ f_1 \Rightarrow 0_{P_3} \circ 0$ that makes the 2-commutative diagram

$$\begin{tikzcd} P_1 \arrow[r, "f_1"] \arrow[d, "0"'] \arrow[dr, phantom, "\swarrow\delta"] & P_2 \arrow[d, "f_2"] \\ \mathbf{0} \arrow[r, "0_{P_3}"] & P_3 \end{tikzcd}$$

2-cartesian in the 2-category of Picard categories.

The main result of this appendix is the following lemma.

Lemma B.1. *Consider a left-exact sequence of Picard categories*

$$\mathbf{0} \xrightarrow{0_{P_1}} P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3.$$

Let $0_{P_2} = f_1(0_{P_1})$ *and* $0_{P_3} = f_2(0_{P_2})$. *Then there is an exact sequence of abelian groups*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Aut}_{P_1}(0_{P_1}) \arrow[r, "(f_1)_*"] & \mathrm{Aut}_{P_2}(0_{P_2}) \arrow[r, "(f_2)_*"] & \mathrm{Aut}_{P_3}(0_{P_3}) \arrow[dll, "\partial"] \\ & \overline{P_1} \arrow[r, "\overline{f_1}"] & \overline{P_2} \arrow[r, "\overline{f_2}"] & \overline{P_3}. \end{tikzcd}$$

Proof. By the explicit description of the 2-fiber product of Picard categories, we may assume that P_1 is expressed in the following way: it is the Picard category with objects pairs (p_2, a), where $p_2 \in P_2$ and $a \colon f_2(p_2) \to 0_{P_3}$ is a morphism in P_3, and morphisms $\phi \colon (p_2, a) \to (p_2', a')$, where $\phi \colon p_2 \to p_2'$ is a morphism such that $a' \circ (f_2)_*(\phi) = \mathrm{Id}_{0_{P_3}}$. The functor $f_1 \colon P_1 \to P_2$ is the forgetful functor: $(p_2, a) \mapsto p_2$. We also take 0_{P_1} to be $(0_{P_2}, \mathrm{Id}_{0_{P_3}})$. Finally, the 2-morphism $\delta \colon f_2 \circ f_1 \Rightarrow 0_{P_3} \circ 0$ sends $(f_2 \circ f_1)(p_2, a) = f_2(p_2)$ to $(0_{P_3} \circ 0)(p_2, a) = 0_{P_3}$ via a.

In this case it is trivial from the definitions that the sequence

$$0 \to \mathrm{Aut}_{P_1}(0_{P_1}) \xrightarrow{(f_1)_*} \mathrm{Aut}_{P_2}(0_{P_2}) \xrightarrow{(f_2)_*} \mathrm{Aut}_{P_3}(0_{P_3})$$

is exact. The morphism ∂ is described as follows: it sends an automorphism $l \colon 0_{P_3} \to 0_{P_3}$ to the isomorphism class of the object $(0_{P_2}, l) \in P_1$. Note that this shows, in particular, that $\partial(l) = 0$ if and only if there is an isomorphism $\phi\colon (0_{P_2}, l) \to (0_{P_2}, \mathrm{Id}_{0_{P_3}})$. That is, $\partial(l) = 0$ if and only if there is an automorphism $\phi\colon 0_{P_2} \to 0_{P_2}$ such that $l \circ (f_2)_*(\phi) = \mathrm{Id}_{P_3}$. It follows that $\partial(l) = 0$ if and only if $l \in \mathrm{im}(f_2)_*$, thus the sequence

$$\mathrm{Aut}_{P_2}(0_{P_2}) \xrightarrow{(f_2)_*} \mathrm{Aut}_{P_3}(0_{P_3}) \xrightarrow{\partial} \overline{P_1}$$

is exact. If $(p_2, a) \in P_1$, then $\overline{f_1}(p_2, a) = 0$ in $\overline{P_2}$ if and only if there is an isomorphism $q\colon p_2 \to 0_{P_2}$. In particular, it follows that q induces an isomorphism

$$(p_2, a) \to \big(0_{P_2}, (f_2)_*(q) \circ a^{-1}\big)$$

in P_1 and so (p_2, a) belongs to $\mathrm{im}\,\partial$ if and only if $\overline{f_1}(p_2, a)$. Finally, if $p_2 \in P_2$, then $\overline{f_2}(p_2) = 0$ in $\overline{P_3}$ if and only if there is an isomorphism $m\colon f_2(p_2) \to 0_{P_3}$. This is manifestly equivalent to p_2 lying in the image of $\overline{f_1}$. We have thus shown that the sequence

$$\mathrm{Aut}_{P_3}(0_{P_3}) \xrightarrow{\partial} \overline{P_1} \xrightarrow{\overline{f_1}} \overline{P_2} \xrightarrow{\overline{f_2}} \overline{P_3}$$

is exact. The result follows. □

C. Local Tor functors on algebraic stacks

The aim of the section is to state some easy generalizations of [17, III.6.5] to algebraic stacks.

