COHOMOLOGY AND BASE CHANGE FOR ALGEBRAIC STACKS

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Abstract. We prove that cohomology and base change holds for algebraic stacks, generalizing work of Brochard in the tame case. We also show that Hom-spaces on algebraic stacks are represented by abelian cones, generalizing results of Grothendieck, Brochard, Olsson, Lieblich, and Roth–Starr. To accomplish all of this, we prove that a wide class of relative Ext-functors in algebraic geometry are coherent (in the sense of M. Auslander).

Introduction

Let $f\colon X \to S$ be a proper morphism of locally noetherian schemes and let $\mathcal{F}$ be a coherent sheaf on X that is flat over S (e.g., f is smooth and $\mathcal{F}$ is a vector bundle). If $s \in S$ is a point, then define X_s to be the fiber of f over s. If s has residue field $\kappa(s)$, then for each integer q there is a natural *base change* morphism of $\kappa(s)$-vector spaces

$$b^q(s)\colon (\mathsf{R}^q f_*\mathcal{F}) \otimes_{\mathcal{O}_S} \kappa(s) \to H^q(X_s, \mathcal{F}_{X_s}).$$

Cohomology and Base Change originally appeared in [EGA, III.7.7.5] in a quite sophisticated form. Mumford [Mum70, §II.5] and Hartshorne [Har77, §III.12], however, were responsible for popularizing a version similar to the following. Let $s \in S$ and let q be an integer.

(1) The following are equivalent.
   (a) The morphism $b^q(s)$ is surjective.
   (b) There exists an open neighbourhood U of s such that $b^q(u)$ is an isomorphism for all $u \in U$.
   (c) There exists an open neighbourhood U of s, a coherent $\mathcal{O}_U$-module $\mathcal{Q}$, and an isomorphism of functors:
   $$\mathsf{R}^{q+1}(f_U)_*(\mathcal{F}_{X_U} \otimes_{\mathcal{O}_{X_U}} f_U^*\mathcal{I}) \cong \mathcal{H}om_{\mathcal{O}_U}(\mathcal{Q}, \mathcal{I}),$$
   where $f_U\colon X_U \to U$ is the pullback of f along $U \subseteq S$.

(2) If the equivalent conditions of (1) hold, then the following are equivalent.
   (a) The morphism $b^{q-1}(s)$ is surjective.
   (b) There exists an open neighbourhood V of s such that the restriction of $\mathsf{R}^q f_*\mathcal{F}$ to V is a vector bundle.

In applications, Cohomology and Base Change is frequently applied to pass from geometric objects over fields (e.g., abelian varieties, curves, closed subschemes, etc.) to families of such objects, that is, the "correct" generalization of these objects over general bases. The construction of the Hilbert and Quot schemes for a projective morphism [FGA, No. 221, p. 12] is an illustrative example. In this particular application of Cohomology and Base Change, and many others, it is important to have the ability to work with base schemes S that are not necessarily reduced.

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Because of the fundamental and pervasive nature of Cohomology and Base Change in algebraic geometry, its extension to algebraic stacks is a natural question to address. As noted by S. Brochard [Bro12, App. A], however, in positive characteristic or when the stack has infinite stabilizers, the usual argument can fail. Indeed, the essential problem is that algebraic stacks in these situations can have infinite cohomological dimension, prohibiting the application of the usual "replace the Čech complex by a perfect complex" argument. Brochard was able to prove Cohomology and Base Change for tame stacks, which includes all Deligne–Mumford stacks in characteristic 0, and when the base is regular, but was unable to address the general case.

We now turn to a discussion of the results of this article. Let $f\colon X \to S$ be a morphism of algebraic stacks. Given complexes of $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{N}$ and an integer q, define the *qth relative Ext sheaf* as the $\mathcal{O}_S$-module:

$$\mathcal{E}xt^q(f;\mathcal{M},\mathcal{N}) = \mathcal{H}^q(\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})),$$

see §1 for precise definitions of all objects involved.

The earliest version of Cohomology and Base Change for relative Ext that the author is aware of is due to Altman–Kleiman [AK80, Thm. 1.9]. There, however, the results are constrained; the complex $\mathcal{M}$ is required to be a sheaf and the base changes in question are non-derived. In fact, with their standing hypotheses and assumptions and no derived categories, it is really a feat that they are able to say so much. Altman–Kleiman went even further, proving some Cohomology and Base Change results in the non-noetherian setting.

Recently, a general version of Cohomology and Base Change for relative Ext for locally noetherian algebraic spaces was incorporated into the Stacks Project [Stacks, Tag 08JR]. This was achieved by combining Grothendieck's original formulation [EGA, III.7.7.5] with the generalization of [LN07, Thm. 4.1] to algebraic spaces. Note, however, that the techniques used in the proof of [LN07, Thm. 4.1] are not yet available for algebraic stacks. Our first main result overcomes this obstacle.

Theorem A. *Let $f\colon X \to S$ be a morphism of locally noetherian algebraic stacks that is locally of finite type. Let $\mathcal{M} \in \mathsf{D}^-_{\mathsf{Coh}}(X)$ and let $\mathcal{N} \in \mathsf{Coh}(X)$ be properly supported and flat over S. For each integer q and morphism of algebraic stacks $\tau\colon T \to S$, there is a natural base change morphism:*

$$b^q(\tau)\colon \tau^*\mathcal{E}xt^q(f;\mathcal{M},\mathcal{N}) \to \mathcal{E}xt^q(f_T;\mathsf{L}(\tau_X)^*_{\mathrm{qc}}\mathcal{M},\tau_X^*\mathcal{N}),$$

where $f_T\colon X_T \to T$ denotes the pullback of f by τ, $\tau_X\colon X_T \to X$ denotes the pullback of τ by f, and $\mathsf{L}(\tau_X)^*_{\mathrm{qc}}\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(X_T)$ is the derived pullback functor (see (1.7)*). Let k be a field, let $s\colon \operatorname{Spec} k \to X$ be a morphism, and let q be an integer.*

1. *The following are equivalent.*
   (a) *The morphism $b^q(s)$ is surjective.*
   (b) *There exists an open neighbourhood $U \subseteq S$ of the image of s such that for every $\tau\colon T \to S$ factoring through U, the map $b^q(\tau)$ is an isomorphism.*
   (c) *There exists an open neighbourhood $U \subseteq S$ of the image of s, a coherent $\mathcal{O}_U$-module $\mathcal{Q}$, and an isomorphism of functors:*
   $$\mathcal{E}xt^{q+1}(f_U;\mathcal{M}_{X_U},\mathcal{N}_{X_U}\otimes_{\mathcal{O}_{X_U}} f_U^*\mathcal{I}) \cong \mathcal{H}om_{\mathcal{O}_U}(\mathcal{Q},\mathcal{I}),$$
   where $f_U\colon X_U \to U$ is the pullback of f along $U \subseteq S$.
2. *If the equivalent conditions of (1) hold, then the following are equivalent.*
   (a) *The morphism $b^{q-1}(s)$ is surjective.*
   (b) *There exists an open neighbourhood $V \subseteq S$ of the image of s such that the restriction of $\mathcal{E}xt^q(f;\mathcal{M},\mathcal{N})$ to V is a vector bundle.*

Note that the usual version of Cohomology and Base Change (i.e., the one discussed at the beginning of the Introduction) follows from Theorem A by taking $\mathcal{M} = \mathcal{O}_X[0]$. We also wish to point out that while we are able to state and prove our results for non-separated morphisms, this a consequence of an easy reduction to the proper case (see Lemma 1.6 and Proposition 1.4). These results are in contrast to the Grothendieck Existence Theorem [EGA, III.5.1.4], perhaps better known as "Formal GAGA", where a reduction to the proper case is rarely possible—the desired result is almost always false (see [LS08] and [HR14]).

Let S be a noetherian scheme. In the previous proofs of Theorem A for schemes and algebraic spaces, a special role is played by those additive functors $\mathsf{QCoh}(S) \to \mathsf{QCoh}(S)$ of the form $\mathcal{H}^0(\mathcal{P} \otimes^{\mathsf{L}}_{\mathcal{O}_S} -)$, where $\mathcal{P}$ is a perfect complex (i.e., Zariski-locally quasi-isomorphic to a bounded complex of vector bundles). Note that because $\mathcal{P}$ is a perfect complex, it follows that $\mathcal{P}^\vee = \mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{P}, \mathcal{O}_S) \in \mathsf{D}^b_{\mathsf{Coh}}(S)$ and that there is a natural quasi-isomorphism $\mathcal{P} \otimes^{\mathsf{L}}_{\mathcal{O}_S} \mathcal{I} \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{P}^\vee, \mathcal{I})$.

In this article, we will consider a different—but closely related—class of functors. An additive functor $\mathsf{QCoh}(S) \to \mathsf{QCoh}(S)$ is *corepresentable by a complex* if it is of the form $\mathcal{I} \mapsto \mathcal{H}^0(\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{E}, \mathcal{I}))$, where $\mathcal{E} \in \mathsf{D}^-_{\mathsf{Coh}}(S)$. Our second main result is the following theorem (see Proposition 2.1 for the most general statement).

Theorem B. *Let S be a scheme that is separated and of finite type over a field or $\mathbb{Z}$. Let $f\colon X \to S$ be a morphism of algebraic stacks that is locally of finite type and let $\mathcal{M} \in \mathsf{D}^-_{\mathsf{Coh}}(X)$. If $\mathcal{N} \in \mathsf{Coh}(X)$ is properly supported and flat over S, then for every integer q the functor*

$$\mathcal{E}xt^q(f; \mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}(-))\colon \mathsf{QCoh}(S) \to \mathsf{QCoh}(S)$$

is corepresentable by a complex.

We will use Theorem B to prove Theorem A, but we cannot do it directly. Indeed, the hypotheses on the base S for Theorem B do not coincide with those of Theorem A. The natural remedy is to approximate everything in Theorem A so that Theorem B again applies. We are working with relative Exts and objects of the derived category, however, and these are not amenable to base changes nor approximations. What further complicates matters is that the collection of functors $\mathsf{QCoh}(S) \to \mathsf{QCoh}(S)$ that are corepresentable by a complex is poorly behaved. Indeed, it is not even closed under direct summands [Har98, Prop. 4.6 & Ex. 5.5].

The following generalization of functors that are corepresentable by a complex was considered by M. Auslander [Aus66] in order to correct such deficiencies. For an affine (not necessarily noetherian) scheme S, a functor $F\colon \mathsf{QCoh}(S) \to \mathsf{Ab}$ is *coherent* if there exists a morphism of quasi-coherent $\mathcal{O}_S$-modules $\mathcal{K}_1 \to \mathcal{K}_2$, such that for all $\mathcal{I} \in \mathsf{QCoh}(S)$ there is a natural isomorphism of abelian groups:

$$F(\mathcal{I}) \cong \operatorname{coker}(\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{K}_2, \mathcal{I}) \to \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{K}_1, \mathcal{I})).$$

More recently, R. Hartshorne [Har98] considered applications of coherent functors to algebraic geometry. Hartshorne's article has an excellent account of the history and development of coherent functors—as well as several new results—and we refer the interested reader there for more background on coherent functors in general. In Hartshorne's article, a different class of coherent functors is considered, however: the standing assumptions are that S is noetherian and $\mathcal{K}_1$ and $\mathcal{K}_2$ are coherent $\mathcal{O}_S$-modules. It is easy to show that Hartshorne's coherent functors are precisely the coherent functors of Auslander that preserve direct limits (Lemma 6.6). In this article, we adopt Auslander's definition of coherent functor.

The collection of coherent functors is very well-behaved: it is an abelian category which is closed under extensions and inverse limits—precisely the sort of properties that are convenient to have at one's disposal when performing induction and approximation arguments. Thus, using Theorem B, we can prove the following theorem.

Theorem C. *Fix an affine scheme S and a morphism of algebraic stacks $f\colon X \to S$ that is locally of finite presentation. Let $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$ and let $\mathcal{N} \in \mathsf{QCoh}(X)$, where $\mathcal{N}$ is of finite presentation. If $\mathcal{N}$ is properly supported and flat over S, then the functor*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}(-))\colon \mathsf{QCoh}(S) \to \mathsf{Ab}$$

is coherent.

If S is noetherian and $\mathcal{M} \in \mathsf{D}^-_{\mathsf{Coh}}(X)$, then Theorem C implies that the functor is coherent in the sense of Hartshorne (see Lemma 1.2).

Theorem A is now rendered a simple consequence of a clever result of A. Ogus and G. Bergman [OB72, Cor. 5.1], Theorem C, and some general vanishing results for coherent functors (Corollary 6.7).

