A RELATIVE GAGA PRINCIPLE FOR FAMILIES OF CURVES

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ABSTRACT. We prove a relative GAGA principle for families of curves, showing: (i) analytic families of pointed curves whose fibers have finite automorphism groups are algebraizable and (ii) analytic birational models of $\mathcal{M}_{g,n}$ possessing modular interpretations with the finite automorphism property are algebraizable. This is accomplished by extending some well-known GAGA results for proper schemes to non-separated Deligne–Mumford stacks.

1. INTRODUCTION

Fundamental relations between analytic and algebraic geometry are described by various GAGA principles. These have been beautifully addressed (in different ways) by many, with major contributions due to W. L. Chow [Cho49], K. Kodaira [Kod54], J. P. Serre [GAGA], A. Grothendieck and M. Raynaud [SGA1, Exp. XII], M. Artin [Art70, Thm. 7.3], and more recently J. Lurie [Lur04] and B. Töen and M. Vaquié [TV08]. For an excellent survey of the classical results, we recommend Hartshorne [Har77, App. B].

There are, however, relative formulations of the GAGA principle which are yet to be addressed. In fact, in the relative situation, even the case of families of curves is subtle.

Fix an integer $n \geq 0$ and an algebraic (resp. analytic) space T. An *n-pointed algebraic* (resp. *analytic*) *T-curve* is a morphism of algebraic (resp. analytic) spaces $\pi : C \to T$ which is proper, flat, and of relative dimension one, together with n algebraic (resp. analytic) sections $\{\sigma_i : T \to C\}_{i=1}^n$ to π. Note that we do not assume that the map π is smooth nor the images of the sections σ_i disjoint. When T is $\operatorname{Spec}\mathbb{C}$ or the punctual analytic space, "T-curve" will be contracted to "curve".

If T is an algebraic space which is locally separated and locally of finite type over $\operatorname{Spec}\mathbb{C}$, then T maybe functorially analytified to an analytic space T_{an}. In particular, an n-pointed algebraic T-curve may be functorially analytified to an n-pointed analytic T_{an}-curve. An n-pointed analytic T_{an}-curve is *algebraizable* if it lies in the essential image of the afforementioned analytification functor.

An automorphism of an n-pointed algebraic (resp. analytic) curve is an algebraic (resp. analytic) automorphism of the underlying curve preserving the sections. An n-pointed algebraic (resp. analytic) T-curve has the *finite automorphism* property if its n-pointed fibers have finite automorphism groups. The Deligne–Mumford stable curves [DM69, Defn. 1.1] and their n-pointed generalizations [Knu83, Defn. 1.1] satisfy the finite automorphism property, though there are many others (e.g. [Sch91, Smy12]). In this paper, we prove the following relative GAGA principle for such families of curves.

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Theorem A. *Fix an algebraic space T, proper over $\operatorname{Spec} \mathbb{C}$, and an integer $n \geq 0$. Then, any n-pointed analytic T_{an}-curve with the finite automorphism property is algebraizable.*

In Example 3.1 we show that Theorem A cannot be strengthened to include smooth families of curves of genus 1 (such families do not have the finite automorphism property).

Another relative GAGA principle for families curves that we prove is related to the Hassett–Keel program [Has05]. This program has recently seen a flurry of activity [AS12, AH12, AFS10, ASv10, Smy11a, Smy11b, FS11, HL10a, HL10b, HL07, HH08], initiated by the work of B. Hassett and D. Hyeon [HH09]. Roughly speaking, this program aims to classify proper and birational models of the stack $\mathcal{M}_{g,n}$ of smooth n-pointed algebraic curves of genus g, whose sections are required to have disjoint images, which admit modular interpretations [ASv10, §1]. One can ask an analogous question for the analytic Deligne–Mumford stack of smooth n-pointed analytic curves of genus g, $\mathcal{M}_{g,n}^{\mathrm{an}}$. We show that these problems are equivalent in the case of modular interpretations with the finite automorphism property (see §7).

Theorem B. *Fix non-negative integers g and n such that $2g-2+n>0$. Then, analytic modular birational models of $\mathcal{M}_{g,n}^{\mathrm{an}}$ with the finite automorphism property are uniquely algebraizable to algebraic modular birational models of $\mathcal{M}_{g,n}$ with the finite automorphism property.*

It is important to note that many of the birational models appearing in the Hassett–Keel program do not have the finite automorphism property. Moreover, at present, there are no counterexamples to an analogue of Theorem B holding in the more general setting of birational models without the finite automorphism property. We also believe that a much deeper understanding of the geometry of Artin stacks—along the lines of [Lur04] and [GZ12]—would be of benefit to attempting such a generalization.

To motivate the proof of Theorem A, fix a scheme T which is proper over $\operatorname{Spec} \mathbb{C}$, and non-negative integers g and n such that $2g-2+n>0$. Consider a smooth n-pointed analytic T_{an}-curve $\varpi : \mathcal{C} \to T_{\mathrm{an}}$ with fibers of genus g such that the sections have disjoint images. By definition of $\mathcal{M}_{g,n}^{\mathrm{an}}$, this is equivalent to a morphism of analytic stacks $T_{\mathrm{an}} \to \mathcal{M}_{g,n}^{\mathrm{an}}$. In §7, we will show that $\mathcal{M}_{g,n}^{\mathrm{an}}$ is the analytification of the stack $\mathcal{M}_{g,n}$. Thus, in the smooth case, it suffices to prove that the morphism f_{ϖ} algebraizes to a map $f_{\pi} : T \to \mathcal{M}_{g,n}$. Indeed, this would give rise to a smooth n-pointed algebraic T-curve $\pi : C \to T$ together with an analytic isomorphism of n-pointed analytic T_{an}-curves $C_{\mathrm{an}} \cong \mathcal{C}$. The Deligne–Mumford stack $\mathcal{M}_{g,n}$ is separated, however, and so the existence of an algebraization follows readily from the GAGA principles for separated Deligne–Mumford stacks (see §2 for precise statements and references).

In the general case, the strategy is similar, except instead of the separated stacks $\mathcal{M}_{g,n}$ and $\mathcal{M}_{g,n}^{\mathrm{an}}$, we use the stacks of all finite automorphism n-pointed algebraic (resp. analytic) curves $\mathcal{U}_n^{\mathrm{FA}}$ (resp. $\mathcal{U}_n^{\mathrm{FA,an}}$). In §7 we show that the stack $\mathcal{U}_n^{\mathrm{FA}}$ is an algebraic Deligne–Mumford stack (see §1.1 for definitions) whose analytification is $\mathcal{U}_n^{\mathrm{FA,an}}$. Note, however, that the algebraic Deligne–Mumford stack $\mathcal{U}_n^{\mathrm{FA}}$ is not separated, but Theorem A now follows from:

Theorem C. *Fix algebraic Deligne–Mumford stacks Z and X. Suppose that Z is proper, then the analytification functor:*

$$\mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z_{\mathrm{an}}, X_{\mathrm{an}})$$

induces an equivalence of categories.

J. Lurie [Lur04, Thm. 1.1] has proved a related result to Theorem C—the stack X is permitted to be algebraic (as opposed to Deligne–Mumford), but the diagonal is assumed to be affine. Note, however, that the diagonal of the Artin stack $\mathcal{U}_n^{\mathrm{FA}}$ is quasi-affine (indeed, it is quasi-finite and separated), but is not known to be affine. Thus, Lurie's result [*loc. cit.*] is currently insufficient to prove Theorem A.

To prove Theorem C, we will prove a generalization of Chow's Theorem [Cho49] for non-separated Deligne–Mumford stacks. The connection here is well-known: to an analytic morphism $\phi : Z_{\mathrm{an}} \to X_{\mathrm{an}}$ we can associate its graph $\Gamma_\phi : Z_{\mathrm{an}} \to Z_{\mathrm{an}} \times X_{\mathrm{an}}$, which is the pullback of the diagonal $\Delta_{X_{\mathrm{an}}} : X_{\mathrm{an}} \to X_{\mathrm{an}} \times X_{\mathrm{an}}$ along $Z_{\mathrm{an}} \times X_{\mathrm{an}} \xrightarrow{(\phi,\mathrm{id})} X_{\mathrm{an}} \times X_{\mathrm{an}}$. If X is a separated scheme, then Γ_ϕ is a closed immersion. If X is a separated Deligne–Mumford stack, then Γ_ϕ can only be assumed to be a finite morphism. Moreover, since algebraizing the graph of a morphism is equivalent to algebraizing the morphism, in the separated case, Theorem C follows from the separated GAGA statements. If X is non-separated, however, the graph Γ_ϕ is no longer finite, but only locally quasi-finite.

For an algebraic (resp. analytic) stack X, let $\mathbf{QF}(X)$ denote the category of 1-morphisms $Z \to X$ which are locally quasi-finite, separated, and representable. Let $\mathbf{QF}_p(X) \subset \mathbf{QF}(X)$ denote the full subcategory consisting of those 1-morphisms $Z \to X$ *with Z proper*. Our main technical result, also instrumental to proving Theorem B, is

Theorem D. *Fix an algebraic Deligne–Mumford stack X. Then, the analytification functor:*

$$\Psi_{X,p} : \mathbf{QF}_p(X) \to \mathbf{QF}_p(X_{\mathrm{an}})$$

induces an equivalence of categories.

Our proof of Theorem D is very similar to the approach of [HalR10, Thm. 3.5]. To motivate this strategy, it is instructive to sketch the proof that there is an equivalence of categories $\mathbf{Coh}(Y) \to \mathbf{Coh}(Y_{\mathrm{an}})$ for proper $\mathbb{C}$-schemes. The technique is via *dévissage* on the category of coherent sheaves $\mathbf{Coh}(Y)$ and we follow [SGA1, XII.4.4]. We say that $\mathcal{F} \in \mathbf{Coh}(Y_{\mathrm{an}})$ is *algebraizable* if it lies in the essential image of the analytification functor $\mathbf{Coh}(Y) \to \mathbf{Coh}(Y_{\mathrm{an}})$. The proof consists of the following steps.

(1) Given coherent sheaves H, H' on Y, the natural map of $\mathbb{C}$-modules:

$$\mathrm{Ext}^i_{\mathcal{O}_Y}(H, H') \to \mathrm{Ext}^i_{\mathcal{O}_{Y_{\mathrm{an}}}}(H_{\mathrm{an}}, H'_{\mathrm{an}})$$

is an isomorphism for all $i \geq 0$. Taking $i = 0$ here shows that the analytification functor $\mathbf{Coh}(Y) \to \mathbf{Coh}(Y_{\mathrm{an}})$ is fully faithful. Hence, it is sufficient to prove that the analytification functor $\mathbf{Coh}(Y) \to \mathbf{Coh}(Y_{\mathrm{an}})$ is essentially surjective.

(2) Show that if we have an exact sequence of coherent sheaves on Y_{an}:

$$0 \longrightarrow \mathcal{H}' \longrightarrow \mathcal{H} \longrightarrow \mathcal{H}'' \longrightarrow 0$$

and two of $\mathcal{H}'$, $\mathcal{H}''$, $\mathcal{H}$ are algebraizable, then the third is. This follows from the exactness of analytification and the $i = 0{,}1$ statements of 1. In particular, given an $\mathcal{O}_{Y_{\mathrm{an}}}$-morphism $\lambda : \mathcal{H} \to \mathcal{H}'$ such that $\mathcal{H}'$, $\ker\lambda$, and $\operatorname{coker}\lambda$ are algebraizable, then $\mathcal{H}$ is algebraizable.

(3) By noetherian induction on the topological space $|Y|$, we may assume that Y is integral and that every $\mathcal{H} \in \mathbf{Coh}\,(Y_{\mathrm{an}})$ with $|\operatorname{supp}\mathcal{H}| \subsetneq |Y|$ is algebraizable.

(4) Combine 2 and 3 to conclude that the essential surjectivity of the analytification functor $\mathbf{Coh}\,(Y) \to \mathbf{Coh}\,(Y_{\mathrm{an}})$ will be shown if we can produce for each $\mathcal{F} \in \mathbf{Coh}\,(Y_{\mathrm{an}})$ an $\mathcal{O}_{Y_{\mathrm{an}}}$-morphism $\mathcal{F} \to H_{\mathrm{an}}$ whose kernel and cokernel is supported on a proper subset of $|Y|$.