Theorem C.1. *Fix a scheme S and a 2-cartesian diagram of algebraic S-stacks:*

$$\begin{array}{ccc} X_3 & \xrightarrow{f_1'} & X_2 \\ {\scriptstyle f_2'}\downarrow & & \downarrow{\scriptstyle f_2} \\ X_1 & \xrightarrow{f_1} & X_0. \end{array}$$

Then for every integer $i \geq 0$ there exists a natural bifunctor

$$\mathcal{T}or_i^{X_0, f_1, f_2}(-,-)\colon \mathbf{QCoh}(X_1) \times \mathbf{QCoh}(X_2) \to \mathbf{QCoh}(X_3),$$

such that the family of bifunctors $\{\mathcal{T}or_i^{X_0, f_1, f_2}(-,-)\}_{i \geq 0}$ forms a ∂-functor in each variable. Moreover, there is a natural isomorphism for all $M \in \mathbf{QCoh}(X_1)$ and $N \in \mathbf{QCoh}(X_2)$,

$$\mathcal{T}or_0^{X_0, f_1, f_2}(M, N) \cong f_2'^{*} M \otimes_{\mathcal{O}_{X_3}} f_1'^{*} N.$$

If M or N is X_0-flat, then for all $i > 0$ we have $\mathcal{T}or_i^{X_0, f_1, f_2}(M, N) = 0$. In addition, if the algebraic stacks X_1 and X_0 are locally noetherian and the morphism f_2 is locally of finite type, then the bifunctor above restricts to a bifunctor:

$$\mathcal{T}or_i^{X_0, f_1, f_2}(-,-)\colon \mathbf{Coh}(X_1) \times \mathbf{Coh}(X_2) \to \mathbf{Coh}(X_3).$$

Proof. We will describe the quasi-coherent sheaves $\mathcal{T}or_i^{X_0,f_1,f_2}(M,N)$ smooth-locally on X_3 and deduce their existence via descent. The other properties will be trivial consequences of this construction. We begin by observing that X_3 admits a smooth cover by affine schemes of the form $\operatorname{Spec}(A_1 \otimes_{A_0} A_2)$, where for each $j = 0$, 1, and 2 there is a smooth morphism $\operatorname{Spec} A_j \to X$. For each integer $i \geq 0$ let

$$\mathcal{T}or_i^{X_0,f_1,f_2}(M,N)|_{\operatorname{Spec}(A_1\otimes_{A_0}A_2)} = \operatorname{Tor}_i^{A_0}\big(\Gamma(\operatorname{Spec} A_1, M), \Gamma(\operatorname{Spec} A_2, N)\big).$$

Clearly, the above is an $(A_1 \otimes_{A_0} A_2)$-module with the relevant properties. The result follows. □

Remark C.2. Unless X_1 and X_2 are tor-independent over X_0, the quasi-coherent $\mathcal{O}_{X_3}$-modules $\mathcal{T}or_i^{X_0,f_1,f_2}(M,N)$ are not isomorphic to $\mathcal{H}^{-i}([\mathsf{L}f_1'^{*}N] \otimes^{\mathsf{L}}_{\mathcal{O}_{X_3}} [\mathsf{L}f_2'M])$.

An immediate consequence of the proof of Theorem C.1 is the following corollary.

Corollary C.3. *Fix a scheme S and a 2-cartesian diagram of algebraic S-stacks*

$$\begin{tikzcd} W \times_Z Y \arrow[r, "h'"] \arrow[d, "g_W"'] & X \times_Z Y \arrow[r, "f_Y"] \arrow[d, "g_X"'] & Y \arrow[d, "g"] \\ W \arrow[r, "h"] & X \arrow[r, "f"] & Z, \end{tikzcd}$$

where the morphism h is affine. Then, for any $M \in \mathbf{QCoh}(W)$, $N \in \mathbf{QCoh}(Y)$, and $i \geq 0$, there is a natural isomorphism of quasi-coherent $\mathcal{O}_{X\times_Z Y}$-modules:

$$\mathcal{T}or_i^{Z,f,g}(h_*M,N) \cong h'_*\mathcal{T}or_i^{Z,f\circ h,g}(M,N).$$

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