An interesting application of Theorem C is the following result. Let $f\colon X \to S$ be a morphism of algebraic stacks and let $\mathcal{M}$ and $\mathcal{N}$ be quasi-coherent $\mathcal{O}_X$-modules. We define the S-presheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_X/S}(\mathcal{M}, \mathcal{N})$ as follows:

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X/S}(\mathcal{M}, \mathcal{N})[T \xrightarrow{\tau} S] = \mathrm{Hom}_{\mathcal{O}_{X_T}}(\tau_X^*\mathcal{M}, \tau_X^*\mathcal{N}),$$

where $\tau_X\colon X \times_S T \to X$ is the projection.

Theorem D. *Let $f\colon X \to S$ be a morphism of algebraic stacks that is locally of finite presentation. Let $\mathcal{M}$ and $\mathcal{N} \in \mathsf{QCoh}(X)$, where $\mathcal{N}$ is of finite presentation. If $\mathcal{N}$ is properly supported and flat over S, then $\underline{\mathrm{Hom}}_{\mathcal{O}_X/S}(\mathcal{M}, \mathcal{N})$ is representable by an abelian cone over S (which is, in particular, affine over S). Moreover, if $\mathcal{M}$ is of finite presentation, then $\underline{\mathrm{Hom}}_{\mathcal{O}_X/S}(\mathcal{M}, \mathcal{N})$ is of finite presentation over S.*

We wish to emphasize that Theorem D is completely elementary once Theorem C is known and does not rely on the algebraicity of Quot functors or Artin's criterion. Using completely different techniques to that employed in Theorems B and C, we also prove a coherence result when nothing is assumed to be flat—at the expense of assuming that X is an algebraic space.

Theorem E. *Fix an affine and noetherian scheme S and a morphism of algebraic spaces $f\colon X \to S$ that is locally of finite type. Let $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$ and let $\mathcal{N} \in \mathsf{D}^b_{\mathrm{qc}}(X)$. If the cohomology sheaves of $\mathcal{N}$ are finitely generated as $\mathcal{O}_X$-modules and properly supported over S, then the functor:*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}(-))\colon \mathsf{QCoh}(S) \to \mathsf{Ab}$$

is coherent.

In particular, for algebraic spaces, one can use Theorem E to prove Theorems A and D. Note that Theorem E does not contradict the work of D. Jaffe [Jaf97, Appendix]. Indeed, there the tensor products are underived, whereas we work systematically with derived tensor products. In [HR12b], Theorem E will be extended to algebraic stacks with finite diagonal.

Relation with other work. In [Hal12], coherent functors featured prominently in a criteria for algebraicity of stacks. Thus Theorem C can be used to show that certain stacks are algebraic [*op. cit.*, §§8–9]. Theorem D can also be used to show that many algebraic stacks of interest have affine diagonals [*loc. cit.*]—generalizing and simplifying the existing work of M. Olsson [Ols06, Prop. 5.10], M. Lieblich [Lie06, Prop. 2.3], and M. Roth and J. Starr [RS09, Thm. 2.1]. In [HR13], Theorem D is combined with the absolute approximation results of D. Rydh [Ryd09] to show that Hilbert stacks and Quot spaces exist without finiteness assumptions.

As noted at the beginning of the Introduction, Grothendieck proved Theorems A and B for a morphism of locally noetherian schemes when $\mathcal{M} \simeq \mathcal{O}_X[0]$ [EGA, III.7.7.5]. In fact, once one reinterprets Grothendieck's formulation in terms of derived categories (which didn't exist at the time), it turns out that the given arguments also cover the situation where $\mathcal{M}$ is a perfect complex. Combining the derived category version of Grothendieck's results with the perfect approximation results of [LN07, Thm. 4.1] easily gives Theorems A and B for a morphism of locally noetherian schemes. For algebraic spaces, as noted earlier, this was written up in the Stacks Project [Stacks, Tag 08JR]—the essential point being the extension of [LN07, Thm. 4.1] to algebraic spaces [Stacks, Tag 08HP]. These results are sufficient to imply Theorem C when S is noetherian, X is an algebraic space, and $\mathcal{M} \in \mathsf{D}^-_{\mathsf{Coh}}(X)$ (Lemma 3.13).

S. Brochard [Bro12, Prop. A.4.3] essentially proved Theorems A and B when $\mathcal{M}$ is perfect, S is a scheme, and either $f\colon X \to S$ is a tame morphism of locally noetherian algebraic stacks or S is regular. In particular, Brochard's results imply Theorem C when $\mathcal{M}$ is perfect, S is noetherian, and either X is tame or S is regular. Furthermore, Brochard's results can be used to prove Theorem D when $\mathcal{M}$ is the cokernel of a morphism of vector bundles and either f is tame or S is regular.

In [EGA, III.7.7.8], Theorem D is proved in the situation that $\mathcal{M}$ is the cokernel of a morphism of vector bundles and f is a morphism schemes. In [Stacks, Tag 08K6], this result is extended to where $\mathcal{M}$ is quasi-coherent and f is a morphism of algebraic spaces. For f a morphism of algebraic stacks, Roth–Starr [RS09, Thm. 2.1] proved Theorem D under the assumption that $\mathcal{M}$ is of finite presentation and assuming that $\underline{\mathrm{Hom}}_{\mathcal{O}_{X/S}}(\mathcal{M},\mathcal{N})$ is representable by an algebraic space. That $\underline{\mathrm{Hom}}_{\mathcal{O}_{X/S}}(\mathcal{M},\mathcal{N})$ is an algebraic space follows from [Lie06, Prop. 2.3] or the algebraicity of the Quot functor for algebraic stacks [OS03, Ols05, Ols06], which all rely on Artin's criterion [Art69].

Altman–Kleiman [AK80, 1.1] worked in the non-noetherian and non-derived context, though their results at the time were new even in the noetherian context. There they prove Theorems A, B, and C when f is proper, flat, and of finite presentation, and $\mathcal{M}$ is an $\mathcal{O}_X$-module of finite presentation and flat over S.

We do not believe that Theorem E has been considered previously.

After completing this paper we also located in the literature two very nice papers of H. Flenner addressing similar results for analytic spaces. In particular, if S is excellent and of finite Krull dimension, then Theorem A follows from the results of [Fle81, §7]. Also, Proposition 2.1 (the strongest form of Theorem B) and Theorem D are the main results of [Fle82]—this is in the analytic category, however, thus $\mathcal{M}$ and $\mathcal{I}$ are also assumed to have coherent cohomology. Without further assumptions on f and $\mathcal{M}$ (e.g., $\mathcal{M}$ and X are flat over S when $q \geq 1$ [AK80, §1]) we cannot see how Theorem A can be easily reduced to the case where S meets Flenner's hypotheses. In fact, we use coherent functors to accomplish this descent, which is effectively the content of Theorem C. This has no counterpart in the analytic category where everything is excellent, coherent, and admits a dualizing complex.

Assumptions, conventions, and notations. For a scheme T, denote by $|T|$ the underlying topological space (with the Zariski topology) and $\mathcal{O}_T$ the (Zariski) sheaf of rings on $|T|$. For $t \in |T|$, let $\kappa(t)$ denote the residue field. Denote by $\mathsf{QCoh}(T)$ (resp. $\mathsf{Coh}(T)$) the abelian category of quasi-coherent (resp. coherent) sheaves on the scheme T. Let Sch/T denote the category of schemes over T. The big étale site over T will be denoted by $(\mathsf{Sch}/T)_{\text{Ét}}$.

For a ring A and an A-module M, denote the quasi-coherent $\mathcal{O}_{\operatorname{Spec} A}$-module associated to M by $\widetilde{M}$. Denote the abelian category of all (resp. coherent) A-modules by $\mathsf{Mod}(A)$ (resp. $\mathsf{Coh}(A)$).

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1. Derived categories of sheaves on algebraic stacks

In this section, we briefly review derived categories of sheaves on algebraic stacks, following [Ols07] (for the bounded below case) and [LO08] (for the unbounded case). For generalities on unbounded derived categories on ringed sites, we refer the reader to [KS06, §18.6].

Fix an algebraic stack X. We take $\mathsf{Mod}(X)$ (resp. $\mathsf{QCoh}(X)$) to denote the abelian category of $\mathcal{O}_X$-modules (resp. quasi-coherent $\mathcal{O}_X$-modules) on the lisse-étale site of X [LMB, 12.1]. Let $\mathsf{D}(X)$ denote the unbounded derived category of $\mathsf{Mod}(X)$. Define $\mathsf{D}_{\mathrm{qc}}(X)$ to be the full subcategory of $\mathsf{D}(X)$ consisting of objects with quasi-coherent cohomology sheaves. Superscripts such as $+$, $-$, $\geq n$, and b decorating $\mathsf{D}(X)$ and $\mathsf{D}_{\mathrm{qc}}(X)$ should be interpreted as usual. In addition, if X is locally noetherian, one may consider the category of coherent sheaves $\mathsf{Coh}(X)$ and the derived category $\mathsf{D}_{\mathsf{Coh}}(X)$ of $\mathcal{O}_X$-modules with coherent cohomology sheaves.

If X is a Deligne–Mumford stack, there is an associated small étale site $X_{\text{ét}}$. We take $\mathsf{Mod}(X_{\text{ét}})$ (resp. $\mathsf{QCoh}(X_{\text{ét}})$) to denote the abelian category of $\mathcal{O}_{X_{\text{ét}}}$-modules (resp. quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-modules). There are naturally induced morphisms of abelian categories $\mathsf{Mod}(X) \to \mathsf{Mod}(X_{\text{ét}})$ and $\mathsf{QCoh}(X) \to \mathsf{QCoh}(X_{\text{ét}})$. Let $\mathsf{D}_{\mathrm{qc}}(X_{\text{ét}})$ denote the triangulated category $\mathsf{D}_{\mathsf{QCoh}(X_{\text{ét}})}(\mathsf{Mod}(X_{\text{ét}}))$. Then the natural functor $\mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(X_{\text{ét}})$ is an equivalence of categories. If X is a scheme, the corresponding statement for the Zariski site is also true.

We now briefly recall cohomological descent (after [LO08, Ex. 2.2.5]). Let X be an algebraic stack and let $p_\bullet \colon U_\bullet \to X$ be a smooth hypercovering. For example, let $p_0 \colon U_0 \to X$ be a smooth surjection, where U_0 is an algebraic space; then the 0-coskeleton, where U_i is the fiber product of U_0 with itself $i+1$ times over X, is such a hypercovering. Let $U^+_{\bullet,\text{lis-ét}}$ and $U^+_{\bullet,\text{ét}}$ be the resulting strictly simplicial topoi [Ols07, §4]. *Cohomological descent* [LO08, Ex. 2.5.5] implies that the natural functors:

$$\mathsf{QCoh}(X) \to \mathsf{QCoh}(U^+_{\bullet,\text{lis-ét}}) \leftarrow \mathsf{QCoh}(U^+_{\bullet,\text{ét}}) \tag{1.1}$$

$$\mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathsf{QCoh}(U^+_{\bullet,\text{lis-ét}})}(\mathsf{Mod}(U^+_{\bullet,\text{lis-ét}})) \leftarrow \mathsf{D}_{\mathsf{QCoh}(U^+_{\bullet,\text{ét}})}(\mathsf{Mod}(U^+_{\bullet,\text{ét}})) \tag{1.2}$$

are all equivalences.

Let X be an algebraic stack. A complex $\mathcal{P} \in \mathsf{D}^-_{\mathrm{qc}}(X)$ is *pseudo-coherent* if locally for the smooth topology on X it is quasi-isomorphic to a bounded above complex of vector bundles of finite rank. Equivalently, $\mathcal{P}$ is pseudo-coherent if its image in $\mathsf{D}(\mathsf{Mod}(U^+_{\bullet,\text{ét}}))$ is $\mathsf{QCoh}(U^+_{\bullet,\text{ét}})$-pseudo-coherent in the sense of [SGA6, Defn. I.2.1], where $U_\bullet \to X$ is some smooth hypercovering by algebraic spaces. In particular, if X is locally noetherian, then $\mathcal{P} \in \mathsf{D}^-_{\mathsf{Coh}}(X)$ if and only if $\mathcal{P}$ is pseudo-coherent.

We now record for future reference some useful formulae. If $\mathcal{M}$ and $\mathcal{N} \in \mathsf{D}(X)$, then there is the derived tensor product $\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N} \in \mathsf{D}(X)$, the derived sheaf Hom functor $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \in \mathsf{D}(X)$ and the derived global Hom functor $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \in \mathsf{D}(\mathsf{Ab})$. For all $\mathcal{P} \in \mathsf{D}(X)$ we have a functorial isomorphism:

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N}, \mathcal{P}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{P})), \tag{1.3}$$

as well as a functorial quasi-isomorphism:

$$\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N}, \mathcal{P}) \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{P})). \tag{1.4}$$

Let $\mathsf{R}\Gamma(X,-) = \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,-)$. Then there is also a natural quasi-isomorphism:

$$\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \simeq \mathsf{R}\Gamma\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}). \tag{1.5}$$

If $\mathcal{M}$ and $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(X)$, then $\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(X)$. Moreover, if $\mathcal{M}$ and $\mathcal{N}$ are pseudo-coherent, then $\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N}$ is pseudo-coherent. Similarly, if $\mathcal{M}$ is pseudo-coherent and $\mathcal{N} \in \mathsf{D}^+_{\mathrm{qc}}(X)$, then $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \in \mathsf{D}^+_{\mathrm{qc}}(X)$. Finally, if X is locally noetherian and $\mathcal{M} \in \mathsf{D}^-_{\mathsf{Coh}}(X)$ and $\mathcal{N} \in \mathsf{D}^+_{\mathsf{Coh}}(X)$, then $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \in \mathsf{D}^+_{\mathsf{Coh}}(X)$. These results are all consequences of [Ols07, §6], [LO08, §2], and [SGA6, II.2.16 & II.3.7].