(5) Prove the result for all projective $\mathbb{C}$-schemes by hand.

(6) If Y is a proper $\mathbb{C}$-scheme, use Chow's Lemma [EGA, II.5.6.1] to construct a projective morphism $p : Y' \to Y$ that is an isomorphism on a dense open subset of Y and such that Y' is projective;

(7) Use 5 for the projective scheme Y' to show that $p_{\mathrm{an}}^*\mathcal{F} \cong G_{\mathrm{an}}$ for some coherent G;

(8) Use [SGA1, XII.4.2] to show that $(p_*G)_{\mathrm{an}} \cong (p_{\mathrm{an}})_*G_{\mathrm{an}}$.

(9) Combine 7 and 8 to see that $(p_{\mathrm{an}})_*p_{\mathrm{an}}^*\mathcal{F} \cong (p_*G)_{\mathrm{an}}$ is algebraizable. Furthermore, we have an adjunction morphism $\eta : \mathcal{F} \to (p_{\mathrm{an}})_*p_{\mathrm{an}}^*\mathcal{F}$. Since p is birational, η satisfies the conditions of step 3 and we conclude that the result has been proven for all proper $\mathbb{C}$-schemes.

Our strategy for proving Theorem D is a reinterpretation of the above steps. Thus, instead of performing a dévissage on the category $\mathbf{Coh}\,(X)$, we perform a dévissage on the category $\mathbf{QF}_p(X)$. In §4, we will reinterpret 1 and 2 in terms of the existence of pushouts in $\mathbf{QF}(X)$ along finite morphisms. The exactness of the analytification functor $\mathbf{Coh}\,(Y) \to \mathbf{Coh}\,(Y_{\mathrm{an}})$ is recast as the preservation of these pushouts under analytification (Lemma 4.2). The analog of 4 is the main result of §4 (Lemma 4.1).

Projective schemes in 5 are replaced by those $\mathbb{C}$-schemes Y' whose structure maps factor as $Y' \to W \to \operatorname{Spec}\mathbb{C}$, where $Y' \to W$ is étale and quasicompact, and W is projective. Proving 5 will make use of a basic case of the Comparison Theorem between étale and complex cohomology [FK88, I.11.5]. The Chow Lemma used in 6 is a generalization due to Raynaud–Gruson [RG71, Cor. 5.7.13]: any quasicompact $\mathbb{C}$-scheme Y admits a blowup $Y' \to Y$ such that $Y' \to \operatorname{Spec}\mathbb{C}$ factors as $Y' \to W \to \operatorname{Spec}\mathbb{C}$, where $Y' \to W$ is étale and W is projective.

In §5 we use Stein factorizations to reinterpret the steps 8 and 9 (Lemma 5.1). This proves Theorem D for all schemes and forms, what we call, the technique of *birational* dévissage (Proposition 5.3). Finally, to prove Theorem D for algebraic Deligne–Mumford stacks, we will require a *finite* dévissage (Proposition 5.4), which is similar to the birational dévissage, but much simpler. The proof of Theorem D is completed in §6.

1.1. **Assumptions and notations.** We will assume that all schemes are locally of finite type over $\operatorname{Spec}\mathbb{C}$. An *algebraic Deligne–Mumford stack* will denote a Deligne–Mumford stack (in the sense of [LMB, 4.1]) which is locally of finite type over $\operatorname{Spec}\mathbb{C}$. Thus, we assume that the diagonals of all algebraic Deligne–Mumford stacks are quasicompact and separated, so are representable by quasicompact, unramified, and separated schemes.

For a ringed space (resp. site) E we denote by $|E|$ its underlying topological space (resp. category) and $\mathcal{O}_E$ its sheaf of functions. For a morphism of ringed spaces (resp. sites) $\phi : E \to F$ we denote by $\phi^\sharp : \mathcal{O}_F \to \phi_*\mathcal{O}_E$ the induced map of functions. Denote by $\mathbf{Mod}(E)$ the abelian category of $\mathcal{O}_E$-modules.

For background material on analytic spaces, we refer the reader to [GR84]. Let **An** denote the category of analytic spaces. A morphism of analytic spaces is étale if it is an isomorphism locally in the analytic topology. Covering families for the étale topology on **An** are given by jointly surjective families of étale morphisms.

Given morphisms $p : U \to V$ and $q : W \to V$, set $U_W = U \times_{p,V,q} W$ and take $p_W : U_W \to W$ to be the projection.

2. Analytic Deligne–Mumford stacks

Analytic Deligne–Mumford stacks have been defined in various levels of generality by multiple authors. In this section, we give a definition which is similar to [Toë99, Ch. 5], but we permit our stacks to be non-separated.

An analytic space $\mathcal{X}$, via its functor of points, gives rise to a stack over **An**. We will not distinguish between the analytic space and its associated stack. A stack $\mathcal{Y}$ over **An** is *representable* if it is isomorphic to an analytic space. A 1-morphism $\mathcal{U} \to \mathcal{V}$ of stacks over **An** is *representable* if for any analytic space $\mathcal{X}$ and any 1-morphism $\mathcal{X} \to \mathcal{V}$, the 2-fiber product $\mathcal{U} \times_{\mathcal{V}} \mathcal{X}$ is representable.

If P is a property of morphism of analytic spaces that is stable under base change (e.g. étale, surjective, separated, flat, proper), then a 1-morphism $\mathcal{U} \to \mathcal{V}$ of stacks over **An** has P if for any analytic space $\mathcal{X}$ and any 1-morphism $\mathcal{X} \to \mathcal{V}$, the morphism of analytic spaces $\mathcal{U} \times_{\mathcal{V}} \mathcal{X} \to \mathcal{X}$ has P.

Definition 2.1. *An* analytic Deligne–Mumford stack $\mathcal{X}$ *is a stack over* **An** *such that:*

(1) *the diagonal morphism* $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ *is representable and separated;*
(2) *there exists an analytic space* $\mathcal{U}$ *and a* 1*-morphism* $\mathcal{U} \to \mathcal{X}$ *which is representable by surjective and étale morphisms.*

Note that in the definition of an analytic Deligne–Mumford stack—unlike the case of an algebraic Deligne–Mumford stack (c.f. §1.1)—we make no compactness assumptions on the diagonal. If we were to do so, all our analytic Deligne–Mumford stacks would be forced to be separated, which is insufficiently general for the purposes of this article.

For an analytic Deligne–Mumford stack $\mathcal{X}$, define $|\mathcal{X}|$ to be the set of isomorphism classes of the groupoid $\mathcal{X}(*)$ (where $*$ denotes the analytic space consisting of a single reduced point). The collection of all open analytic substacks of $\mathcal{X}$ defines the analytic topology on the set $|\mathcal{X}|$.

A morphism of analytic Deligne–Mumford stacks $\mathcal{X} \to \mathcal{Y}$ is *locally quasi-finite* if the continuous morphism $|\mathcal{X}| \to |\mathcal{Y}|$ has discrete fibers.

A morphism of analytic Deligne–Mumford stacks is *separated* if its diagonal 1-morphism is representable by finite morphisms. An analytic Deligne–Mumford stack is *proper* if it is separated and the topological space $|\mathcal{X}|$ is compact.

Let $\mathcal{U}$ be an analytic space. The theory of the abelian category $\mathbf{Coh}(\mathcal{U})$ of coherent analytic sheaves is well-covered in the classic text of Grauert–Remmert [GR84]. We now outline some variations to this theory so that a sliver of it may be applied to the analytic Deligne–Mumford stacks of this article.

An analytic Deligne–Mumford stack $\mathcal{X}$ has an associated small étale site, which we denote as $\mathcal{X}_{\text{ét}}$. The objects of this site are 1-morphisms $\mathcal{U} \to \mathcal{X}$ representable by étale morphisms, where $\mathcal{U}$ is representable; morphisms and covering families in this site are as to be expected. If $x \in |\mathcal{X}|$ and $\mathcal{F}$ is a sheaf on $\mathcal{X}_{\text{ét}}$ we let $\mathcal{F}_x$ denote the *stalk* at x. Set

$$|\operatorname{supp} \mathcal{F}| = \{x \in |\mathcal{X}| \,:\, \mathcal{F}_x \neq 0\}.$$

Note that $\mathcal{F} \cong 0$ if and only if $|\operatorname{supp} \mathcal{F}| = \emptyset$.

The site $\mathcal{X}_{\text{ét}}$ is naturally ringed, and we denote this sheaf of rings by $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$. If $\mathcal{X}$ is representable, then the natural functor $\mathbf{Sh}(\mathcal{X}_{\text{ét}}) \to \mathbf{Sh}(\mathcal{X})$ (resp. $\mathbf{Mod}(\mathcal{X}_{\text{ét}}) \to \mathbf{Mod}(\mathcal{X})$) is an equivalence of categories. We will make use of these equivalences without further mention.

A sheaf of $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$-modules $\mathcal{F}$ is *finitely generated* if there exists a covering family $(j_i : \mathcal{U}_i \to \mathcal{X})_{i \in I}$ such that for each $i \in I$ there is an integer n_i and surjections of $\mathcal{O}_{(\mathcal{U}_i)_{\text{ét}}}$-modules $\mathcal{O}_{(\mathcal{U}_i)_{\text{ét}}}^{\oplus n_i} \to j_i^{-1}\mathcal{F}$. A sheaf of $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$-modules $\mathcal{F}$ is *coherent* if it is finitely generated, and for any $(j : \mathcal{U} \to \mathcal{X}) \in \mathcal{X}_{\text{ét}}$ and integer n, any $\mathcal{O}_{\mathcal{U}_{\text{ét}}}$-module homomorphism $\mathcal{O}_{\mathcal{U}_{\text{ét}}}^{\oplus n} \to j^{-1}\mathcal{F}$ has finitely generated kernel. Let $\mathbf{Coh}(\mathcal{X}_{\text{ét}})$ denote the full subcategory of $\mathbf{Mod}(\mathcal{X}_{\text{ét}})$ having those objects which are coherent. Certainly, coherence is local for the étale topology and is stable under pullbacks. Also, if $\mathcal{X}$ is representable, then the natural functor $\mathbf{Coh}(\mathcal{X}_{\text{ét}}) \to \mathbf{Coh}(\mathcal{X})$ is an equivalence of categories. By [FAC, Thm. 1] we immediately deduce that $\mathbf{Coh}(\mathcal{X}_{\text{ét}}) \subset \mathbf{Mod}(\mathcal{X}_{\text{ét}})$ is a full abelian subcategory closed under extensions. Since there is now no possibility for confusion, we will write $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$ (resp. $\mathbf{Coh}(\mathcal{X}_{\text{ét}})$) as $\mathcal{O}_{\mathcal{X}}$ (resp. $\mathbf{Coh}(\mathcal{X})$) whether $\mathcal{X}$ is representable or not.

Theorem 2.2. *Let $\mathcal{X}$ be an analytic Deligne–Mumford stack.*

(1) *Let $x \in |\mathcal{X}|$, then $\mathcal{O}_{\mathcal{X},x}$ is a noetherian local ring.*
(2) *(Oka's Theorem) $\mathcal{O}_{\mathcal{X}}$ is coherent.*
(3) *(Direct Image Theorem) If $f : \mathcal{X} \to \mathcal{Y}$ is a proper and representable morphism of analytic Deligne–Mumford stacks and $\mathcal{F} \in \mathbf{Coh}(\mathcal{X})$, then $f_*\mathcal{F} \in \mathbf{Coh}(\mathcal{Y})$.*

Proof. Claims 1 and 2 are local for the étale topology on $\mathcal{X}$, thus follow from the corresponding results for analytic spaces [GR84, 2.2.1 and 2.5.3]. Claim 3 is local for the étale topology on $\mathcal{Y}$, so we are now reduced to the case where f is a morphism of analytic spaces, and the claim follows from [GR84, 10.5.6]. □

Just as in the case for analytic spaces [GR84, 1.2.2], for an analytic Deligne–Mumford stack $\mathcal{X}$ and a coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-ideals $\mathcal{I}$, there is an associated closed analytic substack $V(\mathcal{I}) \hookrightarrow \mathcal{X}$ with the property that $|V(\mathcal{I})| = |\operatorname{supp}(\mathcal{O}_{\mathcal{X}}/\mathcal{I})|$ and $\mathcal{O}_{V(\mathcal{I})} = \mathcal{O}_{\mathcal{X}}/\mathcal{I}$. The following Lemma is a Nullstellensatz type result which will be used frequently.