Fix a morphism of algebraic stacks $f\colon X \to Y$. Let $\mathsf{R}f_*\colon \mathsf{D}(X) \to \mathsf{D}(Y)$ be the unbounded derived functor of $f_*\colon \mathsf{Mod}(X) \to \mathsf{Mod}(Y)$. If the morphism f is quasi-compact and quasi-separated, then the restriction of $\mathsf{R}f_*$ to $\mathsf{D}^+_{\mathrm{qc}}(X)$ induces a functor $\mathsf{R}f_*\colon \mathsf{D}^+_{\mathrm{qc}}(X) \to \mathsf{D}^+_{\mathrm{qc}}(Y)$ [Ols07, Lem. 6.20]. If X and Y are Deligne–Mumford stacks, then the restriction of $\mathsf{R}f_*$ to $\mathsf{D}^+_{\mathrm{qc}}(X)$ coincides with the restriction of the derived functor of $(f_{\text{ét}})_*\colon \mathsf{Mod}(X_{\text{ét}}) \to \mathsf{Mod}(Y_{\text{ét}})$ to $\mathsf{D}_{\mathrm{qc}}(X_{\text{ét}})$. That is, the following diagram 2-commutes, and the horizontal functors are equivalences:

$$\begin{array}{ccc} \mathsf{D}^+_{\mathrm{qc}}(X) & \xrightarrow{\sim} & \mathsf{D}^+_{\mathrm{qc}}(X_{\text{ét}}) \\ {\scriptstyle \mathsf{R}f_*}\downarrow & & \downarrow{\scriptstyle \mathsf{R}(f_{\text{ét}})_*} \\ \mathsf{D}^+_{\mathrm{qc}}(Y) & \xrightarrow{\sim} & \mathsf{D}^+_{\mathrm{qc}}(Y_{\text{ét}}) \end{array} \tag{1.6}$$

The following lemma, which is put into a broader context in [HR12b, §2], will be useful.

Lemma 1.1. *Let $f\colon X \to Y$ be a morphism of algebraic stacks that is quasi-compact, quasi-separated, and representable. Then the restriction of the functor $\mathsf{R}f_*\colon \mathsf{D}(X) \to \mathsf{D}(Y)$ to $\mathsf{D}_{\mathrm{qc}}(X)$ factors through $\mathsf{D}_{\mathrm{qc}}(Y)$. In addition, if Y is a Deligne–Mumford stack, then the restriction of $\mathsf{R}(f_{\text{ét}})_*\colon \mathsf{D}(X_{\text{ét}}) \to \mathsf{D}(Y_{\text{ét}})$ to $\mathsf{D}_{\mathrm{qc}}(X_{\text{ét}})$ factors through $\mathsf{D}_{\mathrm{qc}}(Y_{\text{ét}})$ and the following diagram 2-commutes:*

$$\begin{array}{ccc} \mathsf{D}_{\mathrm{qc}}(X) & \xrightarrow{\sim} & \mathsf{D}_{\mathrm{qc}}(X_{\text{ét}}) \\ {\scriptstyle \mathsf{R}f_*}\downarrow & & \downarrow{\scriptstyle \mathsf{R}(f_{\text{ét}})_*} \\ \mathsf{D}_{\mathrm{qc}}(Y) & \xrightarrow{\sim} & \mathsf{D}_{\mathrm{qc}}(Y_{\text{ét}}). \end{array}$$

Proof. Both statements are local on Y, so we may assume that Y is an affine scheme. Since f is representable, it follows that X is an algebraic space. By [Stacks, Tag 08FA] or [LO08, Rem. 2.1.11] combined with a standard Čech complex argument, it follows that the restriction of $\mathsf{R}(f_{\text{ét}})_*\colon \mathsf{D}(X_{\text{ét}}) \to \mathsf{D}(Y_{\text{ét}})$ to $\mathsf{D}_{\mathrm{qc}}(X_{\text{ét}})$ factors through $\mathsf{D}_{\mathrm{qc}}(Y_{\text{ét}})$.

Now the universal properties of derived functors give a natural transformation $\delta(-)\colon (\mathsf{R}f_*(-))|_{Y_{\text{ét}}} \Rightarrow \mathsf{R}(f_{\text{ét}})_*((-)|_{X_{\text{ét}}})$. It remains to show that $\delta(\mathcal{M})$ is a quasi-isomorphism for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$. To verify this, it is sufficient to prove that for every étale morphism $j\colon W \to Y$, where W is an affine scheme, and $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$, the induced morphism:

$$\mathsf{R}\mathrm{Hom}_{\mathcal{O}_{Y_{\text{ét}}}}(j_!\mathcal{O}_{W_{\text{ét}}}, (\mathsf{R}f_*\mathcal{M})_{Y_{\text{ét}}}) \to \mathsf{R}\mathrm{Hom}_{\mathcal{O}_{Y_{\text{ét}}}}(j_!\mathcal{O}_{W_{\text{ét}}}, \mathsf{R}(f_{\text{ét}})_*(\mathcal{M}_{X_{\text{ét}}}))$$

is a quasi-isomorphism. By adjunction, the morphism above is a quasi-isomorphism whenever the natural morphism:

$$\mathsf{R}\Gamma(W, \mathsf{R}f_*\mathcal{M}) \to \mathsf{R}\Gamma(W, \mathsf{R}(f_{\text{ét}})_*(\mathcal{M}_{X_{\text{ét}}}))$$

is a quasi-isomorphism. We are free to replace Y by W, so it is sufficient to prove that $\mathsf{R}\Gamma(Y, \delta(\mathcal{M}))$ is a quasi-isomorphism for every $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$. Note, however, that $\mathsf{R}\Gamma(Y, \mathsf{R}f_*\mathcal{M}) \simeq \mathsf{R}(X, \mathcal{M})$ and $\mathsf{R}\Gamma(Y, \mathsf{R}(f_{\text{ét}})_*(\mathcal{M}_{X_{\text{ét}}})) \simeq \mathsf{R}\Gamma(X_{\text{ét}}, \mathcal{M}_{X_{\text{ét}}})$ and the morphism $\mathsf{R}\Gamma(Y, \delta(\mathcal{M}))$ is equivalent to the natural morphism $\mathsf{R}\Gamma(X, \mathcal{M}) \to \mathsf{R}\Gamma(X_{\text{ét}}, \mathcal{M}_{X_{\text{ét}}})$. By cohomological descent (1.2), this morphism is a quasi-isomorphism, and the result follows. □

A morphism of algebraic stacks $f\colon X \to Y$ does not necessarily induce a left-exact morphism of corresponding lisse-étale sites [Beh03, 5.3.12], thus the construction of the correct derived functors of $f^*\colon \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ is somewhat subtle. There are currently two approaches to constructing these functors. The first, due to Olsson [Ols07] and Laszlo–Olsson [LO08], uses cohomological descent. The other approach appears in the Stacks Project [Stacks], but uses a completely different formulation (big sites) and requires significant amounts of technology that many people may not be familiar with.

In this article, we will follow the approach of Olsson and Laszlo–Olsson. Their crucial observation was that if $f\colon X \to Y$ is a morphism of algebraic stacks, then there exist smooth hypercoverings by algebraic spaces $U_\bullet \to X$ and $V_\bullet \to Y$, together with a lift of f to a morphism of simplicial algebraic spaces $f_\bullet\colon U_\bullet \to V_\bullet$ and that there is an induced morphism of strictly simplicial topoi $f^+_{\bullet,\text{ét}}\colon U^+_{\bullet,\text{ét}} \to V^+_{\bullet,\text{ét}}$ with left-exact inverse image. They then defined $\mathsf{L}f^*_{\text{qc}}$ via the equivalences (1.2) and the functor $\mathsf{L}(f^+_{\bullet,\text{ét}})^*$. In particular, there exists a functor $\mathsf{L}f^*_{\text{qc}}\colon \mathsf{D}_{\text{qc}}(Y) \to \mathsf{D}_{\text{qc}}(X)$ such that $\mathcal{H}^0(\mathsf{L}f^*_{\text{qc}}\mathcal{I}[0]) \cong f^*\mathcal{I}$, whenever $\mathcal{I} \in \mathsf{QCoh}(Y)$. In addition, if f is quasi-compact and quasi-separated, $\mathcal{I} \in \mathsf{D}_{\text{qc}}(Y)$, and $\mathcal{N} \in \mathsf{D}^+_{\text{qc}}(X)$, then there is a natural isomorphism:

$$\operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{I}, \mathsf{R}f_*\mathcal{N}) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathsf{L}f^*_{\text{qc}}\mathcal{I}, \mathcal{N}). \tag{1.7}$$

In the situation where X and Y are Deligne–Mumford stacks, there also exists a derived functor $\mathsf{L}f^*_{\text{ét}}\colon \mathsf{D}_{\text{qc}}(Y_{\text{ét}}) \to \mathsf{D}_{\text{qc}}(X_{\text{ét}})$. The restriction of $\mathsf{L}f^*_{\text{qc}}$ to $\mathsf{D}_{\text{qc}}(Y_{\text{ét}})$ coincides with $\mathsf{L}f^*_{\text{ét}}$.

Let $f\colon X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic stacks. If $\mathcal{I}$ is pseudo-coherent on Y and $\mathcal{N} \in \mathsf{D}^+_{\text{qc}}(X)$, then the isomorphism (1.7) is readily strengthened to a natural quasi-isomorphism in $\mathsf{D}^+_{\text{qc}}(Y)$:

$$\mathsf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}, \mathsf{R}f_*\mathcal{N}) \simeq \mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathsf{L}f^*_{\text{qc}}\mathcal{I}, \mathcal{N}). \tag{1.8}$$

If $f\colon X \to Y$ is a representable, quasi-compact, and quasi-separated morphism of algebraic stacks, then the isomorphism (1.7) and the quasi-isomorphism (1.8) are easily extended to all $\mathcal{N} \in \mathsf{D}_{\text{qc}}(X)$ using the arguments of Lemma 1.1.

Let $f\colon X \to Y$ be a morphism of algebraic stacks and let $\mathcal{N} \in \mathsf{D}^b_{\text{qc}}(X)$. We say that $\mathcal{N}$ has *finite tor-dimension over* Y if there exists a non-negative integer n such that for all integers i and $\mathcal{I} \in \mathsf{D}^{\geq i}_{\text{qc}}(Y)$ we have that $\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\text{qc}}\mathcal{I} \in \mathsf{D}^{\geq i-n}_{\text{qc}}(X)$. Equivalently, if for any morphism of algebraic spaces $\tilde{f}\colon U \to V$ that fits into a 2-commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{p} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{q} & Y, \end{array}$$

where p and q are smooth and surjective, then $\mathcal{N}|_{U_{\text{ét}}}$ is of finite tor-dimension relative to $V_{\text{ét}}$ in the sense of [SGA6, III.3.1]. The following Lemma will be used at several points throughout the article.

Lemma 1.2. *Fix a scheme S and a quasi-compact and quasi-separated morphism of algebraic stacks $f\colon X \to S$. Let $\mathcal{M} \in \mathsf{D}^-_{\text{qc}}(X)$ and $\mathcal{N} \in \mathsf{D}^b_{\text{qc}}(X)$. If $\mathcal{M}$ is pseudo-coherent and $\mathcal{N}$ has finite tor-dimension over S, then the following functor preserves direct limits:*

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\text{qc}}(-))\colon \mathsf{QCoh}(S) \to \mathsf{Ab}.$$

Proof. We may express the functor in question as the following composition:

$$\mathcal{H}^0(\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\text{qc}}(-))).$$

Clearly, $\mathcal{H}^0(-)$ preserves direct limits. Also, by assumption, $\mathcal{N}$ is of finite tor-dimension over S. So there exists a non-negative integer n such that $\mathcal{N}\otimes_{\mathcal{O}_X} \mathsf{L}f_{\mathrm{qc}}^*\mathcal{I} \in \mathsf{D}_{\mathrm{qc}}^{\geq n}(X)$ for all $\mathcal{I} \in \mathsf{QCoh}(S)$. Also, $\mathcal{M}$ is bounded above and pseudo-coherent, so there exists an integer m such that $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f_{\mathrm{qc}}^*\mathcal{I}) \in \mathsf{D}_{\mathrm{qc}}^{\geq m}(X)$ for all $\mathcal{I} \in \mathsf{QCoh}(S)$. Since $\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X} -$ and $\mathsf{L}f_{\mathrm{qc}}^*$ preserve direct limits, it remains to show that the restrictions of $\mathsf{R}f_*$ and $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},-)$ to $\mathsf{D}_{\mathrm{qc}}^{\geq l}(X)$ preserve direct limits for each integer l.