Lemma 2.3. *Fix a proper analytic Deligne–Mumford stack $\mathcal{X}$, a coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ and a coherent $\mathcal{O}_{\mathcal{X}}$-ideal $\mathcal{I}$.*

(1) *If $|\operatorname{supp} \mathcal{F}| \subset |V(\mathcal{I})|$, then there exists $k > 0$ such that $\mathcal{I}^k\mathcal{F} = 0$.*
(2) *Given a coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ such that $|\operatorname{supp} \mathcal{F}'| \subset |V(\mathcal{I})|$, then there exists $k > 0$ such that $(\mathcal{I}^k\mathcal{F}) \cap \mathcal{F}' = 0$.*

Proof. For 1, the Rückert Nullstellansatz [GR84, §3.2] shows that for any $x \in |\mathcal{X}|$, there is an étale neighborhood $(\mathcal{U}_x, u_x) \to (\mathcal{X}, x)$ and a positive integer k_x such that $(\mathcal{I}^{k_x}\mathcal{F})_{\mathcal{U}_x} = 0$. The compactness of $|\mathcal{X}|$ now gives the result. For 2, by 1 we are free to assume that $\mathcal{I}\mathcal{F}' = 0$. Now fix $x \in |\mathcal{X}|$ and observe that the local ring $\mathcal{O}_{\mathcal{X},x}$ is noetherian (Theorem 2.21), so by [AM69, Cor. 10.10] there exists a positive integer k_x such that $(\mathcal{I}_x^{k_x}\mathcal{F}_x) \cap \mathcal{F}'_x = 0$. Note that for any $x \in |\mathcal{X}|$, the sheaf of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{G}^x = (\mathcal{I}^{k_x}\mathcal{F}) \cap \mathcal{F}'$ is coherent, and has closed support. As $(\mathcal{G}^x)_x = 0$, there is an open neighborhood $\mathcal{U}_x$ of $x \in |\mathcal{X}|$ such that $\mathcal{G}^x \mid_{\mathcal{U}_x} = 0$. The compactness of $|\mathcal{X}|$ now gives the claim. □

Fix an algebraic Deligne–Mumford stack X and let $U \to X$ be an étale cover by a scheme. Set $R = U \times_X U$, and define X_{an} to be the quotient stack $[R_{\mathrm{an}} \rightrightarrows U_{\mathrm{an}}]$ in the 2-category of stacks over $\mathbf{An}$. Arguing as in [LMB, 4.3.1], one readily deduces that X_{an} is an analytic Deligne–Mumford stack and is independent of the covering $U \to X$. We call X_{an} the *analytification* of X and this assignment can be made functorial. On the level of sets we have $|X_{\mathrm{an}}| = |X(\operatorname{Spec}\mathbb{C})|$. It is readily seen—using arguments similar to [SGA1, XII.3.1–2], with the aid of [LMB, 16.6]—that an algebraic Deligne–Mumford stack is proper if and only if its analytification is so. Similarly, a morphism of algebraic Deligne–Mumford stacks is locally quasi-finite (resp. separated, representable, surjective, etc.) if and only if its analytification is so.

If X is an algebraic Deligne–Mumford stack, there is also an analytification functor:

$$\mathbf{Coh}\,(X) \to \mathbf{Coh}\,(X_{\mathrm{an}}) : F \mapsto F_{\mathrm{an}}$$

Observe that if $I \lhd \mathcal{O}_X$ is a coherent sheaf of ideals, then $(V(I))_{\mathrm{an}} = V(I_{\mathrm{an}})$. For an algebraic (resp. analytic) Deligne–Mumford stack X which is separated, let $\mathbf{Coh}_p(X)$ denote the category coherent $\mathcal{O}_X$-modules with proper support. If $M \in \mathbf{Coh}_p(X)$, then $M_{\mathrm{an}} \in \mathbf{Coh}_p(X_{\mathrm{an}})$. We conclude this section with some GAGA results for separated Deligne–Mumford stacks, which are marginally stronger than [Toë99, 5.10] (where they are proved when X is proper).

Theorem 2.4. *Let X be an algebraic Deligne–Mumford stack. If X is separated, then the analytification functor:*

$$\mathbf{Coh}_p(X) \to \mathbf{Coh}_p(X_{\mathrm{an}})$$

is an equivalence of categories.

Proof. We give the necessary modifications to the arguments of [Toë99, 5.10]. Fix $G \in \mathbf{Coh}_p(X)$, then there is a closed immersion $\imath : W \hookrightarrow X$ with W proper such that the adjunction $G \to i_*i^*G$ is an isomorphism. Thus, given $F \in \mathbf{Coh}_p(X)$, we have natural bijections:

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{O}_X}(F,G) &\cong \operatorname{Hom}_{\mathcal{O}_X}(F, i_*i^*G) \cong \operatorname{Hom}_{\mathcal{O}_W}(i^*F, i^*G) \\
&\cong \operatorname{Hom}_{\mathcal{O}_{W_{\mathrm{an}}}}((i^*F)_{\mathrm{an}}, (i^*G)_{\mathrm{an}}) && \text{[Toë99, 5.10]} \\
&\cong \operatorname{Hom}_{\mathcal{O}_{W_{\mathrm{an}}}}(i_{\mathrm{an}}^*F_{\mathrm{an}}, (i^*G)_{\mathrm{an}}) && \text{[Toë99, 5.12]} \\
&\cong \operatorname{Hom}_{\mathcal{O}_{X_{\mathrm{an}}}}(F_{\mathrm{an}}, (i_{\mathrm{an}})_*(i^*G)_{\mathrm{an}}) \\
&\cong \operatorname{Hom}_{\mathcal{O}_{X_{\mathrm{an}}}}(F_{\mathrm{an}}, (i_*i^*G)_{\mathrm{an}}) && \text{[Toë99, 5.11]} \\
&\cong \operatorname{Hom}_{\mathcal{O}_{X_{\mathrm{an}}}}(F_{\mathrm{an}}, G_{\mathrm{an}}).
\end{aligned}$$

Thus, we have proved that the analytification functor is fully faithful. For the essential surjectivity, let $\mathcal{F} \in \mathbf{Coh}_p(X_{\mathrm{an}})$ and take O_X to be the category of quasi-compact open subsets of X. We note that $\{U\}_{U \in O_X}$ is an open cover of X and so

$\{|U_{\mathrm{an}}| \cap |\operatorname{supp}\mathcal{F}|\}_{U\in O_X}$ is an open cover of $|\operatorname{supp}\mathcal{F}|$. By assumption $|\operatorname{supp}\mathcal{F}|$ is compact, so $|\operatorname{supp}\mathcal{F}| \subset |U_{\mathrm{an}}|$ for some open immersion $j : U \to X$ with U quasicompact. Thus the adjunction $(j_{\mathrm{an}})_! j_{\mathrm{an}}^* \mathcal{F} \to \mathcal{F}$ is an isomorphism, $j_{\mathrm{an}}^*\mathcal{F} \in \mathbf{Coh}_p(U_{\mathrm{an}})$, and as $(j_{\mathrm{an}})_! H_{\mathrm{an}} = (j_! H)_{\mathrm{an}}$ for all $\mathcal{O}_U$-modules H (easily verified on stalks), we deduce that it is sufficient to prove the essential surjectivity of the analytification functor $\mathcal{A}_{X,p}$ when X is, in addition, quasicompact. The Chow Lemma for separated Deligne–Mumford stacks [LMB, 16.6.1] and the arguments of [Toë99, 5.10] immediately reduce us to the situation where X is a quasiprojective scheme. In [SGA1, XII.4.4], the analytification functor $\mathcal{A}_{X,p}$ is proved to be an equivalence when X is a proper scheme. To deduce the quasiprojective case from the projective case, one argues analogously to [EGA, III.5.2.6]. □

We now obtain two easy corollaries. First, we have Theorem D in the separated case.

Corollary 2.5. *Let X be an algebraic Deligne–Mumford stack which is separated. If $\sigma : \mathcal{Z} \to X_{\mathrm{an}}$ is finite with $\mathcal{Z}$ proper, then there exists a finite morphism of algebraic Deligne–Mumford stacks $s : Z \to X$, a 1-isomorphism $f : Z_{\mathrm{an}} \to \mathcal{Z}$, and a 2-morphism $\alpha_f : s_{\mathrm{an}} \to \sigma \circ f$.*

Second, we have Theorem C in the separated case.

Corollary 2.6. *Fix algebraic Deligne–Mumford stacks Z and X over $\mathbb{C}$. Suppose that Z is proper and X is separated. Then, the analytification functor:*

$$\operatorname{Hom}(Z, X) \to \operatorname{Hom}(Z_{\mathrm{an}}, X_{\mathrm{an}})$$

induces an equivalence of categories.

With Corollary 2.6 at our disposal, we can also prove the full-faithfulness of Theorem D. Before we do this, however, we will need to give some precise definitions.

For an algebraic (resp. analytic) Deligne–Mumford stack X, let $\mathbf{QF}(X)$ denote the category of 1-morphisms $(Z \xrightarrow{s} X)$ which are locally quasi-finite, separated, and representable. A morphism $(f, \alpha_f) : (Z' \xrightarrow{s'} X) \to (Z \xrightarrow{s} X)$ in $\mathbf{QF}(X)$ consists of a 1-morphism $f : Z' \to Z$ together with a 2-morphism $\alpha_f : s' \Rightarrow s \circ f$ over X. We will frequently contract this to "$f : Z' \to Z$ is a morphism in $\mathbf{QF}(X)$". Let $\mathbf{QF}_p(X)$ denote the full subcategory $\mathbf{QF}(X)$ with objects those $(Z \to X)$ such that *Z is proper.* If X is an algebraic Deligne–Mumford stack, there is an analytification functor:

$$\Psi_X : \mathbf{QF}(X) \to \mathbf{QF}(X_{\mathrm{an}}).$$

The analytification functor Ψ_X sends $\mathbf{QF}_p(X)$ to $\mathbf{QF}_p(X_{\mathrm{an}})$, and we denote this restriction by $\Psi_{X,p}$. An object $(\mathcal{Z} \xrightarrow{\sigma} X_{\mathrm{an}}) \in \mathbf{QF}(X_{\mathrm{an}})$ (resp. $\mathbf{QF}_p(X_{\mathrm{an}})$) is *algebraizable* if it lies in the essential image of the analytification functor Ψ_X (resp. $\Psi_{X,p}$).

Lemma 2.7. *Fix an algebraic Deligne–Mumford stack X. Then, the analytification functor:*

$$\Psi_{X,p} : \mathbf{QF}_p(X) \to \mathbf{QF}_p(X_{\mathrm{an}})$$

is fully faithful.

Proof. For $i = 1, 2$ let $(Z^i \xrightarrow{s^i} X) \in \mathbf{QF}_p(X)$, then:

$$\operatorname{Hom}_{\mathbf{QF}(X)}((Z^1 \to X), (Z^2 \to X)) = \operatorname{Hom}_X(Z^1, Z^2) = \{Z^1 \xrightarrow{t} Z^1 \times_X Z^2 \,:\, (s^1, s^2) \circ t = \mathrm{id}_{Z^1}\}.$$

For $i = 1,2$ the morphisms s^i are separated and representable, thus $Z^1 \times_X Z^2$ is separated. Moreover, Z^1 is proper, so Corollary 2.6 now demonstrates that:

$$\{Z^1 \xrightarrow{t} Z^1 \times_X Z^2 \,:\, (s^1, s^2) \circ t = \mathrm{id}_{Z^1}\} = \{Z^1_{\mathrm{an}} \xrightarrow{\tau} Z^1_{\mathrm{an}} \times_{X_{\mathrm{an}}} Z^2_{\mathrm{an}} \,:\, (s^1_{\mathrm{an}}, s^2_{\mathrm{an}}) \circ \tau = \mathrm{id}_{Z^1_{\mathrm{an}}}\}.$$

As before, however:

$$\{Z^1_{\mathrm{an}} \xrightarrow{\tau} Z^1_{\mathrm{an}} \times_{X_{\mathrm{an}}} Z^2_{\mathrm{an}} \,:\, (s^1_{\mathrm{an}}, s^2_{\mathrm{an}}) \circ \tau = \mathrm{id}_{Z^1_{\mathrm{an}}}\} = \mathrm{Hom}_{X_{\mathrm{an}}}(Z^1_{\mathrm{an}}, Z^2_{\mathrm{an}}),$$

and we deduce the claim. □

3. Counterexamples

First, we give an example showing that Theorem A is false for families of curves of genus 1.

Example 3.1. *Let H be the Hopf surface: it is the quotient of $\mathbb{C}^2 - \{0\}$ by the free $\mathbb{Z}$-action $(z_1, z_2) \mapsto (\frac{1}{2}z_1, \frac{1}{2}z_2)$. The surface H is proper and is an elliptic fiber space over $\mathbb{CP}^1$, but is not algebraizable* [BHPV04, V.18]. *In particular, this gives a family of smooth analytic curves over a projective base which is not algebraizable.*

Our next example shows that the separatedness assumption in Theorem 2.4 is essential—even for smooth and universally closed schemes. The argument given is a variation of [Har77, B.2.0.1].

Example 3.2. *Let E be an elliptic curve. Since $\mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{O}_E, \mathcal{O}_E) \cong H^1(E, \mathcal{O}_E) \neq 0$, there exists a non-split extension of $\mathcal{O}_E$ by $\mathcal{O}_E$, which we denote as $\mathcal{E}$. Set $Y = \mathbb{P}(\mathcal{E})$, then* [Har77, V.2.3] *shows that $\mathrm{Pic}\, E$ is a direct summand of $\mathrm{Pic}\, Y$. Since E is an elliptic curve, $\mathrm{Pic}\, E$ is uncountable, thus $\mathrm{Pic}\, Y$ is uncountable.*

Now let $E_0 \subset Y$ denote the unique section of $Y \to E$ with self-intersection number $E_0^2 = 0$ [Har77, V.2.8.1]. *Take $U = Y \setminus E_0$ and denote the resulting open immersion by $i : U \hookrightarrow Y$. By* [Har77, II.6.6 & II.6.16], *$\mathrm{Pic}\, Y \to \mathrm{Pic}\, U$ is surjective with countable kernel, so $\mathrm{Pic}\, U$ is also uncountable. It is shown in* [Har70, pp. 232–234], *however, that there is an analytic isomorphism $U_{\mathrm{an}} \cong \mathbb{C}^\times \times \mathbb{C}^\times$, thus $\mathrm{Pic}\, U_{\mathrm{an}} \cong \mathbb{Z} \times \mathbb{Z}$ is countable. We deduce that the morphism of abelian groups $\mathrm{Pic}\, U \to \mathrm{Pic}\, U_{\mathrm{an}}$ has uncountable kernel and is consequently non-empty. Since $\mathrm{Pic}\, Y \to \mathrm{Pic}\, U$ is surjective, it follows that there are two algebraic line bundles L and M on Y, such that the analytic line bundles $i^*_{\mathrm{an}} L_{\mathrm{an}}$ and $i^*_{\mathrm{an}} M_{\mathrm{an}}$ are analytically isomorphic, but the algebraic line bundles $i^* L$ and $i^* M$ are not algebraically isomorphic.*

Define the smooth, universally closed, and finite type $\mathbb{C}$-scheme X by gluing two copies of Y along U. Let $j_1,\ j_2 : Y \rightrightarrows X$ denote the two different inclusions of Y into X. By gluing, we obtain an analytic line bundle $\mathcal{F}$ on X_{an} such that $(j_1)^*_{\mathrm{an}} \mathcal{F} \cong \mathcal{L}_{\mathrm{an}}$ and $(j_2)^*_{\mathrm{an}} \mathcal{F} \cong \mathcal{M}_{\mathrm{an}}$. If $\mathcal{F}$ is algebraizable, then there is a coherent $\mathcal{O}_Y$-module F together with an analytic isomorphism $F_{\mathrm{an}} \cong \mathcal{F}$. In particular, we see that there are induced analytic isomorphisms of coherent sheaves on Y_{an}:

$$(j_1^* F)_{\mathrm{an}} \cong (j_1)^*_{\mathrm{an}} \mathcal{F} \cong L_{\mathrm{an}} \qquad \textit{and} \qquad (j_2^* F)_{\mathrm{an}} \cong (j_2)^*_{\mathrm{an}} \mathcal{F} \cong M_{\mathrm{an}}.$$

Since Y is projective, by GAGA [SGA1, XII.4.4], *the induced isomorphism $(j_1^* F)_{\mathrm{an}} \cong L_{\mathrm{an}}$ (resp. $(j_1^* F)_{\mathrm{an}} \cong L_{\mathrm{an}}$) is uniquely algebraizable to an algebraic isomorphism $j_1^* F \cong L$ (resp. $j_2^* F \cong M$). However, this implies there is an algebraic isomorphism:*

$$i^* L \cong i^* j_1^* F \cong i^* j_2^* F \cong i^* M,$$

which is a contradiction. Hence, the analytic line bundle $\mathcal{F}$ is not algebraizable. Thus, the analytification functor $\mathbf{Coh}(X) \to \mathbf{Coh}(X_{\mathrm{an}})$ is not necessarily essentially surjective for universally closed but not separated schemes.

Our last example shows that Corollary 2.5 cannot be extended to non-separated schemes.

Example 3.3. *Let X and $\mathcal{F}$ be as in Example 3.2. Set $\mathcal{Z}$ to be the analytic space with $|\mathcal{Z}| = |X_{\mathrm{an}}|$ and $\mathcal{O}_{\mathcal{Z}} = \mathcal{O}_{X_{\mathrm{an}}} \oplus \mathcal{F}$, with the multiplication map $(x,f)(x',f') = (xx', xf' + x'f)$. The finite map of analytic spaces $\mathcal{Z} \to X_{\mathrm{an}}$ is not algebraizable.*

4. Pushouts

The main result of this is the following dévissage lemma (Lemma 4.1) which is fundamental to our proof of Theorem D.

Lemma 4.1. *Fix an algebraic Deligne–Mumford stack X. Consider a finite and surjective $\mathbf{QF}_p(X_{\mathrm{an}})$-morphism $f : (\mathcal{Z}' \xrightarrow{\sigma'} X_{\mathrm{an}}) \to (\mathcal{Z} \xrightarrow{\sigma} X_{\mathrm{an}})$ with $(\mathcal{Z}' \xrightarrow{\sigma'} X_{\mathrm{an}}) \in \mathbf{QF}_p(X_{\mathrm{an}})$ algebraizable. Fix a Zariski closed subset $|Q| \subset |X|$ and suppose that any closed analytic substack of $\mathcal{Z}$ supported in $\sigma^{-1}|Q_{\mathrm{an}}|$ is algebraizable. If $\ker f^\sharp$ and $\operatorname{coker} f^\sharp$ have support contained in $\sigma^{-1}|Q_{\mathrm{an}}|$, then $(\mathcal{Z} \xrightarrow{\sigma} X_{\mathrm{an}}) \in \mathbf{QF}_p(X_{\mathrm{an}})$ is algebraizable.*

We will prove Lemma 4.1 by forming finite colimits in the category $\mathbf{QF}(X)$ along finite morphisms. In other words, we will glue stacks which are locally quasi-finite and representable over an algebraic or analytic Deligne–Mumford stack along finite morphisms. For this, we require several preliminaries.

Let X be an algebraic Deligne–Mumford stack and fix $(Z^i \xrightarrow{s^i} X) \in \mathbf{QF}(X)$ for $i = 1, 2, 3$. In addition, fix finite morphisms $t^j : Z^3 \to Z^j$ for $j = 1, 2$ in $\mathbf{QF}(X)$. It was shown in [HalR10, Thm. 2.10], that there is a cocartesian diagram in $\mathbf{QF}(X)$:

$$\begin{tikzcd} Z^3 \arrow[r, "t^1"] \arrow[d, "t^2"'] & Z^1 \arrow[d, "m^1"] \\ Z^2 \arrow[r, "m^2"] & Z^4, \end{tikzcd} \tag{1}$$

and the morphisms $m^i : Z^i \to Z^4$ in $\mathbf{QF}(X)$ are finite. It was also shown [*loc. cit.*], that the Zariski topological space $|Z^4|$ is the colimit of the diagram of topological spaces $[|Z^1| \xleftarrow{t^1} |Z^3| \xrightarrow{t^2} |Z^2|]$, and that there is an isomorphism of coherent sheaves $\mathcal{O}_{Z^4} \to m^1_*\mathcal{O}_{Z^1} \times_{m^3_*\mathcal{O}_{Z^3}} m^2_*\mathcal{O}_{Z^2}$. Also, if for $i = 1, 2$ we have $(Z^i \xrightarrow{s^i} X) \in \mathbf{QF}_p(X)$, then $(Z^4 \xrightarrow{s^4} X) \in \mathbf{QF}_p(X)$.

We would now like to investigate the behaviour of the diagram (1) under analytification. This requires some deliberations on pushouts in the category of analytic spaces.

Given a diagram of ringed spaces $E := [V^1 \leftarrow V^3 \to V^2]$, let the topological space $|V^4|$ be the colimit of the induced diagram $|E| := [|V^1| \leftarrow |V^3| \to |V^2|]$ in the category of topological spaces. We have induced maps $n^i : |V^i| \to |V^4|$, and the colimit of the diagram E in the category of ringed spaces is the ringed space $V^4 := (|V^4|, n^1_*\mathcal{O}_{V^1} \times_{n^3_*\mathcal{O}_{V^3}} n^2_*\mathcal{O}_{V^2})$. The morphisms of topological spaces

$n^i : |V^i| \to |V^4|$ together with the projections $\mathcal{O}_{Z_4} \to n^i_* \mathcal{O}_{V^i}$ induce morphisms of ringed spaces $n^i : V^i \to V^4$.

If the ringed spaces V^i are locally ringed, and the maps n^i are morphisms of locally ringed spaces, then V^4 is the colimit of the diagram E in the category of *locally* ringed spaces. Indeed, given a locally ringed space W and for $i = 1, 2, 3$ compatible morphisms of locally ringed spaces $w^i : V^i \to W$, there is a uniquely induced morphism of *ringed* spaces $w^4 : V^4 \to W$. It remains to show that w^4 is a morphism of locally ringed spaces. Let $z \in |V^4|$ have image $w \in |W|$, then we have an induced morphism of local rings $\mathcal{O}_{W,w} \to \mathcal{O}_{V^4,z}$ which we must show is local. The morphism $|V^1| \amalg |V^2| \to |V^4|$ is surjective, so there exists some $i \in \{1, 2\}$ and $z^i \in |V^i|$ such that $z = n^i(z^i)$. We now have a triple of morphisms of local rings $\mathcal{O}_{W,w} \to \mathcal{O}_{V^4,z} \to \mathcal{O}_{V^i,z^i}$ and by hypothesis, the morphisms $\mathcal{O}_{W,w} \to \mathcal{O}_{V^i,z^i}$ and $\mathcal{O}_{V^4,z} \to \mathcal{O}_{V^i,z^i}$ are local. The claim is now a consequence of the following general observation: given a triple of morphisms of local rings $(A, \mathfrak{m}_A) \xrightarrow{f} (B, \mathfrak{m}_B) \xrightarrow{g} (C, \mathfrak{m}_C)$ such that gf and g are local, then f is local.

Thus, to show that a scheme (resp. analytic space) is a pushout of some schemes (resp. analytic spaces), it will suffice to show that it is the pushout in the category of ringed spaces, and the maps involved are all maps of schemes (resp. analytic spaces). In the lemma that follows we retain the notation of diagram (1).

Lemma 4.2. *Let X be an algebraic Deligne–Mumford stack. Then, the commutative diagram in $\mathbf{QF}(X_{\mathrm{an}})$:*

$$\begin{tikzcd} Z^3_{\mathrm{an}} \arrow[r, "t^1_{\mathrm{an}}"] \arrow[d, "t^2_{\mathrm{an}}"'] & Z^1_{\mathrm{an}} \arrow[d, "m^1_{\mathrm{an}}"] \\ Z^2_{\mathrm{an}} \arrow[r, "m^2_{\mathrm{an}}"] & Z^4_{\mathrm{an}}, \end{tikzcd}$$

is cocartesian and remains so after flat and representable base change on X_{an}. Thus all *finite colimits along finite morphisms exist in $\mathbf{QF}(X)$, are preserved under analytification, and remain colimits under flat and representable base change on X_{an}.*

Proof. The latter claim is consequence of the fact that finite colimits can be built from finite disjoint unions and pushouts, so it suffices to prove the former.