For $\mathsf{R}f_*$, we first apply the hypercohomology spectral sequence:

$$\mathcal{H}^p(\mathsf{R}f_*\mathcal{H}^q(\mathcal{G})) \Rightarrow \mathcal{H}^{p+q}(\mathsf{R}f_*\mathcal{G})$$

to reduce the claim to proving that the functors $\mathsf{R}^pf_*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(S)$ preserve direct limits. This is local on S for the smooth topology, so we may assume henceforth that S is an affine scheme. It remains to prove that the cohomology functors $H^p(X_{\text{lis-ét}}, -)\colon \mathsf{Mod}(X) \to \mathsf{Ab}$ preserve direct limits, which is well-known (see [Stacks, Tag 0739] for instance).

For $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},-)$, we note that the claim is local on X for the smooth topology, so we may consequently assume that X is an affine scheme. Since $\mathcal{M}$ is pseudo-coherent, for every integer q there exists a morphism $v_q\colon \mathcal{P}_q \to \mathcal{M}$, where $\mathcal{P}_q$ is a perfect complex, such that the induced morphism $\tau^{\geq q}v_q\colon \tau^{\geq q}\mathcal{P}_q \to \tau^{\geq q}\mathcal{M}$ is a quasi-isomorphism. If $q' < l-q$, then the natural maps:

$$\mathcal{H}^{q'}(\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{I})) \to \mathcal{H}^{q'}(\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}_q,\mathcal{I})) \leftarrow \mathcal{H}^{q'}(\mathcal{P}_q^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{I})$$

are isomorphisms. The result follows. □

The following variants of the well-known Projection Formula will also be useful.

Proposition 1.3. *Let $f\colon X \to Y$ be a morphism of algebraic stacks that is quasi-compact and quasi-separated. Let $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(X)$ and let $\mathcal{I} \in \mathsf{D}_{\mathrm{qc}}(Y)$. Assume that one of the following conditions is satisfied.*

1. *$\mathcal{N}$ is bounded below and $\mathcal{I}$ has finite tor-dimension over Y.*
2. *$\mathcal{N}$ has finite tor-dimension over Y and $\mathcal{I}$ is bounded below.*
3. *f is representable.*

Then there is a natural quasi-isomorphism:

$$(\mathsf{R}f_*\mathcal{N})\otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathcal{I} \simeq \mathsf{R}f_*(\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathsf{L}f_{\mathrm{qc}}^*\mathcal{I}).$$

Proof. For (1) and (2) (resp. (3)), let $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}^+(X)$ (resp. $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(X)$). Since f is quasi-compact and quasi-separated (resp. and representable), by trivial duality (1.7), there is a natural morphism $\mathsf{L}f_{\mathrm{qc}}^*\mathsf{R}f_*\mathcal{N} \to \mathcal{N}$. Let $\mathcal{I} \in \mathsf{D}_{\mathrm{qc}}(Y)$. Applying the functor $-\otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f_{\mathrm{qc}}^*\mathcal{I}$ to this morphism produces a natural morphism:

$$(\mathsf{L}f_{\mathrm{qc}}^*\mathsf{R}f_*\mathcal{N})\otimes^{\mathsf{L}}_{\mathcal{O}_X} (\mathsf{L}f_{\mathrm{qc}}^*\mathcal{I}) \to \mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X} (\mathsf{L}f_{\mathrm{qc}}^*\mathcal{I}).$$

Pre-composing this morphism with the inverse of the natural quasi-isomorphism:

$$(\mathsf{L}f_{\mathrm{qc}}^*\mathsf{R}f_*\mathcal{N})\otimes^{\mathsf{L}}_{\mathcal{O}_X} (\mathsf{L}f_{\mathrm{qc}}^*\mathcal{I}) \simeq \mathsf{L}f_{\mathrm{qc}}^*(\mathsf{R}f_*\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathcal{I})$$

gives a natural morphism:

$$\mathsf{L}f_{\mathrm{qc}}^*(\mathsf{R}f_*\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathcal{I}) \to \mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f_{\mathrm{qc}}^*\mathcal{I}.$$

In case (3), f is representable so trivial duality (1.7) produces the claimed natural morphism. In the cases (1) and (2) the complex $\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f_{\mathrm{qc}}^*\mathcal{I}$ is bounded below, so trivial duality applies in this case too and we obtain the claimed morphism.

Verifying that the natural morphism is quasi-isomorphism is local on Y for the smooth topology, so we may assume that Y is an affine scheme. In the cases (1) and (2) the arguments of [SGA6, III.3.7] apply without change (which are just "way-out right" arguments [Har66, I.7.1]). For the case (3), the result follows from the commutativity of the diagram in Lemma 1.1 and [Stacks, Tag 08IN], which is similar to [Nee96, Prop. 5.3]. □

Consider a diagram of algebraic stacks:

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

If $\mathcal{N} \in \mathsf{QCoh}(X)$ and $\mathcal{F} \in \mathsf{QCoh}(Y')$, then they are *tor-independent over Y* if for any 2-commutative diagram:

$$\begin{array}{ccccc} & & \operatorname{Spec} B & & \\ & & {\scriptstyle \tilde{f}}\downarrow & \searrow{\scriptstyle p} & \\ \operatorname{Spec} A' & \xrightarrow{\tilde{g}} & \operatorname{Spec} A & & X \\ & \searrow{\scriptstyle q'} & & \searrow{\scriptstyle q} & \downarrow{\scriptstyle f} \\ & & Y' & \xrightarrow{g} & Y, \end{array}$$

where p, q, and q' are smooth, then

$$\operatorname{Tor}_i^A(\mathcal{F}(\operatorname{Spec} A' \xrightarrow{q'} Y'), \mathcal{N}(\operatorname{Spec} B \xrightarrow{p} X)) = 0$$

for all integers $i > 0$. We say that $\mathcal{N}$ and $g\colon Y' \to Y$ are *tor-independent over Y* if $\mathcal{N}$ and $\mathcal{O}_{Y'}$ are tor-independent over Y. Note that if $\mathcal{N}$ is flat over Y, then it is tor-independent of every $\mathcal{F} \in \mathsf{QCoh}(Y')$ and so is tor-independent of every $g\colon Y' \to Y$. In the following Corollary, we further refine Proposition 1.3(2).

Corollary 1.4. *Fix a 2-cartesian of diagram of algebraic stacks:*

$$\begin{array}{ccc} X_T & \xrightarrow{\tau_X} & X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\tau} & S, \end{array}$$

where τ is quasi-compact and quasi-separated. Let $\mathcal{N} \in \mathsf{QCoh}(X)$ and suppose that $\mathcal{N}$ and τ are tor-independent.

(1) *If f is affine, then there is a natural quasi-isomorphism:*

$$\mathsf{L}\tau^*_{\mathrm{qc}}\mathsf{R}f_*\mathcal{N} \simeq \mathsf{R}(f_T)_*\tau_X^*\mathcal{N}.$$

(2) *If $\mathcal{N}$ is of finite tor-dimension over S and $\mathcal{I} \in \mathsf{D}^+_{\mathrm{qc}}(T)$, then there is a natural quasi-isomorphism:*

$$\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}\tau_*\mathcal{I} \simeq \mathsf{R}(\tau_X)_*(\tau_X^*\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_{X_T}} \mathsf{L}(f_T)^*_{\mathrm{qc}}\mathcal{I}).$$

(3) *If $\mathcal{J} \in \mathsf{QCoh}(T)$ and $\mathcal{N}$ is flat over S, then there is a natural quasi-isomorphism:*

$$\tau_X^*\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_{X_T}} \mathsf{L}(f_T)^*_{\mathrm{qc}}\mathcal{J} \simeq \tau_X^*\mathcal{N} \otimes_{\mathcal{O}_{X_T}} (f_T)^*\mathcal{J}.$$

Proof. For (1), by trivial duality (1.7), there are natural morphisms:

$$\mathsf{L}\tau^*_{\mathrm{qc}}\mathsf{R}f_*\mathcal{N} \to \mathsf{R}(f_T)_*\mathsf{L}(\tau_X)^*_{\mathrm{qc}}\mathcal{N} \to \mathsf{R}(f_T)_*\tau_X^*\mathcal{N}.$$

Proving that the above morphism is a quasi-isomorphism is local on S and T for the smooth topology, so we may assume that $S = \operatorname{Spec} A$, $T = \operatorname{Spec} B$, $X = \operatorname{Spec} C$, $X_T = \operatorname{Spec}(B \otimes_A C)$, and $\mathcal{N}$ arises from a C-module N such that $\operatorname{Tor}_i^A(N, B) = 0$ for all $i > 0$. The morphism in question factors as the following composition of quasi-isomorphisms:

$$B \otimes^{\mathsf{L}}_A N \simeq B \otimes_A N \simeq (B \otimes_A C) \otimes_C N,$$

and the result follows.

For (2), tensoring the natural map $\mathsf{L}(\tau_X)^*_{\mathrm{qc}}\mathcal{N} \to \tau_X^*\mathcal{N}$ by $\mathsf{L}(\tau_X)^*_{\mathrm{qc}}\mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}\tau_*\mathcal{I}$ induces by functoriality natural maps:

$$\begin{aligned}\mathsf{L}(\tau_X)^*_{\mathrm{qc}}(\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}\tau_*\mathcal{I}) &\to (\tau_X^*\mathcal{N}) \otimes^{\mathsf{L}}_{\mathcal{O}_{X_T}} \mathsf{L}(\tau_X)^*_{\mathrm{qc}}\mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}\tau_*\mathcal{I} \\ &\simeq (\tau_X^*\mathcal{N}) \otimes^{\mathsf{L}}_{\mathcal{O}_{X_T}} \mathsf{L}(f_T)^*_{\mathrm{qc}}\mathsf{L}\tau^*_{\mathrm{qc}}\mathsf{R}\tau_*\mathcal{I} \\ &\to (\tau_X^*\mathcal{N}) \otimes^{\mathsf{L}}_{\mathcal{O}_{X_T}} \mathsf{L}(f_T)^*_{\mathrm{qc}}\mathcal{I} \qquad \text{(by (1.7))}.\end{aligned}$$

Since $\mathcal{N}$ is of finite tor-dimension over S and is tor-independent of τ, it follows that $\tau_X^*\mathcal{N}$ is of finite tor-dimension over T. Indeed, this is smooth local on S, T, and X, so we may assume that $S = \operatorname{Spec} A$, $T = \operatorname{Spec} B$, $X = \operatorname{Spec} C$, $X_T = \operatorname{Spec}(B \otimes_A C)$, and $\mathcal{N}$ arises from a C-module N of finite tor-dimension over A such that $\operatorname{Tor}_i^A(N, B) = 0$ for all $i > 0$. To prove that τ_X^*N, which arises from the $C \otimes_A B$-module $N \otimes_C (C \otimes_A B)$, is of finite tor-dimension over T, we simply observe that if $I \in \mathsf{D}(B)$, then there are natural quasi-isomorphisms:

$$[N \otimes_C (C \otimes_A B)] \otimes^{\mathsf{L}}_B I \simeq [N \otimes_A B] \otimes^{\mathsf{L}}_B I \simeq [N \otimes^{\mathsf{L}}_A B] \otimes^{\mathsf{L}}_B I \simeq N \otimes^{\mathsf{L}}_A I,$$

and the claim follows. In particular, $\tau_X^*\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_{X_T}} \mathsf{L}(f_T)^*_{\mathrm{qc}}\mathcal{I}$ is bounded below. Thus, by trivial duality (1.7), we obtain a natural morphism:

$$\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}\tau_*\mathcal{I} \to \mathsf{R}(\tau_X)_*(\tau_X^*\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_{X_T}} \mathsf{L}(f_T)^*_{\mathrm{qc}}\mathcal{I}). \tag{1.9}$$

Verifying that the map (1.9) is a quasi-isomorphism is smooth local on S and X, so we may assume that they are both affine. Hence f is affine and it suffices verify that the map (1.9) is a quasi-isomorphism after applying the functor $\mathsf{R}f_*$. We now have the following commutative diagram of quasi-isomorphisms:

$$\begin{array}{ccc}\mathsf{R}f_*(\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}\tau_*\mathcal{I}) & \xrightarrow{\mathsf{R}f_*(1.9)} & \mathsf{R}f_*\mathsf{R}(\tau_X)_*(\tau_X^*\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_{X_T}} \mathsf{L}(f_T)^*_{\mathrm{qc}}\mathcal{I}) \\ \sim\uparrow & & \downarrow\sim \\ \mathsf{R}f_*\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_S} \mathsf{R}\tau_*\mathcal{I} & & \mathsf{R}\tau_*\mathsf{R}(f_T)_*(\tau_X^*\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_{X_T}} \mathsf{L}(f_T)^*_{\mathrm{qc}}\mathcal{I}) \\ \sim\downarrow & & \downarrow\sim \\ \mathsf{R}\tau_*(\mathsf{L}\tau^*_{\mathrm{qc}}\mathsf{R}f_*\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_T} \mathcal{I}) & \xrightarrow{\sim} & \mathsf{R}\tau_*(\mathsf{R}(f_T)_*\tau_X^*\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_T} \mathcal{I}),\end{array}$$

where the quasi-isomorphism along the bottom is from (1), and the vertical morphisms are from Proposition 1.3. The result follows.

For (3), the morphism in question is:

$$\tau_X^*\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_{X_T}} \mathsf{L}(f_T)^*_{\mathrm{qc}}\mathcal{J} \to \tau^{\geq 0}(\tau_X^*\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_{X_T}} \mathsf{L}(f_T)^*_{\mathrm{qc}}\mathcal{J}) \simeq \tau_X^*\mathcal{N} \otimes_{\mathcal{O}_{X_T}} (f_T)^*\mathcal{J}.$$

That this is a quasi-isomorphism is smooth local on S, T, and X so we may assume that $S = \operatorname{Spec} A$, $T = \operatorname{Spec} B$, $X = \operatorname{Spec} C$, $X_T = \operatorname{Spec}(B \otimes_A C)$, $\mathcal{N}$ arises from a C-module N that is flat over A, and $\mathcal{J}$ arises from a B-module J. Note that $N \otimes_A B$ is a flat B-module and since the morphism factors as the following sequence of quasi-isomorphisms:

$$\begin{aligned}&[N \otimes_C (C \otimes_A B)] \otimes^{\mathsf{L}}_{C \otimes_A B} [(C \otimes_A B) \otimes^{\mathsf{L}}_B J] \\ &\qquad \simeq (N \otimes_A B) \otimes^{\mathsf{L}}_B J \simeq (N \otimes_A B) \otimes_B J \\ &\qquad \simeq [N \otimes_C (C \otimes_A B)] \otimes_{C \otimes_A B} [(C \otimes_A B) \otimes_B J],\end{aligned}$$

the result follows. □

We now briefly review homotopy limits in a triangulated category $\mathcal{T}$ admitting countable products. Fix for each $i \geq 0$ a morphism in $\mathcal{T}$, $t_i \colon T_{i+1} \to T_i$. Set $t \colon \Pi_{i\geq 0} T_i \to \Pi_{i\geq 0} T_i$ to be the composition of the product of the morphisms t_i with the projection $\Pi_{i\geq 0} T_i \to \Pi_{i\geq 1} T_i$. We define $\operatorname{holim}_i T_i$ via the following distinguished triangle:

$$\operatorname{holim}_i T_i \longrightarrow \prod_{i\geq 0} T_i \xrightarrow{\mathrm{Id}-t} \prod_{i\geq 0} T_i \ .$$

The category of lisse-étale $\mathcal{O}_X$-modules is a Grothendieck abelian category, thus $\mathsf{D}(X)$ admits small products (combine [Nee01a, Thm. 0.2] with [Nee01b, Cor. 1.18]). Note that for each $\mathcal{M} \in \mathsf{D}(X)$ the functor $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, -)$ preserves homotopy limits. Indeed, it preserves products and distinguished triangles. We now prove the following lisse-étale variant of [Stacks, Tag 08IY], where the result is proved in the context of the big fppf site.

Lemma 1.5. *Let X be an algebraic stack and fix $\mathcal{N} \in \mathsf{D}(X)$. The projections $\mathcal{N} \to \tau^{\geq -i}\mathcal{N}$ induce a non-canonical morphism:*

$$\phi \colon \mathcal{N} \to \operatorname{holim}_i \tau^{\geq -i}\mathcal{N}.$$

If $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(X)$, then any such ϕ is a quasi-isomorphism.

Proof. The arguments proving [Stacks, Tag 08IY] also apply in this setting, provided one substitutes "big fppf site" with "lisse-étale site" throughout. □

Note that the main result of [Nee11] produces—in positive characteristic—complexes $\mathcal{N} \in \mathsf{D}(\mathsf{QCoh}(B\mathbb{G}_a))$ with the property that there are no quasi-isomorphisms:

$$\mathcal{N} \to \operatorname{holim}_i \tau^{\geq -i}\mathcal{N}.$$

We wish to emphasize that this does not contradict Lemma 1.5. Indeed, while the categories $\mathsf{D}^+(\mathsf{QCoh}(B\mathbb{G}_a))$ and $\mathsf{D}^+_{\mathrm{qc}}(B\mathbb{G}_a)$ are equivalent [Lur04, Thm. 3.8], this equivalence does not extend to the unbounded derived categories. This will be clarified in joint work with D. Rydh and A. Neeman [HNR13]. We conclude this section with a simple, but general, Lemma that will be used several times throughout the article.

Lemma 1.6. *Let X be an algebraic stack and let $\mathcal{N} \in \mathsf{QCoh}(X)$. If $\mathcal{N}$ is of finite presentation, then there exists a closed immersion $i \colon Z \to X$ of finite presentation with the following two properties:*

1. *the adjunction morphism $\mathcal{N} \to i_*i^*\mathcal{N}$ is an isomorphism and*
2. $\operatorname{supp}(\mathcal{N}) = |Z|$.

Proof. If X is locally noetherian, the result is trivial. Indeed, take Z to be the closed immersion defined by the ideal

$$\operatorname{Ann}_{\mathcal{O}_X}(\mathcal{N}) = \ker(\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{N})).$$

If X is not locally noetherian, we instead take Z to be the closed immersion defined by the 0th Fitting ideal of $\mathcal{N}$ [Stacks, Tag 07Z9]. □

2. Corepresentability of $\mathsf{R}\mathcal{H}om$-functors

Let S be a noetherian algebraic stack. A complex $\mathcal{K} \in \mathsf{D}^b_{\mathsf{Coh}}(S)$ is *dualizing* if it is locally of finite injective dimension and for every $\mathcal{F} \in \mathsf{D}_{\mathsf{Coh}}(S)$ the natural map:

$$\mathcal{F} \to \mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{K}), \mathcal{K})$$

is a quasi-isomorphism. For notational convenience we set $\mathfrak{D}(-) = \mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(-, \mathcal{K})$. Two useful facts about dualizing complexes are the following:

(D1) the functor $\mathfrak{D}(-)$ interchanges $\mathsf{D}^-_{\mathsf{Coh}}(S)$ and $\mathsf{D}^+_{\mathsf{Coh}}(S)$;

(D2) for $\mathcal{F}$, $\mathcal{G} \in \mathsf{D}_{\mathsf{Coh}}(S)$, there is a natural quasi-isomorphism:

$$\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F},\mathcal{G}) \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathfrak{D}(\mathcal{G}),\mathfrak{D}(\mathcal{F})).$$

For background material on dualizing complexes on noetherian schemes we refer the reader to [Har66, V.2]. Spectra of Gorenstein rings (e.g., $\operatorname{Spec}\mathbb{Z}$ or a field) and maximal-adically complete local rings admit dualizing complexes. Moreover, by [Hin93], the spectrum of a henselian noetherian local ring admits a dualizing complex if and only if it is excellent. Also, if a noetherian scheme Z admits a dualizing complex and $h\colon Y \to Z$ is a morphism of finite type, then Y admits a dualizing complex [Har66, VI.3.5]. The existence of dualizing complexes on general noetherian schemes, algebraic spaces, and algebraic stacks is a subtle problem and is not something that we will attempt to address.

The main result of this section is the following Proposition.

Proposition 2.1. *Fix a noetherian algebraic stack S that admits a dualizing complex $\mathcal{K}$ and a morphism of algebraic stacks $f\colon X \to S$ that is proper. Let $\mathcal{M} \in \mathsf{D}^-_{\mathsf{Coh}}(X)$ and $\mathcal{N} \in \mathsf{D}^b_{\mathsf{Coh}}(X)$. If $\mathcal{N}$ is of finite tor-dimension over S, then there exists a quasi-isomorphism:*

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{I}) \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{E}_{\mathcal{M},\mathcal{N}},\mathcal{I}) \quad \forall \mathcal{I}\in \mathsf{D}^+_{\mathrm{qc}}(S),$$

natural in $\mathcal{I}$, where:

$$\mathcal{E}_{\mathcal{M},\mathcal{N}} \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{K}),\mathcal{K}) \in \mathsf{D}^-_{\mathsf{Coh}}(S).$$

In addition, if $\mathcal{N}\in\mathsf{Coh}(X)$, then the formation of $\mathcal{E}_{\mathcal{M},\mathcal{N}}$ is compatible with base changes $\tau\colon T\to S$ that are tor-independent of $\mathcal{N}$ over S.

We wish to point out that Proposition 2.1 covers most situations encountered in practice (in particular, it generalizes [BF97, Lem. 6.1]). Using Proposition 2.1 we can quickly dispatch Theorem B.

Proof of Theorem B. By [Har66, V.2], if S is a separated scheme of finite type over a field or $\mathbb{Z}$, then S admits a dualizing complex. Also, since $\mathcal{N}$ is properly supported over S, then there is a closed immersion $i\colon X_0 \to X$ such that $f_0 = f\circ i$ is proper and the natural map $\mathcal{N}\to i_*i^*\mathcal{N}$ is an isomorphism. Since $\mathcal{N}$ is flat over S, $i^*\mathcal{N}$ is flat over S. In particular, $i^*\mathcal{N}$ is of finite tor-dimension over S and so by Proposition 2.1, we deduce that for every integer q the functor

$$\mathcal{E}xt^q(f_0;\mathsf{L}i^*_{\mathrm{qc}}\mathcal{M},(i^*\mathcal{N})\otimes^{\mathsf{L}}_{\mathcal{O}_{X_0}}\mathsf{L}(f_0)^*_{\mathrm{qc}}(-))\colon \mathsf{QCoh}(S)\to\mathsf{QCoh}(S)$$

is corepresentable by a complex. By trivial duality (1.8) and the natural isomorphisms of functors $\mathsf{R}(f_0)_* \simeq \mathsf{R}f_*\circ\mathsf{R}i_*$ and $\mathsf{L}(f_0)^*_{\mathrm{qc}} \simeq \mathsf{L}i^*_{\mathrm{qc}}\circ\mathsf{L}f^*_{\mathrm{qc}}$, there is a natural quasi-isomorphism for every $\mathcal{I}\in\mathsf{QCoh}(S)$:

$$\begin{aligned}\mathsf{R}(f_0)_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_{X_0}}&(\mathsf{L}i^*_{\mathrm{qc}}\mathcal{M},(i^*\mathcal{N})\otimes^{\mathsf{L}}_{\mathcal{O}_{X_0}}\mathsf{L}(f_0)^*_{\mathrm{qc}}\mathcal{I})\\ &\simeq \mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathsf{R}i_*(i^*\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_{X_0}}\mathsf{L}i^*_{\mathrm{qc}}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{I})).\end{aligned}$$

By Proposition 1.3(3), there are natural quasi-isomorphisms:

$$\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{I} \simeq (\mathsf{R}i_*i^*\mathcal{N})\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{I} \simeq \mathsf{R}i_*(i^*\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_{X_0}}\mathsf{L}i^*_{\mathrm{qc}}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{I}).$$

The result follows. □

Prior to proving Proposition 2.1 it will be necessary to prove the following lemma.

Lemma 2.2. *Let $f\colon X\to S$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{N}\in\mathsf{D}^b_{\mathrm{qc}}(X)$ be pseudo-coherent and have finite tor-dimension over S, let $\mathcal{F}\in\mathsf{D}^-_{\mathrm{qc}}(S)$ be pseudo-coherent, and let $\mathcal{G}\in\mathsf{D}^+_{\mathrm{qc}}(S)$.*

(1) *There is a natural morphism in* $\mathsf{D}^+_{\mathrm{qc}}(S)$*:*

$$\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F},\mathcal{G}) \to \mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{F},\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G}).$$

(2) *There is a natural quasi-isomorphism in* $\mathsf{D}^+_{\mathrm{qc}}(X)$*:*

$$\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F},\mathcal{G}) \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathsf{L}f^*_{\mathrm{qc}}\mathcal{F},\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G}).$$

Proof. For (1), we begin by observing that since $\mathcal{N}$ is of finite tor-dimension over S, it follows that $\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G}\in\mathsf{D}^+_{\mathrm{qc}}(X)$. Next, by (1.3), the identity morphism $\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G}\to\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G}$ gives rise to a morphism $\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G}\to\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G})$. Since $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G})\in\mathsf{D}^+_{\mathrm{qc}}(X)$, trivial duality (1.7) thus produces a morphism $\mathcal{N}\to\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G})$. Applying the functor $\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F},-)$ to this gives rise to a morphism:

$$\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F},\mathcal{G})\to\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F},\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G})).$$

By (1.8) this gives rise to a natural morphism:

$$\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F},\mathcal{G})\to\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathsf{L}f^*_{\mathrm{qc}}\mathcal{F},\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G})).$$

Now apply (1.4) to obtain the asserted map.