Set $m^3 := m^1 \circ t^1$. First, we assume that X is a scheme. For $l = 1, 2, 3$ observe that because the morphism $m^l : Z^l \to Z^4$ is finite, the natural morphism of $\mathcal{O}_{Z^4_{\mathrm{an}}}$-modules $(m^l_* \mathcal{O}_{Z^l})_{\mathrm{an}} \to (m^l_{\mathrm{an}})_* \mathcal{O}_{Z^l_{\mathrm{an}}}$ is an isomorphism [SGA1, XII.4.2]. By [SGA1, XII.1.3.1], the analytification functor $\mathbf{Coh}\,(Z^4) \to \mathbf{Coh}\,(Z^4_{\mathrm{an}})$ is also exact. Thus we have a natural sequence of isomorphisms of coherent $\mathcal{O}_{Z^4_{\mathrm{an}}}$-modules:

$$\begin{aligned} \mathcal{O}_{Z^4_{\mathrm{an}}} &\cong (\mathcal{O}_{Z^4})_{\mathrm{an}} \cong (m^1_* \mathcal{O}_{Z^1} \times_{m^3_* \mathcal{O}_{Z^3}} m^2_* \mathcal{O}_{Z^2})_{\mathrm{an}} \cong (m^1_* \mathcal{O}_{Z^1})_{\mathrm{an}} \times_{(m^3_* \mathcal{O}_{Z^3})_{\mathrm{an}}} (m^2_* \mathcal{O}_{Z^2})_{\mathrm{an}} \\ &\cong (m^1_{\mathrm{an}})_* \mathcal{O}_{Z^1_{\mathrm{an}}} \times_{(m^3_{\mathrm{an}})_* \mathcal{O}_{Z^3_{\mathrm{an}}}} (m^2_{\mathrm{an}})_* \mathcal{O}_{Z^2_{\mathrm{an}}}. \end{aligned}$$

Hence, we conclude that Z^4_{an} is the colimit of the diagram in the category of ringed spaces, and remains so after flat base change on X. It is clear that this implies that Z^4_{an} is the colimit in the category of analytic spaces thus the diagram is also cocartesian in $\mathbf{QF}(X_{\mathrm{an}})$.

Next, we assume that X is an algebraic Deligne–Mumford stack. Let $X_1 \to X$ be an étale cover by a scheme and set $X_2 = X_1 \times_X X_1$ (which is also a scheme).

Also, set $Z_i^k = Z^k \times_X X_i$ for $i = 1$, 2 and $k = 1$, 2, 3, 4. By the case of schemes already considered, we know for $i = 1$ and 2 that the analytic space $(Z_i^4)_{\mathrm{an}}$ is the colimit of the diagram $[(Z_i^1)_{\mathrm{an}} \leftarrow (Z_i^3)_{\mathrm{an}} \to (Z_i^2)_{\mathrm{an}}]$ in the category of analytic spaces. The universal properties furnish us with an étale groupoid of analytic spaces $[(Z_2^4)_{\mathrm{an}} \rightrightarrows (Z_1^4)_{\mathrm{an}}]$, with quotient in the category of analytic Deligne–Mumford stacks Z_{an}^4. The universal property of the pushouts and étale descent immediately imply that $(s_{\mathrm{an}}^4 : Z_{\mathrm{an}}^4 \to X_{\mathrm{an}}) \in \mathbf{QF}(X_{\mathrm{an}})$ is the relevant colimit and is also preserved after representable and flat base change on X_{an}. □

We now recall some general facts about conductors. Fix a morphism of rings $\phi : A \to B$, define the *conductor* $\mathfrak{C}_\phi$ of ϕ to be the image of the A-ideal $\mathrm{Ann}_A(\operatorname{coker}\phi)$ in B. A trivial calculation shows $\mathfrak{C}_\phi$ is a B-ideal. Also, if I is an A-ideal such that $\widetilde{I} := IB \subset \mathfrak{C}_\phi$, then another easy calculation proves that $\widetilde{I} \subseteq \operatorname{im}\phi$. In particular, if ϕ is injective, then $\widetilde{I}$ is also an A-ideal and the resulting *conductor square*:

$$\begin{tikzcd} B/\widetilde{I} & B \arrow[l] \\ A/\widetilde{I} \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd} \tag{2}$$

is cartesian. The power of the conductor square lies in the fact that the rings B, $A/\widetilde{I}$, and $B/\widetilde{I}$ can frequently be chosen to be better behaved than A.

The above deliberations can be variedf to handle a finite morphism of algebraic or analytic Deligne–Mumford stacks $f : Z' \to Z$. Define the *conductor* $\mathfrak{C}_f$ of f as the image of the $\mathcal{O}_Z$-ideal $\mathrm{Ann}_{\mathcal{O}_Z}(\operatorname{coker} f^\sharp)$ in $f_*\mathcal{O}_{Z'}$. As before, $\mathfrak{C}_f$ is also an $f_*\mathcal{O}_{Z'}$-ideal. If $I \lhd \mathcal{O}_Z$ is a coherent ideal such that $I \subset \mathfrak{C}_f$, then the image of I in $f_*\mathcal{O}_{Z'}$ generates a $f_*\mathcal{O}_{Z'}$-ideal $\widetilde{I} \subset \mathfrak{C}_f$, contained in the image of $\mathcal{O}_Z$. In particular, if $f^\sharp$ is injective, then $\widetilde{I}$ is, in addition, a coherent $\mathcal{O}_Z$-ideal containing I. For a coherent ideal $J \lhd \mathcal{O}_Z$, let $J_{Z'}$ denote the $\mathcal{O}_{Z'}$-ideal generated by f^*J (if f is flat, then $f^*J = J_{Z'}$). Since f is finite, the inclusion of $\mathcal{O}_Z$-ideals $I \subset \widetilde{I}$ induces an equality of $\mathcal{O}_{Z'}$-ideals $I_{Z'} = \widetilde{I}_{Z'}$. We now extend the conductor square (2) to analytic Deligne–Mumford stacks.

Lemma 4.3. *Let $\mathcal{X}$ be an analytic Deligne–Mumford stack and consider a morphism $f : \mathcal{Z}' \to \mathcal{Z}$ in $\mathbf{QF}(\mathcal{X})$ which is finite and $f^\sharp : \mathcal{O}_{\mathcal{Z}} \to f_*\mathcal{O}_{\mathcal{Z}'}$ is injective. Fix a coherent ideal $\mathcal{I} \lhd \mathcal{O}_{\mathcal{Z}}$ such that $\mathcal{I} \subset \mathfrak{C}_f$. Then, the following diagram in $\mathbf{QF}(\mathcal{X})$ is cocartesian:*

$$\begin{tikzcd} V(\mathcal{I}_{\mathcal{Z}'}) \arrow[r, hook] \arrow[d] & \mathcal{Z}' \arrow[d, "f"] \\ V(\widetilde{\mathcal{I}}) \arrow[r, hook] & \mathcal{Z}. \end{tikzcd}$$

Proof. First, we assume that $\mathcal{X}$ is an analytic space, then we will show that the diagram above is cocartesian in the category of locally ringed spaces, thus in the category of analytic spaces. We will use the criterion of [Fer03, Sc. 4.3(b)]. Note that from the associated cartesian conductor square for *rings* (2), it suffices to show that $\mathcal{Z}$ has the correct topological space. Since $f^\sharp$ is injective and f is finite, then f is surjective and closed, thus submersive. Let $\mathcal{U} = \mathcal{Z} - V(\widetilde{\mathcal{I}})$ and $\mathcal{U}' = \mathcal{Z}' - V(\mathcal{I}_{\mathcal{Z}'})$. It remains to show that f induces a bijection of sets $\mathcal{U}' \to \mathcal{U}$. Since $\widetilde{\mathcal{I}} \subset \mathfrak{C}_f$, then for $u \in \mathcal{U}$ we have that the map $f_u^\sharp : \mathcal{O}_{\mathcal{Z},u} \to (f_*\mathcal{O}_{\mathcal{Z}'})_u$ is a bijection. Thus, since f

is finite, we may conclude that the induced surjective morphism $f^{-1}(\mathcal{U}) \to \mathcal{U}$ has connected fibers—thus is a bijection of sets. Hence, we are reduced to showing that the inclusion $\mathcal{U}' \hookrightarrow f^{-1}(\mathcal{U})$ is surjective. This follows from $(\mathcal{Z} - V(\widetilde{\mathcal{I}})) \cap f(V(\mathcal{I}_{\mathcal{Z}'})) = \emptyset$, which is obvious.

In the case where $\mathcal{X}$ is an analytic Deligne–Mumford stack, since all of these constructions commute with étale base change on $\mathcal{Z}$, we may work étale locally on $\mathcal{X}$ and deduce the result from the case of analytic spaces already proved. □

Combining Lemmata 4.2 and 4.3 we can now prove Lemma 4.1.

of Lemma 4.1. Let $J \lhd \mathcal{O}_X$ be a coherent ideal such that $|V(J)| = |Q|$. By assumption, $|\operatorname{supp}\ker f^\sharp| \subset |V((J_{\mathrm{an}})_{\mathcal{Z}})|$ so by Lemma 2.32 we may replace J by some power J^k so that $(J_{\mathrm{an}})_{\mathcal{Z}} \cap \ker f^\sharp = 0$. Now consider the diagram of analytic X_{an}-stacks $[V((J_{\mathrm{an}})_{\mathcal{Z}}) \leftarrow V((J_{\mathrm{an}})_{\mathcal{Z}'}) \to \mathcal{Z}']$. By hypothesis, these analytic Deligne–Mumford stacks are all algebraizable. By Lemma 4.2, the colimit of this diagram in $\mathbf{QF}(X_{\mathrm{an}})$, which we denote as $\mathcal{Z}''$, is algebraizable. In particular, we obtain a factorization of the map $f : \mathcal{Z}' \to \mathcal{Z}$ as $\mathcal{Z}' \to \mathcal{Z}'' \xrightarrow{\beta} \mathcal{Z}$. Trivially, β is finite and surjective. We claim, in addition, that $\beta^\sharp$ is injective and $\mathfrak{C}_\beta \subset \mathfrak{C}_f$. Note that because β_* is left-exact we have the following isomorphism:

$$\beta_*\mathcal{O}_{\mathcal{Z}''} \cong \mathcal{O}_{\mathcal{Z}}/(J_{\mathrm{an}})_{\mathcal{Z}} \times_{f_*[\mathcal{O}_{\mathcal{Z}'}/(J_{\mathrm{an}})_{\mathcal{Z}'}]} f_*\mathcal{O}_{\mathcal{Z}'}.$$

Thus, it remains to prove that if $\rho : A \to B$ is a ring homomorphism and $M \lhd A$ is an ideal such that $M \cap \ker\rho = 0$, then the map $\rho' : A \to A/M \times_{B/MB} B$ is injective and $\mathfrak{C}_{\rho'} \subseteq \mathfrak{C}_\rho$. The first claim is trivial and the latter follows readily from the Snake Lemma.

We may now conclude that $|\operatorname{supp}(\ker\beta^\sharp)| = \emptyset$ and $|\operatorname{supp}(\operatorname{coker}\beta^\sharp)| \subseteq |\operatorname{supp}(\operatorname{coker} f^\sharp)| \subseteq \sigma^{-1}|Q_{\mathrm{an}}|$. So we are now reduced to the case where $f^\sharp$ is assumed to be injective.

By hypothesis, $(V((J_{\mathrm{an}})_{\mathcal{Z}}) \to X_{\mathrm{an}})$ is algebraizable. By Lemma 2.31 we may replace J by some power such that $(J_{\mathrm{an}})_{\mathcal{Z}}\operatorname{coker} f^\sharp = 0$, thus $\mathfrak{C}_f \supseteq (J_{\mathrm{an}})_{\mathcal{Z}}$. As $\mathcal{Z}'$ is proper and algebraizable, then $(V((J_{\mathrm{an}})_{\mathcal{Z}'}) \to X_{\mathrm{an}})$ is also algebraizable. The diagram:

$$\begin{tikzcd} V((J_{\mathrm{an}})_{\mathcal{Z}'}) \arrow[r] \arrow[d] & \mathcal{Z}' \arrow[d] \\ V((\widetilde{J}_{\mathrm{an}})_{\mathcal{Z}}) \arrow[r] & \mathcal{Z} \end{tikzcd}$$

is cocartesian in $\mathbf{QF}(X_{\mathrm{an}})$ (Lemma 4.3), thus $(\mathcal{Z} \xrightarrow{\sigma} X_{\mathrm{an}})$ is algebraizable (Lemma 4.2) . □

5. Dévissage

First we prove the birational dévissage on the category $\mathbf{QF}_p(X)$ for schemes. Before we get to this, we require the following Lemma which does most of the work.