For (2), using (1) as the starting point, we apply (1.4) to obtain a natural morphism:

$$\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F},\mathcal{G})\to\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathsf{L}f^*_{\mathrm{qc}}\mathcal{F},\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G})).$$

Applying (1.7) followed by (1.3) gives rise to a natural map

$$\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F},\mathcal{G})\to\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathsf{L}f^*_{\mathrm{qc}}\mathcal{F},\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G}),$$

which we will show is a quasi-isomorphism. This is smooth local on X and S, thus we may assume that these are both affine schemes. For $\mathcal{F}$ pseudo-coherent on S, let

$$F_1(\mathcal{F})=\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F},\mathcal{G})\quad\text{and}$$
$$F_2(\mathcal{F})=\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathsf{L}f^*_{\mathrm{qc}}\mathcal{F},\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathsf{L}f^*_{\mathrm{qc}}\mathcal{G}).$$

Then, there is a natural transformation $F_1(\mathcal{F})\to F_2(\mathcal{F})$ and it remains to show that for all integers k the induced morphism $\mathcal{H}^k(F_1(\mathcal{F}))\to\mathcal{H}^k(F_2(\mathcal{F}))$ is an isomorphism. Note that this is certainly the case whenever $\mathcal{F}$ is a perfect complex. Moreover, the hypotheses on $\mathcal{N}$ and $\mathcal{G}$ are precisely those which guarantee that there exists a $n\geq 0$ such that for every integer i and every $\mathcal{F}\in\mathsf{D}^-_{\mathsf{Coh}}(S)$ with $\tau^{\geq i}\mathcal{F}\simeq 0$, we have for $j=1$ and 2 that $\tau^{\leq -i-n}(F_j(\mathcal{F}))\simeq 0$.

Now fix an integer k. Given $\mathcal{F}\in\mathsf{D}^-_{\mathsf{Coh}}(S)$, since S is affine, we may find a perfect complex $\mathcal{Q}$ and a morphism $\mathcal{Q}\to\mathcal{F}$ whose cone $\mathcal{C}$ has the property that $\tau^{\geq -k-n}\mathcal{C}\simeq 0$. We thus have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathcal{H}^{k-1}(F_1(\mathcal{C})) & \longrightarrow & \mathcal{H}^k(F_1(\mathcal{F})) & \longrightarrow & \mathcal{H}^k(F_1(\mathcal{Q})) & \longrightarrow & \mathcal{H}^k(F_1(\mathcal{C}))\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\mathcal{H}^{k-1}(F_2(\mathcal{C})) & \longrightarrow & \mathcal{H}^k(F_2(\mathcal{F})) & \longrightarrow & \mathcal{H}^k(F_2(\mathcal{Q})) & \longrightarrow & \mathcal{H}^k(F_2(\mathcal{C})).
\end{array}$$

Since $\mathcal{H}^k(F_1(\mathcal{Q}))\to\mathcal{H}^k(F_2(\mathcal{Q}))$ is an isomorphism and $\tau^{\leq k}(F_j(\mathcal{C}))\simeq 0$ for $j=1$ and 2, we deduce the claim. ☐

We can now prove Proposition 2.1.

Proof of Proposition 2.1. If $\mathcal{I} \in \mathsf{D}^+_{\mathsf{Coh}}(S)$, then we have the following sequence of natural quasi-isomorphisms:

$$\begin{aligned}
&\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{I}) \\
&\simeq \mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathfrak{D}(\mathcal{I}), \mathcal{K})) && (\mathcal{K} \text{ is dualizing}) \\
&\simeq \mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathsf{L}f^*_{\mathrm{qc}}\mathfrak{D}(\mathcal{I}), \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{K})) && (\text{by Lemma 2.2(2)}) \\
&\simeq \mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathfrak{D}(\mathcal{I}), \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{K}) && (\text{by (1.4)}) \\
&\simeq \mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathsf{L}f^*_{\mathrm{qc}}\mathfrak{D}(\mathcal{I}), \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{K})) && (\text{by (1.4)}) \\
&\simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathfrak{D}(\mathcal{I}), \mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{K})) && (\text{by (1.8)}) \\
&\simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathfrak{D}(\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{K})), \mathcal{I}).
\end{aligned}$$

The final quasi-isomorphism is a consequence of the following sequence of observations. First, $\mathcal{N} \in \mathsf{D}^b_{\mathsf{Coh}}(X)$ has finite tor-dimension over S and $\mathcal{K} \in \mathsf{D}^b_{\mathsf{Coh}}(S)$, so $\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{K} \in \mathsf{D}^b_{\mathsf{Coh}}(X)$. Also, f is proper so [Ols07, 6.4.4 & 10.13] implies that $\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{K}) \in \mathsf{D}^+_{\mathsf{Coh}}(S)$. Thus property (D2) of dualizing complexes applies. Hence, we have produced a natural quasi-isomorphism for all $\mathcal{I} \in \mathsf{D}^+_{\mathsf{Coh}}(S)$:

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{I}) \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{E}_{\mathcal{M},\mathcal{N}}, \mathcal{I}), \tag{2.1}$$

where

$$\mathcal{E}_{\mathcal{M},\mathcal{N}} \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{K}), \mathcal{K}) \in \mathsf{D}^-_{\mathsf{Coh}}(S).$$

We now need to extend the quasi-isomorphism (2.1) to $\mathcal{I} \in \mathsf{D}^+_{\mathrm{qc}}(S)$. First, we note that because $\mathcal{N}$ is bounded, there exists an r such that for all integers n and $\mathcal{I} \in \mathsf{D}_{\mathrm{qc}}(S)$ the natural map:

$$\tau^{\geq n+r}(\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{I}) \to \tau^{\geq n+r}(\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}[\tau^{\geq n}\mathcal{I}]) \tag{2.2}$$

is a quasi-isomorphism. Hence there exist maps for every $\mathcal{I} \in \mathsf{D}_{\mathsf{Coh}}(S)$:

$$\begin{aligned}
\mathsf{R}\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{E}_{\mathcal{M},\mathcal{N}}, \mathcal{I}) &\simeq \operatorname*{holim}_n \mathsf{R}\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{E}_{\mathcal{M},\mathcal{N}}, \tau^{\geq -n}\mathcal{I}) && (\text{Lemma 1.5}) \\
&\simeq \operatorname*{holim}_n \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}[\tau^{\geq -n}\mathcal{I}]) && (\text{by (2.1)}) \\
&\to \operatorname*{holim}_n \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \tau^{\geq -n+r}(\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}[\tau^{\geq -n}\mathcal{I}])) \\
&\simeq \operatorname*{holim}_n \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \tau^{\geq -n+r}(\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{I})) && (\text{by (2.2)}) \\
&\simeq \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{I}).
\end{aligned}$$

Note, however, that the maps above depend on $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{I}$ in a non-natural way (this is because $\operatorname*{holim}_n$ is constructed as a cone, thus is not functorial). In any case, corresponding to the identity map $\mathcal{E}_{\mathcal{M},\mathcal{N}} \to \mathcal{E}_{\mathcal{M},\mathcal{N}}$ there is a morphism $\psi_{\mathcal{M},\mathcal{N}} \colon \mathcal{M} \to \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{E}_{\mathcal{M},\mathcal{N}}$ (which is not necessarily functorial in $\mathcal{M}$ or $\mathcal{N}$). Now let $\mathcal{I} \in \mathsf{D}^+_{\mathrm{qc}}(S)$. By Lemma 2.2(1), there is a natural sequence of maps:

$$\begin{aligned}
\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{E}_{\mathcal{M},\mathcal{N}}, \mathcal{I}) &\to \mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{E}_{\mathcal{M},\mathcal{N}}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{I}) \\
&\to \mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}\mathcal{I}).
\end{aligned}$$

By (2.1), the map above is certainly a quasi-isomorphism for all $\mathcal{I} \in \mathsf{D}^+_{\mathsf{Coh}}(S)$. To show that it is a quasi-isomorphism for all $\mathcal{I} \in \mathsf{D}^+_{\mathrm{qc}}(S)$, by the "way-out right" results of [Har66, I.7.1], it is sufficient to prove that it is a quasi-isomorphism for all quasi-coherent $\mathcal{O}_S$-modules. We may now reduce to the case where S is an affine and noetherian scheme. Hence, it is sufficient to prove that the natural transformation of functors from $\mathsf{QCoh}(S) \to \mathsf{Ab}$:

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{E}_{\mathcal{M},\mathcal{N}}, (-)[0]) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}(-)[0])$$

is an isomorphism. By Lemma 1.2, both functors preserve direct limits and the exhibited natural transformation is an isomorphism for all $\mathcal{I} \in \mathsf{Coh}(S)$. Since every quasi-coherent $\mathcal{O}_S$-module is a direct limit of its coherent submodules, we deduce the result.

It now remains to address the compatibility of $\mathcal{E}_{\mathcal{M},\mathcal{N}}$ with base change. So, we fix a morphism of noetherian algebraic stacks $\tau\colon T \to S$ such that T admits a dualizing complex and form the 2-cartesian square of noetherian algebraic stacks:

$$\begin{array}{ccc} X_T & \xrightarrow{\tau_X} & X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\tau} & S. \end{array}$$

Let $\mathcal{N} \in \mathsf{Coh}(X)$ and suppose that it is tor-independent of τ over S. By Corollary 1.4 and what we have proven so far, there is a quasi-isomorphism, natural in $\mathcal{I} \in \mathsf{D}^+_{\mathrm{qc}}(T)$:

$$\mathsf{R}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{E}_{\mathcal{M},\mathcal{N}[0]}, \mathsf{R}\tau_*\mathcal{I}) \simeq \mathsf{R}\tau_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_T}(\mathcal{E}_{\mathsf{L}(\tau_X)^*_{\mathrm{qc}}\mathcal{M},(\tau_X^*\mathcal{N})[0]}, \mathcal{I}).$$

By trivial duality (1.8) we thus obtain a quasi-isomorphism, natural in $\mathcal{I} \in \mathsf{D}^+_{\mathrm{qc}}(T)$:

$$\mathsf{R}\tau_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_T}(\mathsf{L}\tau^*_{\mathrm{qc}}\mathcal{E}_{\mathcal{M},\mathcal{N}[0]}, \mathcal{I}) \simeq \mathsf{R}\tau_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_T}(\mathcal{E}_{\mathsf{L}(\tau_X)^*_{\mathrm{qc}}\mathcal{M},(\tau_X^*\mathcal{N})[0]}, \mathcal{I}).$$

By (1.5), we thus see that we have a quasi-isomorphism, natural in $\mathcal{I} \in \mathsf{D}^+_{\mathrm{qc}}(T)$:

$$\mathsf{R}\mathrm{Hom}_{\mathcal{O}_T}(\mathsf{L}\tau^*_{\mathrm{qc}}\mathcal{E}_{\mathcal{M},\mathcal{N}[0]}, \mathcal{I}) \simeq \mathsf{R}\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{E}_{\mathsf{L}(\tau_X)^*_{\mathrm{qc}}\mathcal{M},(\tau_X^*\mathcal{N})[0]}, \mathcal{I}).$$

By Lemma 1.5 and the above we obtain a sequence of quasi-isomorphisms:

$$\begin{aligned} \mathsf{L}\tau^*_{\mathrm{qc}}\mathcal{E}_{\mathcal{M},\mathcal{N}[0]} &\simeq \operatorname*{holim}_n \tau^{\geq -n}\mathsf{L}\tau^*_{\mathrm{qc}}\mathcal{E}_{\mathcal{M},\mathcal{N}[0]} \simeq \operatorname*{holim}_n \tau^{\geq -n}\mathcal{E}_{\mathsf{L}(\tau_X)^*_{\mathrm{qc}}\mathcal{M},(\tau_X^*\mathcal{N})[0]} \\ &\simeq \mathcal{E}_{\mathsf{L}(\tau_X)^*_{\mathrm{qc}}\mathcal{M},(\tau_X^*\mathcal{N})[0]}. \end{aligned}$$

□

3. Finitely generated and coherent functors

Fix a ring A. Let $\mathbf{Fun}^\times_A$ denote the category of functors $\mathsf{Mod}(A) \to \mathsf{Sets}$ which commute with finite products. This is a full subcategory of the category of all functors $\mathsf{Mod}(A) \to \mathsf{Sets}$. Denote by $\mathbf{Lin}_A$ the category of A-linear functors $\mathsf{Mod}(A) \to \mathsf{Mod}(A)$.