Lemma 5.1. *Fix a proper and surjective morphism of schemes $q : Y \to X$. Let $(\mathcal{Z} \xrightarrow{\sigma} X_{\mathrm{an}}) \in \mathbf{QF}_p(X_{\mathrm{an}})$ be such that $(\mathcal{Z}_{Y_{\mathrm{an}}} \to Y_{\mathrm{an}}) \in \mathbf{QF}_p(Y_{\mathrm{an}})$ is algebraizable. Then, there exists a finite and surjective $\mathbf{QF}_p(X_{\mathrm{an}})$-morphism $f : (\mathcal{Z}' \xrightarrow{\sigma'} X_{\mathrm{an}}) \to (\mathcal{Z} \xrightarrow{\sigma} X_{\mathrm{an}})$ with $(\mathcal{Z}' \xrightarrow{\sigma'} X_{\mathrm{an}}) \in \mathbf{QF}_p(X_{\mathrm{an}})$ algebraizable. In addition, if q is an isomorphism over an open subscheme $U \subset X$, then f maybe chosen to be such that* $\ker f^\sharp$ *and* $\operatorname{coker} f^\sharp$ *have support contained in $\sigma^{-1}(|X_{\mathrm{an}}| \setminus |U_{\mathrm{an}}|)$.*

Proof. The morphism $(q_{\mathrm{an}})_{\mathcal{Z}} : \mathcal{Z}_{Y_{\mathrm{an}}} \to \mathcal{Z}$ is a proper morphism of analytic spaces, thus admits a Stein factorization $\mathcal{Z}_{Y_{\mathrm{an}}} \xrightarrow{\beta} \mathcal{Z}' \xrightarrow{f} \mathcal{Z}$ [GR84, §10.6.1]. That is, β is proper and surjective with connected fibers, f is finite and surjective, and the natural map $\beta^\sharp : \mathcal{O}_{\mathcal{Z}'} \to \beta_* \mathcal{O}_{\mathcal{Z}_{Y_{\mathrm{an}}}}$ is an isomorphism. If q is an isomorphism over an open subscheme $U \subset X$, then $(q_{\mathrm{an}})_{\mathcal{Z}}$ is an isomorphism over $\sigma^{-1}(U_{\mathrm{an}})$, whence f is an isomorphism over $\sigma^{-1}(U_{\mathrm{an}})$. We deduce immediately that this implies that $\ker f^\sharp$ and $\operatorname{coker} f^\sharp$ have support contained in $\sigma^{-1}(|X_{\mathrm{an}}| \setminus |U_{\mathrm{an}}|)$. Thus it remains to show that the resulting object $(\mathcal{Z}' \xrightarrow{\sigma'} X_{\mathrm{an}}) \in \mathbf{QF}_p(X_{\mathrm{an}})$ is algebraizable. To do this we will give an alternate description of $\mathcal{Z}'$.

By hypothesis, $\mathcal{Z}_{Y_{\mathrm{an}}} \cong W_{\mathrm{an}}$ for some $W \in \mathbf{QF}_p(Y)$. Note that W is proper over $\operatorname{Spec} \mathbb{C}$ and since the diagonal of Y is quasicompact, it follows that the morphism $W \to Y$ is also quasicompact. Hence, $W \to Y$ is quasi-finite, separated, and representable. Zariski's Main Theorem [EGA, IV.18.12.13] now provides a finite Y-scheme E and a dense open immersion $j : W \hookrightarrow E$. Let r be the composition $E \to Y \to X$, then r is a proper morphism of locally noetherian schemes and thus admits a Stein factorization $E \xrightarrow{r'} E' \xrightarrow{t} X$ [EGA, III.4.3.1]. That is, $\mathcal{O}_{E'} \cong r'_* \mathcal{O}_E$, r' is proper and surjective with geometrically connected fibers, and t is finite.

We now proceed to show that $|Z'| = r'(|W|) \subseteq |E'|$ is Zariski open and that $\mathcal{Z}'$ is isomorphic to the analytification of the resulting open subscheme $Z' \subseteq E'$. The morphism r' is proper, so is of finite type, hence by Chevalley's Theorem [EGA, IV.1.8.4] $|Z'| \subseteq |E'|$ is constructible. In addition to being proper, r' is surjective, thus is universally submersive [SGA1, IX.2.2], so $|Z'| \subset |E'|$ is Zariski open if and only if $r'^{-1}|Z'| \subset |E|$ is so. By [EGA, IV.1.8.2], $r'^{-1}|Z'|$ is also constructible. Since E is Jacobson [EGA, IV.10.4.6], $r'^{-1}|Z'| \subset |E|$ is Zariski open if and only if $|E(\mathbb{C})| \cap r'^{-1}|Z'| \subset |E(\mathbb{C})|$ is Zariski open [EGA, IV.10.4.8] (where we give these sets the subspace topology). Note, however, that $|E(\mathbb{C})| \cap r'^{-1}|Z'| = r'^{-1}(|Z'| \cap |E'(\mathbb{C})|)$, hence it is sufficient to show that the natural map $|W(\mathbb{C})| \to r'^{-1}(|Z'| \cap |E'(\mathbb{C})|)$ is bijective. To prove this will require an analysis of the fibers of W over $|X(\mathbb{C})|$ and $|E'(\mathbb{C})|$.

Let $x \in |X_{\mathrm{an}}| = |X(\mathbb{C})|$, then $|\mathcal{Z}_x| = \sigma^{-1}(x)$ is a closed and discrete subset of $|\mathcal{Z}|$. Since $|\mathcal{Z}|$ is compact, it follows that $|\mathcal{Z}_x|$ is a finite set, and so $\mathcal{Z}_x$ is a finite analytic space. Set $(q_{\mathrm{an}})_{\mathcal{Z}} : \mathcal{Z}_{Y_{\mathrm{an}}} \to \mathcal{Z}$ to be the pullback of $q_{\mathrm{an}} : Y_{\mathrm{an}} \to X_{\mathrm{an}}$ by $\sigma : \mathcal{Z} \to X_{\mathrm{an}}$. Observe that we have a natural morphism of fibers $(q_{\mathrm{an}})_{\mathcal{Z},x} : (\mathcal{Z}_{Y_{\mathrm{an}}})_x \to \mathcal{Z}_x$ over x. The morphism q is proper, thus so is the morphism $(q_{\mathrm{an}})_{\mathcal{Z},x}$ and as $\mathcal{Z}_x$ is finite, we conclude that $(\mathcal{Z}_{Y_{\mathrm{an}}})_x$ is a proper analytic space. Since properness of a $\mathbb{C}$-scheme can be verified on its analytification, we deduce that W_x is a proper $\mathbb{C}$-scheme.

Now let $v \in |E'(\mathbb{C})|$ have image $x \in |X(\mathbb{C})|$. Since W_v is a closed subscheme of $W_x \times_X E'$, $W_x \times_X E' \to W_x$ is finite, and W_x is proper, then W_v is a proper $\mathbb{C}$-scheme. In particular, the open immersion $W_v \to E_v$ is also proper, thus its image is open and closed. But E_v is connected (by construction), so the image of W_v in E_v is either everything or empty. We deduce immediately that the map $|W(\mathbb{C})| \to r'^{-1}(|Z'| \cap |E'(\mathbb{C})|)$ is bijective. Thus, we have proved that $|Z'| \subset |E'|$ is a Zariski open subset.

We now let $(Z' \xrightarrow{s'} X) \in \mathbf{QF}_p(X)$ denote the open subscheme of E' defined by $|Z'|$. Let $b : W \to Z'$ denote the induced map. Observe that b is proper and surjective with geometrically connected fibers, and $b^\sharp : \mathcal{O}_{Z'} \to b_* \mathcal{O}_W$ is an isomorphism. By [SGA1, XII.4.2], the resulting morphism $b_{\mathrm{an}} : W_{\mathrm{an}} \to Z'_{\mathrm{an}}$ has the same properties. That $(Z'_{\mathrm{an}} \xrightarrow{s'_{\mathrm{an}}} X_{\mathrm{an}})$ is isomorphic to $(\mathcal{Z}' \xrightarrow{\sigma'} X_{\mathrm{an}})$ in $\mathbf{QF}_p(X_{\mathrm{an}})$ now follows from Lemma 5.2 by taking $\mathcal{X} = X_{\mathrm{an}}$, $\mathcal{W} = W_{\mathrm{an}}$, $\mathcal{Z}_1 = \mathcal{Z}'$, $\mathcal{Z}_2 = Z'_{\mathrm{an}}$, $\beta_1 = \beta$, $\beta_2 = b_{\mathrm{an}}$, $\sigma_1 = \sigma'$, and $\sigma_2 = s'_{\mathrm{an}}$. ☐

Lemma 5.2. *Fix an analytic space $\mathcal{X}$ and a commutative diagram:*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "\beta_1"] \arrow[d, "\beta_2"'] & \mathcal{Z}_1 \arrow[d, "\sigma_1"] \\ \mathcal{Z}_2 \arrow[r, "\sigma_2"] & \mathcal{X}, \end{tikzcd}$$

where for $i = 1$ and 2 we have $(\mathcal{Z}_i \xrightarrow{\sigma_i} \mathcal{X}) \in \mathbf{QF}_p(\mathcal{X})$, β_i is proper and surjective with connected fibers, and $\beta_i^\sharp : \mathcal{O}_{\mathcal{Z}_i} \to (\beta_i)_*\mathcal{O}_{\mathcal{Y}}$ is an isomorphism. Then, there exists a unique isomorphism $\mathcal{Z}_1 \to \mathcal{Z}_2$ in $\mathbf{QF}_p(\mathcal{X})$ completing the diagram.

Proof. It is sufficient to construct a unique morphism of analytic spaces $\gamma : \mathcal{Z}_1 \to \mathcal{Z}_2$ completing the diagram. For $z \in |\mathcal{Z}_1|$, the subset $(\beta_1)^{-1}(z) \subset |\mathcal{W}|$ is closed and connected and so $G_z := \beta_2 \circ (\beta_1)^{-1}(z) \subset |\mathcal{Z}_2|$ is also closed and connected. Note, however, that

$$\sigma_2(G_z) = \sigma_2 \circ \beta_2 \circ (\beta_1)^{-1}(z) = \sigma_1 \circ \beta_1 \circ (\beta_1)^{-1}(z) = \sigma_1(z).$$

Hence, we conclude that $G_z \subset \sigma_2^{-1} \circ \sigma_1(z)$. The morphism $\sigma_2 : \mathcal{Z}_2 \to \mathcal{X}$ is locally quasi-finite, thus we conclude that G_z is discrete. Since G_z is also connected, G_z consists of a single point. Thus we see that there is a unique map of sets $\gamma : |\mathcal{Z}_1| \to |\mathcal{Z}_2| : z \mapsto G_z$ which is compatible with the remainder of the data. We now claim that γ is continuous. To see this, fix a closed subset $\mathcal{V} \subset |\mathcal{Z}_2|$, then because β_1 is submersive (it is proper and surjective), $\gamma^{-1}(\mathcal{V}) \subset |\mathcal{Z}_1|$ is closed if and only if $(\beta_1)^{-1}\gamma^{-1}(\mathcal{V}) \subset |\mathcal{W}|$ is closed. By construction, however, $(\beta_1)^{-1}\gamma^{-1}(\mathcal{V}) = (\beta_2)^{-1}(\mathcal{V}) \subset |\mathcal{W}|$, which is closed because β_2 is continuous, thus γ is continuous. We define the morphism $\gamma^\sharp : \mathcal{O}_{\mathcal{Z}_2} \to \gamma_*\mathcal{O}_{\mathcal{Z}_1}$ on functions as the following composition:

$$\mathcal{O}_{\mathcal{Z}_2} \xrightarrow{\beta_2^\sharp} (\beta_2)_*\mathcal{O}_{\mathcal{W}} = (\gamma \circ \beta_1)_*\mathcal{O}_{\mathcal{W}} \cong \gamma_*(\beta_1)_*\mathcal{O}_{\mathcal{W}} \xrightarrow{\gamma_*[(\beta_1^\sharp)^{-1}]} \gamma_*\mathcal{O}_{\mathcal{Z}_1}.$$

In particular, this morphism is uniquely defined by the data, and we deduce the result. ☐

With Lemma 5.1 at our disposal, we now move to proving the birational dévissage.

Proposition 5.3. *Fix a scheme X which is quasicompact. Suppose that for any closed immersion $V \hookrightarrow X$, there is a proper and birational morphism $V' \to V$ of schemes such that the analytification functor $\Psi_{V',p}$ is an equivalence. Then, the analytification functor $\Psi_{X,p}$ is an equivalence.*

Proof. The analytification functor $\Psi_{X,p}$ is fully faithful by Lemma 2.7, so it remains to prove its essential surjectivity, which we do by noetherian induction on the closed subsets of X. Thus, we are immediately reduced to the situation where the analytification functor $\Psi_{V,p}$ is already known to be an equivalence for all closed subschemes $V \hookrightarrow X$ with $|V| \subsetneq |X|$.

By hypothesis, there is a proper and birational morphism $q : Y \to X$ such that the analytification functor $\Psi_{Y,p}$ is an equivalence. Fix a dense open subscheme $U \subset X$ for which $q^{-1}U \to U$ is an isomorphism, set $|Q| = |X \setminus U|$, and let $(\mathcal{Z} \xrightarrow{\sigma} X_{\mathrm{an}}) \in \mathbf{QF}_p(X_{\mathrm{an}})$. By assumption, $(\mathcal{Z}_{Y_{\mathrm{an}}} \xrightarrow{\sigma_{Y_{\mathrm{an}}}} Y_{\mathrm{an}}) \in \mathbf{QF}_p(Y_{\mathrm{an}})$ is algebraizable.

By Lemma 5.1 there is a finite and surjective $\mathbf{QF}_p(X_{\mathrm{an}})$-morphism $f : (\mathcal{Z}' \xrightarrow{\sigma'} X_{\mathrm{an}}) \to (\mathcal{Z} \xrightarrow{\sigma} X_{\mathrm{an}})$, such that $\ker f^\sharp$ and $\operatorname{coker} f^\sharp$ have support contained in $\sigma^{-1}|Q_{\mathrm{an}}|$, and $(\mathcal{Z}' \xrightarrow{\sigma'} X_{\mathrm{an}}) \in \mathbf{QF}_p(X_{\mathrm{an}})$ algebraizable. By noetherian induction, any closed analytic subspace of $\mathcal{Z}$ supported in $\sigma^{-1}|Q_{\mathrm{an}}|$ is algebraizable. Thus, by Lemma 4.1, $(\mathcal{Z} \xrightarrow{\sigma} X_{\mathrm{an}}) \in \mathbf{QF}_p(X_{\mathrm{an}})$ is algebraizable. Hence, the analytification functor $\Psi_{X,p}$ is essentially surjective. □

Finally, we have the finite dévissage on $\mathbf{QF}_p(X)$ for algebraic Deligne–Mumford stacks X.

Proposition 5.4. *Fix an algebraic Deligne–Mumford stack X which is quasicompact. Suppose that for any closed immersion $V \hookrightarrow X$, there is a finite and generically étale map $V' \to V$ such that the analytification functors $\Psi_{V',p}$ and $\Psi_{V'\times_V V',p}$ are equivalences, then the analytification functor $\Psi_{X,p}$ is an equivalence.*

Proof. By Lemma 2.7, the analytification functor $\Psi_{X,p}$ is fully faithful, thus it remains to treat the essential surjectivity. We now prove the result by noetherian induction on the closed substacks of X. Hence, it suffices to assume that the analytification functor $\Psi_{V,p}$ is an equivalence for any closed substack $V \hookrightarrow X$ such that $|V| \subsetneq |X|$. By assumption, there is a finite and generically étale map $\pi : X^1 \to X$ such that the analytification functors $\Psi_{X^1,p}$ and $\Psi_{X^1\times_X X^1,p}$ are equivalences. Fix a dense open substack $U \hookrightarrow X$ such that $\pi^{-1}U \to U$ is étale and set $Q = X \setminus U$. Let $(\mathcal{Z} \xrightarrow{\sigma} X_{\mathrm{an}}) \in \mathbf{QF}_p(X_{\mathrm{an}})$. Set $X^2 = X^1 \times_X X^1$ and for $i = 1$ and 2 define $\mathcal{Z}^i = \mathcal{Z} \times_{X_{\mathrm{an}}} X^i_{\mathrm{an}}$. By the hypotheses on the analytification functors $\Psi_{X^1,p}$ and $\Psi_{X^2,p}$, the diagram $[\mathcal{Z}^2 \rightrightarrows \mathcal{Z}^1]$ is algebraizable in $\mathbf{QF}_p(X_{\mathrm{an}})$. Hence, by Lemma 4.2, we conclude that the coequalizer in the category $\mathbf{QF}_p(X_{\mathrm{an}})$ exists, and is algebraizable. We denote this coequalizer by $(\sigma' : \mathcal{Z}' \to X_{\mathrm{an}})$ and the universal properties give a finite morphism $f : \mathcal{Z}' \to \mathcal{Z}$ in $\mathbf{QF}(X_{\mathrm{an}})$.

Consider the open analytic substack $\mathcal{U} := \sigma^{-1}(U_{\mathrm{an}})$ of $\mathcal{Z}$. We claim that the induced map $f^{-1}\mathcal{U} \to \mathcal{U}$ is an analytic isomorphism. By Lemma 4.2, this may be verified after pulling back everything by the morphism $U^1 := \pi^{-1}(U) \to U$. In particular, we are free to assume that $(\pi_{\mathrm{an}})_{\mathcal{U}} : \mathcal{U}^1 := (\pi_{\mathrm{an}})^{-1}_{\mathcal{Z}}(\mathcal{U}) \to \mathcal{U}$ admits a section s. It is now a trivial calculation (using the Yoneda embedding) to verify that the coequalizer of $[\mathcal{U}^1 \times_{\mathcal{U}} \mathcal{U}^1 \rightrightarrows \mathcal{U}^1]$ in $\mathbf{QF}(U_{\mathrm{an}})$ is $\mathcal{U}$. But the coequalizer of $[\mathcal{U}^1 \times_{\mathcal{U}} \mathcal{U}^1 \rightrightarrows \mathcal{U}^1]$, by Lemma 4.2, is uniquely isomorphic to $f^{-1}\mathcal{U}$. Hence, the universal properties guarantee that $f^{-1}\mathcal{U} \to \mathcal{U}$ is an isomorphism.

In particular, we deduce that $\ker f^\sharp$ and $\operatorname{coker} f^\sharp$ have support contained in $\sigma^{-1}|Q_{\mathrm{an}}|$. By noetherian induction, any closed analytic substack of $\mathcal{Z}$ supported in $\sigma^{-1}|Q_{\mathrm{an}}|$ is algebraizable. Thus, by Lemma 4.1, we deduce that $(\mathcal{Z} \xrightarrow{\sigma} X_{\mathrm{an}}) \in \mathbf{QF}_p(X_{\mathrm{an}})$ is algebraizable. Thus, the analytification functor $\Psi_{X,p}$ is essentially surjective. □

6. Proof of Theorems C and D

We now use the main results of § 5 to prove Theorem D.

of Theorem D. Given Lemma 2.7 it is sufficient to prove the essential surjectivity. **Special case.** First we prove that the analytification functor $\Psi_{X,p}$ is essentially surjective in the case where X is a quasicompact scheme such that the structure morphism factors as $X \xrightarrow{f} Y \xrightarrow{g} \operatorname{Spec}\mathbb{C}$, where f is étale and g is projective. Fix

$(\mathcal{Z} \xrightarrow{\sigma} X_{\mathrm{an}}) \in \mathbf{QF}_p(X_{\mathrm{an}})$. The analytic space Y_{an} is separated, so the composition $f \circ \sigma : \mathcal{Z} \to Y_{\mathrm{an}}$ is a locally quasi-finite and proper morphism of analytic spaces. By [GR84, Thm. XII.4.2] such a morphism is finite. Since Y is also projective, by [SGA1, XII.4.6], there is a finite morphism of schemes $Z \to Y$ with Z proper, and $Z_{\mathrm{an}} \cong \mathcal{Z}$ over Y_{an}. To complete the proof in this setting, it suffices to produce a quasi-finite morphism of schemes $t : Z \to X$ such that $t_{\mathrm{an}} = \sigma$. Note that it is equivalent to produce a section to the quasicompact étale morphism $h : W := X \times_Y Z \to Z$ which agrees with the analytic section to the morphism of analytic spaces h_{an} induced by σ. Thus, it remains to show that $\mathrm{Hom}_Z(Z, W) = \mathrm{Hom}_{Z_{\mathrm{an}}}(Z_{\mathrm{an}}, W_{\mathrm{an}})$. By [FK88, I.11.3], the analytification of the sheaf $\mathrm{Hom}_Z(-, W)$ on the small étale site of Z is representable by the analytic space W_{an} on the small étale site of Z_{an}. Moreover, since $h : W \to Z$ is quasicompact and étale, the sheaf $\mathrm{Hom}_Z(-, W)$ on the small étale site of Z is constructible. By [FK88, I.11.5], $\mathrm{Hom}_Z(Z, W) = \mathrm{Hom}_{Z_{\mathrm{an}}}(Z_{\mathrm{an}}, W_{\mathrm{an}})$.

Quasicompact schemes. Now we prove that the analytification functor $\Psi_{X,p}$ is essentially surjective in the case where X is a quasicompact schemes. Thus, any closed immersion $V \hookrightarrow X$ is of finite type over $\mathbb{C}$, so by [RG71, Cor. 5.7.13], there is a schematic and birational morphism $V' \to V$ such that the structure morphism of V' over $\mathbb{C}$ factors as $V' \xrightarrow{f_V} Y_V \xrightarrow{g_V} \operatorname{Spec} \mathbb{C}$, where f_V is étale and g_V is projective. By the special case considered above, the analytification functor $\Psi_{V',p}$ is an equivalence and so by Proposition 5.3 the analytification functor $\Psi_{X,p}$ is an equivalence.

Quasicompact algebraic Deligne–Mumford stacks. Now we prove that the analytification functor $\Psi_{X,p}$ is essentially surjective in the case where X is a quasicompact algebraic Deligne–Mumford stack. Note that any closed immersion $V \hookrightarrow X$ is of finite type over $\mathbb{C}$, so by [LMB, Thm. 16.6], there is a finite and generically étale morphism $V' \to V$, where V' is a scheme. Since V' is a quasicompact scheme, the case previously considered shows that the analytification functors $\Psi_{V',p}$ and $\Psi_{V' \times_V V',p}$ are equivalences. By Proposition 5.4 the analytification functor $\Psi_{X,p}$ is an equivalence.

General case. We finally prove that the analytification functor $\Psi_{X,p}$ is essentially surjective for all algebraic Deligne–Mumford stacks X. Fix $(\mathcal{Z} \xrightarrow{\sigma} X_{\mathrm{an}}) \in \mathbf{QF}_p(X_{\mathrm{an}})$. Let O_X denote the category of quasicompact open subsets of X. We note that $\{U\}_{U \in O_X}$ is an open cover of X and so $\{\sigma^{-1}(U_{\mathrm{an}})\}_{U \in O_X}$ is an open cover of $\mathcal{Z}$. Since $\mathcal{Z}$ is a compact topological space, and the exhibited cover is closed under finite unions, there is an open immersion $U \hookrightarrow X$ such that the map $\mathcal{Z} \to X_{\mathrm{an}}$ factors uniquely through U_{an}. By the previous case considered the analytification functor $\Psi_{X,p}$ is an equivalence. □

We now prove Theorem C.

of Theorem C. For the full faithfulness, fix morphisms f and $g : Z \to X$ such that $f_{\mathrm{an}} = g_{\mathrm{an}}$. Let $E = X \times_{\Delta, X \times X, (f,g)} Z$ be the equalizer of f and g. Since the diagonal of X is quasi-finite, separated, and representable by schemes, the same is true of the induced morphism $e : E \to Z$. To prove that $f = g$, it suffices to show that the map e is an isomorphism. This is a local problem on Z for the étale topology, so we are reduced to the case where Z and E are schemes. Finally, because analytification preserves fiber products, we see that the map $e_{\mathrm{an}} : E_{\mathrm{an}} \to Z_{\mathrm{an}}$ is an isomorphism, whence e is an isomorphism [SGA1, XII.3.3(a)].