Lemma 3.1. *Let A be a ring. Then the forgetful functor:*

$$\mathbf{Lin}_A \to \mathbf{Fun}^\times_A$$

is an equivalence of categories.

Proof. We leave the verification that the functor is fully faithful to the reader. For the essential surjectivity, let $F\colon \mathsf{Mod}(A) \to \mathsf{Sets}$ be a functor that preserves finite products. That is, for each A-module M and N, the natural homomorphism

$$\Pi^F_{M,N}\colon F(M \times N) \to F(M) \times F(N)$$

that is induced by the universal property defining the product, is bijective. Let L be an A-module and let $\Sigma_L\colon L \times L \to L$ be the addition morphism. The composition:

$$F(L) \times F(L) \xrightarrow{(\Pi^F_{L,L})^{-1}} F(L \times L) \xrightarrow{F(\Sigma_L)} F(L)$$

is easily verified to endow $F(L)$ with a structure of an abelian group that is natural in L and F. If $a \in A$, then there is an induced A-module homomorphism $a\cdot\colon L \to L$ that defines the action of a on L. By functoriality, we obtain a naturally induced map $F(a\cdot)\colon F(L) \to F(L)$, and it is also easily verified that this endows $F(L)$ with a natural structure as an A-module. The result follows. □

Let A be a ring and set $T = \operatorname{Spec} A$. Lemma 3.1 shows that the category of additive functors $\mathsf{QCoh}(T) \to \mathsf{Ab}$ is equivalent to the category of A-linear functors $\mathsf{Mod}(A) \to \mathsf{Mod}(A)$. We will use this equivalence without further mention to translate definitions between the two categories.

An A-linear functor Q is *finitely generated* if there exists an A-module I and an object $\eta \in Q(I)$ such that for all A-modules M, the induced morphism of sets $\operatorname{Hom}_A(I, M) \to Q(M)\colon f \mapsto f_*\eta$ is surjective. We call the pair (I, η) a *generator* for the functor Q. The notion of finite generation of a functor is due to M. Auslander [Aus66].

Example 3.2. Fix a ring A. For every A-module I, the functor $M \mapsto \operatorname{Hom}_A(I, M)$ is finitely generated. The functor $M \mapsto I \otimes_A M$ is finitely generated if and only if the A-module I is finitely generated as an A-module. Indeed, a generator is obtained by choosing a surjection $A^n \to I$, and noting that for every A-module M, there is a surjection $A^n \otimes_A M \to I \otimes_A M$, and an isomorphism $A^n \otimes_A M \to \operatorname{Hom}_A(A^n, M)$.

Example 3.3. Fix a ring A and a collection of A-linear functors $\{F_\lambda\}_{\lambda\in\Lambda}$ indexed by a set Λ. If for every $\lambda \in \Lambda$ the functor F_λ is finitely generated, then the A-linear functor $M \mapsto \Pi_{\lambda\in\Lambda} F_\lambda(M)$ is finitely generated. Indeed, for each $\lambda \in \Lambda$ let $(I_\lambda, \eta_\lambda)$ be a generator for F_λ, which by definition induces a surjection for every $\lambda \in \Lambda$: $\operatorname{Hom}_A(M_\lambda, -) \to F_\lambda$. There is an induced surjection $\Pi_\lambda \operatorname{Hom}_A(M_\lambda, -) \to \Pi_\lambda F_\lambda$. The result follows from the observation that $\operatorname{Hom}_A(\oplus_\lambda M_\lambda, -) \cong \Pi_\lambda \operatorname{Hom}_A(M_\lambda, -)$.

For an A-algebra B, and a functor $Q\colon \mathsf{Mod}(A) \to \mathsf{Sets}$, there is an induced functor $Q_B\colon \mathsf{Mod}(B) \to \mathsf{Sets}$ given by regarding a B-module as an A-module. Note that since the forgetful functor $\mathsf{Mod}(B) \to \mathsf{Mod}(A)$ commutes with all limits and colimits, it follows that if the functor Q preserves certain limits or colimits, so does the functor Q_B. In particular, if the functor Q commutes with finite products, then the functor $Q_B\colon \mathsf{Mod}(B) \to \mathsf{Sets}$ preserves finite products. By Lemma 3.1, it follows that Q_B may be regarded as a B-linear functor $Q_B\colon \mathsf{Mod}(B) \to \mathsf{Mod}(B)$. We will use this fact frequently and without further comment.

Example 3.4. Let A be a ring, let B be an A-algebra, and let Q be an A-linear functor. If Q is finitely generated, then so is Q_B. Indeed, if (I, η) generates Q, let $p\colon I \to I \otimes_A B$ be the natural A-module homomorphism, then $(I \otimes_A B, p_*\eta)$ generates Q_B.

An A-linear functor F is *coherent*, if there exists an A-module homomorphism $f\colon I \to J$ and an element $\eta \in F(I)$, inducing an exact sequence for every A-module M:

$$\operatorname{Hom}_A(J, M) \longrightarrow \operatorname{Hom}_A(I, M) \longrightarrow F(M) \longrightarrow 0.$$

We refer to the data $(f\colon I \to J, \eta)$ as a *presentation* for F. For accounts of coherent functors, we refer the interested reader to [Aus66, Har98]. We now have a number of examples.

Example 3.5. Let A be a ring and consider an exact sequence of A-linear functors:

$$H_1 \longrightarrow H_2 \longrightarrow H_3 \longrightarrow H_4 \longrightarrow H_5.$$

If for each $i \neq 3$ we have that H_i is coherent, then H_3 is coherent. In particular, the category of coherent functors is stable under kernels, cokernels, subquotients, and extensions. This follows from [Aus66, Prop. 2.1].

Example 3.6. Fix a ring A, an A-algebra B, and an A-linear functor F. If F is coherent, then analogously to Example 3.4, the restriction F_B is also coherent.

Fix a ring A. An A-linear functor F is *half-exact* if for every short exact sequence of A-modules $0 \to M' \to M \to M'' \to 0$, the sequence $F(M') \to F(M) \to F(M'')$ is exact.

Example 3.7. Let A be a ring. If $Q^\bullet$ is a complex of A-modules, then the A-linear functor $M \mapsto H^i(\operatorname{Hom}_A(Q^\bullet, M))$ is coherent for every integer i. If the complex $Q^\bullet$ is term-by-term projective, the functor $M \mapsto H^i(\operatorname{Hom}_A(Q^\bullet, M))$ is also half-exact.

An A-linear functor of the form $M \mapsto \operatorname{Ext}^i_A(Q^\bullet, M)$ is said to be *corepresentable by a complex*. By Example 3.7, such functors are coherent and half-exact. These functors were intially studied by M. Auslander [Aus66], with stronger results—in the noetherian setting—obtained R. Hartshorne [Har98]. In [HR12a], it is shown that étale-locally every half-exact coherent functor is corepresentable by a complex.

Example 3.8. Fix a ring A and an A-linear functor F. If F is coherent, then F preserves small products. It was shown by H. Krause [Kra03, Prop. 3.2] that the preservation of small products characterizes coherent functors.

Example 3.9. Fix a ring A. Example 3.3 extends to show that the category of A-linear coherent functors is closed under small products. By Example 3.5, the category of A-linear coherent functors is also closed under equalizers. Thus the category of A-linear coherent functors is closed under small limits.

Example 3.10. Fix a ring A and an A-linear coherent functor F that is left exact. By Example 3.8, F also preserves small products, thus F preserves small limits. The Eilenberg-Watts Theorem [Wat60, Thm. 6] now implies that F is corepresentable. That is, there exists an A-module Q such that $F(-) \cong \operatorname{Hom}_A(Q, -)$. If, in addition, the functor F preserves direct limits, then Q is of finite presentation. By Example 3.7, this observation generalizes [EGA, III.7.4.6].

Example 3.11. Let A be a ring. If N is an A-module, then the A-linear functor $M \mapsto M \otimes_A N$ is coherent if and only if N is of finite presentation.

Example 3.12. Let R be a noetherian ring. If $Q^\bullet$ is a complex of finitely generated R-modules, then for every integer i the functor $M \mapsto H^i(Q^\bullet \otimes_R M)$ is coherent and preserves direct limits. If, in addition, $Q^\bullet$ is flat term-by-term, then the functor $M \mapsto H^i(Q^\bullet \otimes_R M)$ is also half-exact.

Example 3.13. Let R be a noetherian ring. If $Q^\bullet$ is a bounded above complex of R-modules with coherent cohomology, then the functors $M \mapsto \operatorname{Tor}^R_i(Q^\bullet, M)$ and $M \mapsto \operatorname{Ext}^i_R(Q^\bullet, M)$ are coherent, half-exact, and preserve direct limits.

4. Coherence of Hom-functors: flat case

The proof of Theorem C will be via an induction argument that permits us to reduce to the case of Theorem B. The following notation will be useful.

Notation 4.1. For a morphism of algebraic stacks $f\colon X \to S$ and $\mathcal{M}$ and $\mathcal{N}$ belonging to $\mathsf{D}_{\mathrm{qc}}(X)$, let

$$\mathbb{H}_{\mathcal{M},\mathcal{N}} = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}(-))\colon \mathsf{QCoh}(S) \to \mathsf{Ab}.$$

If $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(X)$, let $\mathcal{T}^{\mathcal{N}}_{X/S} \subseteq \mathsf{D}_{\mathrm{qc}}(X)$ be the full subcategory whose objects are those $\mathcal{M}$ such that $\mathbb{H}_{\mathcal{M},\mathcal{N}[n]}$ is coherent for all integers n.

We begin with two general reductions.

Lemma 4.2. *Fix an affine scheme S and a morphism of algebraic stacks $f\colon X \to S$. If $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(X)$, then the subcategory $\mathcal{T}^{\mathcal{N}}_{X/S} \subseteq \mathsf{D}_{\mathrm{qc}}(X)$ is triangulated and closed under small direct sums. In particular, if*

(1) $\mathsf{D}^-_{\mathrm{QCoh}}(X) \subseteq \mathcal{T}^{\mathcal{N}}_{X/S}$, *or*

(2) $\mathcal{N}$ *has finite tor-dimension over* S *and* $\mathcal{M} \in \mathcal{T}^{\mathcal{N}}_{X/S}$ *for all* $\mathcal{M} \in \mathsf{QCoh}(X)$,

then $\mathcal{T}^{\mathcal{N}}_{X/S} = \mathsf{D}_{\mathrm{qc}}(X)$.

Proof. Certainly, $\mathcal{T}^{\mathcal{N}}_{X/S}$ is closed under shifts. Next, given a triangle $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3$ in $\mathsf{D}_{\mathrm{qc}}(X)$ with $\mathcal{M}_1$, $\mathcal{M}_2 \in \mathcal{T}^{\mathcal{N}}_{X/S}$, we obtain an exact sequence of functors:

$$\mathbb{H}_{\mathcal{M}_2[1],\mathcal{N}} \to \mathbb{H}_{\mathcal{M}_1[1],\mathcal{N}} \to \mathbb{H}_{\mathcal{M}_3,\mathcal{N}} \to \mathbb{H}_{\mathcal{M}_2,\mathcal{N}} \to \mathbb{H}_{\mathcal{M}_1,\mathcal{N}}.$$

By Example 3.5, $\mathbb{H}_{\mathcal{M}_3,\mathcal{N}} \in \mathcal{T}^{\mathcal{N}}_{X/S}$ and so $\mathcal{T}^{\mathcal{N}}_{X/S}$ is a triangulated subcategory of $\mathsf{D}_{\mathrm{qc}}(X)$. Let $\{\mathcal{M}_i\}_{i\in I}$ be a set of elements from $\mathcal{T}^{\mathcal{N}}_{X/S}$. Let $\mathcal{M} = \oplus_{i\in I}\mathcal{M}_i$. Then for all integers n there is an isomorphism of functors $\mathbb{H}_{\mathcal{M},\mathcal{N}[n]} \cong \Pi_{i\in I}\mathbb{H}_{\mathcal{M}_i,\mathcal{N}[n]}$. By Example 3.9 we conclude that $\mathcal{M} \in \mathcal{T}^{\mathcal{N}}_{X/S}$.

For (1), by [LO08, Lem. 4.3.2], given $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$, there is a triangle:

$$\bigoplus_{n\geq 0} \tau^{\leq n}\mathcal{M} \to \bigoplus_{n\geq 0} \tau^{\leq n}\mathcal{M} \to \mathcal{M}.$$

By hypothesis, $\tau^{\leq n}\mathcal{M} \in \mathcal{T}^{\mathcal{N}}_{X/S}$ for all $n \geq 0$, and so $\mathcal{M} \in \mathcal{T}^{\mathcal{N}}_{X/S}$.