For the essential surjectivity, let $\phi : Z_{\mathrm{an}} \to X_{\mathrm{an}}$ be an analytic map and take $\Gamma_\phi : Z_{\mathrm{an}} \to (Z \times X)_{\mathrm{an}}$ to be its graph. Since the diagonal of X is quasi-finite, separated, and representable it follows that the same is true of Γ_ϕ. By Theorem D, $(Z_{\mathrm{an}} \xrightarrow{\Gamma_\phi} (Z \times X)_{\mathrm{an}}) \in \mathbf{QF}_p((Z \times X)_{\mathrm{an}})$ is uniquely algebraizable to $(Z' \to Z \times X) \in \mathbf{QF}_p(Z \times X)$ thus the $\mathbf{QF}_p(Z_{\mathrm{an}} \times X_{\mathrm{an}})$-isomorphism $Z_{\mathrm{an}} \to (Z')_{\mathrm{an}}$ is algebraizable and the induced composition $Z \to Z' \to Z \times X \to X$ defines a morphism f such that $f_{\mathrm{an}} \cong \phi$. □

7. Proof of Theorems A and B

Fix an integer $n \geq 0$. Let $\mathcal{U}_n$ (resp. $\mathcal{U}_n^{\mathrm{an}}$) denote the moduli stack of all n-pointed algebraic (resp. analytic) curves. That is, a morphism from a scheme (resp. analytic space) T to $\mathcal{U}_n$ (resp. $\mathcal{U}_n^{\mathrm{an}}$) is equivalent to a morphism of algebraic (resp. analytic) spaces $C \to T$ which is proper and flat with one-dimensional fibers, together with n sections to the map $C \to T$. As in the Introduction, these curves are not assumed to be smooth nor are the sections assumed to be disjoint. In [Smy12, B.1] it proved that $\mathcal{U}_n$ is an Artin stack, locally of finite presentation, with quasicompact and separated diagonal.

Let $\mathcal{U}_n^{\mathrm{FA}}$ (resp. $\mathcal{U}_n^{\mathrm{FA,an}}$) denote the substack of $\mathcal{U}_n$ (resp. $\mathcal{U}_n^{\mathrm{an}}$) consisting of those n-pointed curves with the finite automorphism property. Fix a non-negative integer g and let $\mathcal{M}_{g,n}$ (resp. $\mathcal{M}_{g,n}^{\mathrm{an}}$) denote the substack of $\mathcal{U}_n$ (resp. $\mathcal{U}_n^{\mathrm{an}}$) consisting of those smooth n-pointed curves of genus g whose n sections have disjoint images. If $2g-2+n>0$, then we have inclusions $\mathcal{M}_{g,n} \subset \mathcal{U}_n^{\mathrm{FA}} \subset \mathcal{U}_n$ which are representable by open immersions. Since Artin stacks with quasi-finite diagonal in characteristic zero are Deligne–Mumford, $\mathcal{U}_n^{\mathrm{FA}}$ is an algebraic Deligne–Mumford stack. There is an induced morphism of analytic Deligne–Mumford stacks $(\mathcal{U}_n^{\mathrm{FA}})_{\mathrm{an}} \to \mathcal{U}_n^{\mathrm{FA,an}}$. We now have three lemmata, which are likely known to experts.

Lemma 7.1. *Fix a* 1*-morphism of stacks over* **An***,* $f : \mathcal{X} \to \mathcal{G}$*. Suppose that* $\mathcal{X}$ *is an analytic Deligne–Mumford stack and the diagonal of* $\mathcal{G}$ *is representable by analytic spaces. If for any local artinian* $\mathbb{C}$*-scheme* S *the functor* $f(S_{\mathrm{an}}) : \mathcal{X}(S_{\mathrm{an}}) \to \mathcal{G}(S_{\mathrm{an}})$ *is an equivalence, then* f *is an equivalence.*

Proof. It is sufficient to show that for each analytic space $\mathcal{W}$ and 1-morphism $\mathcal{W} \to \mathcal{G}$, the resulting 1-morphism $\mathcal{X} \times_{\mathcal{G}} \mathcal{W} \to \mathcal{W}$ is an equivalence of stacks over **An**. Note, however, that because the diagonal of $\mathcal{G}$ is representable by analytic spaces, the 1-morphism $\mathcal{X} \times_{\mathcal{G}} \mathcal{W} \to \mathcal{X} \times \mathcal{W}$ is representable by analytic spaces (it is the pullback of $\Delta_{\mathcal{G}} : \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ along $\mathcal{X} \times \mathcal{W} \to \mathcal{G} \times \mathcal{G}$). By assumption $\mathcal{X}$ is an analytic Deligne–Mumford stack, thus $\mathcal{X} \times_{\mathcal{G}} \mathcal{W} \to \mathcal{W}$ is a 1-morphism of analytic Deligne–Mumford stacks with the property that for all local artinian $\mathbb{C}$-schemes S, the functor $(\mathcal{X} \times_{\mathcal{G}} \mathcal{W})(S_{\mathrm{an}}) \to \mathcal{W}(S_{\mathrm{an}})$ is an equivalence.

Thus it remains to prove that if $g : \mathcal{V}' \to \mathcal{V}$ is a 1-morphism of analytic Deligne–Mumford stacks such that $g(S_{\mathrm{an}}) : \mathcal{V}'(S_{\mathrm{an}}) \to \mathcal{V}(S_{\mathrm{an}})$ is an equivalence for all local artinian $\mathbb{C}$-schemes S, then g is an isomorphism. First assume that g is representable, then it is representable by bijective étale morphisms of analytic spaces, thus is an isomorphism. For the general case, observe that $\Delta_g : \mathcal{V}' \to \mathcal{V}' \times_{\mathcal{V}} \mathcal{V}'$ is representable and $\Delta_g(S_{\mathrm{an}})$ is an equivalence for all local artinian S-schemes S. Thus, by the case already considered, Δ_g is an isomorphism. In particular, g is a monomorphism, thus is representable, and we deduce the result. □

Lemma 7.2. *The morphism of* **An***-stacks* $(\mathcal{U}_n^{\mathrm{FA}})_{\mathrm{an}} \to \mathcal{U}_n^{\mathrm{FA,an}}$ *is an equivalence; hence,* $\mathcal{U}_n^{\mathrm{an}}$ *is an analytic Deligne–Mumford stack. Moreover, this equivalence sends* $(\mathcal{M}_{g,n})_{\mathrm{an}}$ *to* $\mathcal{M}_{g,n}^{\mathrm{an}}$*.*

Proof. A simple direct argument (or Theorem C to be sure) shows that for any algebraic Deligne–Mumford stack X the natural functor $X(S) \to X_{\mathrm{an}}(S_{\mathrm{an}})$ is an equivalence for all local artinian $\mathbb{C}$-schemes S. Taking $X = \mathcal{U}_n^{\mathrm{FA}}$ we see that the functor $\mathcal{U}_n^{\mathrm{FA}}(S) \to (\mathcal{U}_n^{\mathrm{FA}})_{\mathrm{an}}(S_{\mathrm{an}})$ is an equivalence for all local artinian $\mathbb{C}$-schemes S. Thus, by Lemma 7.1, the 1-morphism $(\mathcal{U}_n^{\mathrm{FA}})_{\mathrm{an}} \to \mathcal{U}_n^{\mathrm{FA,an}}$ is an equivalence if the functor $\mathcal{U}_n^{\mathrm{FA}}(S) \to \mathcal{U}_n^{\mathrm{FA,an}}(S_{\mathrm{an}})$ is an equivalence for any local artinian $\mathbb{C}$-scheme S and the diagonal of $\mathcal{U}_n^{\mathrm{FA,an}}$ is representable by analytic spaces.

By separated GAGA (Corollary 2.6) the functor $\mathcal{U}_n^{\mathrm{FA}}(S) \to \mathcal{U}_n^{\mathrm{FA,an}}(S_{\mathrm{an}})$ is easily discerned to be fully faithful for all local artinian $\mathbb{C}$-schemes S. For the essential surjectivity, a proper and flat analytic curve $\mathcal{C} \to S_{\mathrm{an}}$, $\mathcal{C}$ is a one-dimensional proper analytic space, thus is algebraizable. The flat structure map $\mathcal{C} \to S_{\mathrm{an}}$ algebraizes, as do the sections [*loc. cit.*]. Combining [Pou69, Thm. 2] with [Dou66, Thm. 10.2.1], shows that Isom-spaces between n-pointed proper and flat relative analytic spaces are representable by analytic spaces, whence we have proved that $(\mathcal{U}_n^{\mathrm{FA}})_{\mathrm{an}} \to \mathcal{U}_n^{\mathrm{FA,an}}$ is an equivalence. Finally, since smoothness of a morphism of algebraic $\mathbb{C}$-spaces and disjointness of sections to a morphism of algebraic $\mathbb{C}$-space can be tracked by their analytifications, it follows immediately that $(\mathcal{M}_{g,n})_{\mathrm{an}} \cong \mathcal{M}_{g,n}^{\mathrm{an}}$. □

We may now prove Theorem A.

of Theorem A. By Lemma 7.2, an analytic map $\mathcal{W} \to Z_{\mathrm{an}}$ is equivalent to an analytic map $Z_{\mathrm{an}} \to \mathcal{U}_n^{\mathrm{FA,an}} \cong (\mathcal{U}_n^{\mathrm{FA}})_{\mathrm{an}}$. By Theorem C, this map is algebraizable to a morphism of Deligne–Mumford stacks $Z \to \mathcal{U}_n^{\mathrm{FA}}$, giving rise to a flat family of curves $W \to Z$ such that $W_{\mathrm{an}} \cong \mathcal{W}$. □

We now address Theorem B. First, a definition.

Definition 7.3. *Fix non-negative integers g and n satisfying $2g-2+n>0$. An* algebraic *(resp.* analytic*)* modular birational model *of $\mathcal{M}_{g,n}$ (resp. $\mathcal{M}_{g,n}^{\mathrm{an}}$) with* the finite automorphism property *is a proper algebraic (resp. analytic) Deligne–Mumford stack N (resp. $\mathcal{N}$), fitting into a 2-fiber diagram:*

$$\begin{tikzcd} V \arrow[r, hook, "j'"] \arrow[d, "i'"'] & \mathcal{M}_{g,n} \arrow[d, "i"] \\ N \arrow[r, hook, "j"] & \mathcal{U}_n^{\mathrm{FA}} \end{tikzcd} \qquad \text{resp.} \qquad \begin{tikzcd} \mathcal{V} \arrow[r, hook, "j'"] \arrow[d, "i'"'] & \mathcal{M}_{g,n}^{\mathrm{an}} \arrow[d, "i"] \\ \mathcal{N} \arrow[r, hook, "j"] & \mathcal{U}_n^{\mathrm{FA,an}}, \end{tikzcd}$$

where the map j is an open immersion, and the maps i' and j' have dense image.

of Theorem B. Let $\mathcal{N} \hookrightarrow \mathcal{U}_n^{\mathrm{FA,an}}$ be an analytic modular birational model of $\mathcal{M}_{g,n}^{\mathrm{an}}$ with the finite automorphism property. Lemma 7.2 and Theorem D imply that the open immersion of analytic stacks $\mathcal{N} \hookrightarrow \mathcal{U}_n^{\mathrm{FA,an}}$ is algebraizable to a quasi-finite and separated morphism $j : N \to \mathcal{U}_n$, where N is proper. Since j_{an} is an open immersion, it follows that j is also. Next, form the 2-fiber square:

$$\begin{tikzcd} V \arrow[r, "j'"] \arrow[d, "i'"'] & \mathcal{M}_{g,n} \arrow[d, "i"] \\ N \arrow[r, "j"] & \mathcal{U}_n, \end{tikzcd}$$

Clearly, all maps in the above diagram are open immersions, and it remains to show that i' and j' have dense image. This may be checked after passing to the analytifications, and since analytification commutes with 2-fiber products, we deduce the claim. □

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