For (2) by (1), it is sufficient to prove that $\mathsf{D}^-_{\mathrm{qc}}(X) \subseteq \mathcal{T}^{\mathcal{N}}_{X/S}$. Since $\mathcal{N}$ has finite tor-dimension over S, there exists an integer l such that the natural map $\mathcal{N}\otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}I \in \mathsf{D}^{\geq l}_{\mathrm{qc}}(X)$ for all $I \in \mathsf{QCoh}(S)$. Thus, if $\mathcal{M} \in \mathsf{D}^-_{\mathrm{qc}}(X)$, then for every integer n we have a natural isomorphism of functors: $\mathbb{H}_{\mathcal{M},\mathcal{N}[n]} \cong \mathbb{H}_{\tau^{\geq l-n}\mathcal{M},\mathcal{N}[n]}$. Hence, it is sufficient to prove that $\mathsf{D}^b_{\mathrm{qc}}(X) \subseteq \mathcal{T}^{\mathcal{N}}_{X/S}$.

If $\mathcal{M} \in \mathsf{D}^b_{\mathrm{qc}}(X)$, define $t(\mathcal{M}) = \max\{i : \mathcal{H}^i(\mathcal{M}) \neq 0\}$, $b(\mathcal{M}) = \min\{i : \mathcal{H}^i(\mathcal{M}) \neq 0\}$, and $\ell(\mathcal{M}) = t(\mathcal{M}) - b(\mathcal{M}) \geq 0$. We will prove by induction on $\ell(\mathcal{M}) \geq 0$ that $\mathcal{M} \in \mathcal{T}^{\mathcal{N}}_{X/S}$. The base case, where $\ell(\mathcal{M}) = 0$, is the hypothesis. So it remains to prove that if $\mathcal{M} \in \mathsf{D}^b_{\mathrm{qc}}(X)$, where $\ell(\mathcal{M}) \geq 1$, and $\mathcal{Q} \in \mathcal{T}^{\mathcal{N}}_{X/S}$ for all $\mathcal{Q} \in \mathsf{D}^b_{\mathrm{qc}}(X)$ with $\ell(\mathcal{Q}) < \ell(\mathcal{M})$, then $\mathcal{M} \in \mathcal{T}^{\mathcal{N}}_{X/S}$. To see this, we consider the distinguished triangle:

$$\tau^{<t(\mathcal{M})}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \mathcal{H}^{t(\mathcal{M})}(\mathcal{M})[-t(\mathcal{M})].$$

By the inductive hypothesis, $\tau^{<t(\mathcal{M})}\mathcal{M}$ and $\mathcal{H}^{t(\mathcal{M})}(\mathcal{M})[-t(\mathcal{M})]$ both belong to $\mathcal{T}^{\mathcal{N}}_{X/S}$. Since $\mathcal{T}^{\mathcal{N}}_{X/S}$ is a triangulated subcategory of $\mathsf{D}_{\mathrm{qc}}(X)$, it follows that $\mathcal{M}$ also belongs to $\mathcal{T}^{\mathcal{N}}_{X/S}$. □

Lemma 4.3. *Fix an affine scheme* S *and a representable morphism of algebraic* S*-stacks* $p\colon X' \to X$ *that is quasi-compact and quasi-separated. If* $\mathcal{G}' \in \mathsf{D}_{\mathrm{qc}}(X')$, *then* $\mathbb{H}_{\mathcal{M},\mathsf{R}p_*\mathcal{G}'} = \mathbb{H}_{\mathsf{L}p^*_{\mathrm{qc}}\mathcal{M},\mathcal{G}'}$. *In particular, if* $\mathcal{T}^{\mathcal{G}'}_{X'/S} = \mathsf{D}_{\mathrm{qc}}(X')$, *then* $\mathcal{T}^{\mathsf{R}p_*\mathcal{G}'}_{X/S} = \mathsf{D}_{\mathrm{qc}}(X)$.

Proof. Fix $\mathcal{M} \in \mathsf{D}_{\mathrm{qc}}(X)$, then:

$$\begin{aligned}\mathbb{H}_{\mathcal{M},\mathsf{R}p_*\mathcal{G}'} &= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, (\mathsf{R}p_*\mathcal{G}') \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}(-)) \\ &\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathsf{R}p_*(\mathcal{G}' \otimes^{\mathsf{L}}_{\mathcal{O}_{X'}} \mathsf{L}p^*_{\mathrm{qc}}\mathsf{L}f^*_{\mathrm{qc}}(-))) \qquad \text{(Proposition 1.3(3))} \\ &\cong \mathrm{Hom}_{\mathcal{O}_X}((\mathsf{L}p^*_{\mathrm{qc}}\mathcal{M}), \mathcal{G}' \otimes^{\mathsf{L}}_{\mathcal{O}_{X'}} \mathsf{L}g^*_{\mathrm{qc}}(-)) = \mathbb{H}_{\mathsf{L}p^*_{\mathrm{qc}}\mathcal{M},\mathcal{G}'}.\end{aligned}$$ □

We now have our first induction result.

Lemma 4.4. *Fix an affine scheme* S, *a morphism of algebraic stacks* $f\colon X \to S$ *that is of finite presentation,* $\mathcal{N} \in \mathsf{D}_{\mathrm{qc}}(X)$, *and an integer* n. *Suppose that the functor* $\mathbb{H}_{\mathcal{M},\mathcal{N}[r]}$ *is coherent in the following situations:*

(1) *for all* $r < n$ *and* $\mathcal{M} \in \mathsf{QCoh}(X)$;

(2) $r = n$ *and* $\mathcal{M} \in \mathsf{QCoh}(X)$ *of finite presentation.*

Then the functor $\mathbb{H}_{\mathcal{M},\mathcal{N}[n]}$ *is coherent for all* $\mathcal{M} \in \mathsf{QCoh}(X)$*.*

Proof. Let $\mathcal{M} \in \mathsf{QCoh}(X)$. The morphism f is of finite presentation, so by [Ryd09, Thm. A], the quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ is a direct limit of $\mathcal{O}_X$-modules $\mathcal{M}_\lambda$ of finite presentation. Let $\mathcal{Q}_1 = \oplus_\lambda \mathcal{M}_\lambda$, $\mathcal{Q}_2 = \oplus_{\lambda \leq \lambda'} \mathcal{M}_\lambda$ and take $\theta\colon \mathcal{Q}_2 \to \mathcal{Q}_1$ to be the natural map with $\operatorname{coker}\theta \cong \mathcal{M}$. Take $\mathcal{Q}$ to be the cone of θ in $\mathsf{D}_{\mathrm{qc}}(X)$, for all integers r we obtain an exact sequence in $\mathbf{Fun}_S^{\times}$:

$$\mathbb{H}_{\mathcal{Q}_1,\mathcal{N}[r-1]} \longrightarrow \mathbb{H}_{\mathcal{Q}_2,\mathcal{N}[r-1]} \longrightarrow \mathbb{H}_{\mathcal{Q},\mathcal{N}[r]} \longrightarrow \mathbb{H}_{\mathcal{Q}_1,\mathcal{N}[r]} \longrightarrow \mathbb{H}_{\mathcal{Q}_2,\mathcal{N}[r]}.$$

For all integers r we also have isomorphisms

$$\mathbb{H}_{\mathcal{Q}_1,\mathcal{N}[r]} \cong \prod_{\lambda} \mathbb{H}_{\mathcal{M}_\lambda,\mathcal{N}[r]} \quad \text{and} \quad \mathbb{H}_{\mathcal{Q}_2,\mathcal{N}[r]} \cong \prod_{\lambda \leq \lambda'} \mathbb{H}_{\mathcal{M}_\lambda,\mathcal{N}[r]}.$$

By Examples 3.9 and 3.5, together with our hypotheses, we deduce that $\mathbb{H}_{\mathcal{Q},\mathcal{N}[r]}$ is a coherent functor for all $r \leq n$. Now, there is a distinguished triangle $\mathcal{Q} \to \mathcal{M}[-1] \to (\ker\theta)[1]$, thus we obtain an exact sequence in $\mathbf{Fun}_S^{\times}$ for all integers r:

$$\mathbb{H}_{\mathcal{Q},\mathcal{N}[r-1]} \longrightarrow \mathbb{H}_{(\ker\theta),\mathcal{N}[r-2]} \longrightarrow \mathbb{H}_{\mathcal{M},\mathcal{N}[r]} \longrightarrow \mathbb{H}_{\mathcal{Q},\mathcal{N}[r]} \longrightarrow \mathbb{H}_{(\ker\theta),\mathcal{N}[r-1]}.$$

By hypothesis, $\mathbb{H}_{(\ker\theta),\mathcal{N}[r]}$ is coherent for all $r < n$. Taking $r = n$ in the exact sequence above and applying Example 3.5, we deduce that $\mathbb{H}_{\mathcal{M},\mathcal{N}[n]}$ is coherent. □

Our next result forms the second part of the induction process. We wish to emphasize that in the following lemma, some of the pullbacks are underived. This is not a typographical error and is essential to the argument.

Lemma 4.5. *Fix a 2-cartesian diagram of algebraic stacks:*

$$\begin{array}{ccc} X & \xrightarrow{h} & X_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_0} \\ S & \xrightarrow{g} & S_0 \end{array}$$

where S and S_0 are affine schemes. Fix an integer n and let $\mathcal{M}_0$, $\mathcal{N}_0 \in \mathsf{QCoh}(X_0)$. Assume, in addition, that $\mathcal{N}_0$ is of finite presentation, flat over S_0, and that the functors:

$$\mathbb{H}_{\mathcal{M}_0,\mathcal{N}_0[l]} \quad \textit{and} \quad \mathbb{H}_{\mathcal{F},(h^*\mathcal{N}_0)[r]}$$

are coherent for all integers l and for all integers $r < n$ and all $\mathcal{F} \in \mathsf{QCoh}(X)$. Then, the functors $\mathbb{H}_{(h^*\mathcal{M}_0),(h^*\mathcal{N}_0)[r]}$ are coherent for all integers $r \leq n$.

Proof. Set $\mathcal{M} = \mathsf{L}h_{\mathrm{qc}}^*\mathcal{M}_0$ and $\mathcal{N} = h^*\mathcal{N}_0$. Given integers r and l set:

$$V_{r,l} = \mathbb{H}_{\tau^{\geq -l}\mathcal{M},\mathcal{N}[r+l]} \quad \text{and} \quad W_{r,l} = \mathbb{H}_{\mathcal{H}^{-l}(\mathcal{M}),\mathcal{N}[r]}.$$

It is sufficient to prove that $V_{r,l}$ is coherent for all $r \leq n$ and all l (the result follows by taking $l = 0$). Now, we have a distinguished triangle in $\mathsf{D}_{\mathrm{qc}}(X)$:

$$\mathcal{H}^{-l-1}(\mathcal{M})[l+1] \longrightarrow \tau^{\geq -l-1}\mathcal{M} \longrightarrow \tau^{\geq -l}\mathcal{M}.$$

Applying to this triangle the functor $\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{N}[r+l] \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f_{\mathrm{qc}}^*(-))$ we obtain an exact sequence in $\mathbf{Fun}_S^{\times}$:

$$V_{r-2,l+1} \longrightarrow W_{r-2,l+1} \longrightarrow V_{r,l} \longrightarrow V_{r-1,l+1} \longrightarrow W_{r-1,l+1}. \tag{4.1}$$

By hypothesis, $W_{r,l}$ is coherent for all l and all $r < n$. Thus, by Example 3.5, the result will follow from the assertion that $V_{r,l}$ is coherent for all l and all $r < n$. By induction on r, the exact sequence (4.1) shows that it is sufficient to prove that $V_{r,l}$ is coherent for all integers l and all $r \leq 0$.

Since $\mathcal{N}_0$ is S_0-flat, the natural transformation of functors $\mathbb{H}_{\tau^{\geq 0}\mathcal{L},\mathcal{N}} \to \mathbb{H}_{\mathcal{L},\mathcal{N}}$ is an isomorphism for all $\mathcal{L} \in \mathsf{D}_{\mathrm{qc}}(X)$. Hence, if $l \geq 0$ and $r \leq 0$:

$$V_{r,l} = \mathbb{H}_{\tau^{\geq -l}\mathcal{M},\mathcal{N}[r+l]} \cong \mathbb{H}_{\mathcal{M},\mathcal{N}[r+l]}.$$

Moreover, if $l < 0$, then $\tau^{\geq -l}\mathcal{M} \simeq 0$, so $V_{r,l} \equiv 0$. Thus, it remains to show that $\mathbb{H}_{\mathcal{M},\mathcal{N}[l]}$ is coherent for all l. If $I \in \mathsf{QCoh}(S)$, then Corollary 1.4 gives a natural isomorphism $\mathbb{H}_{\mathcal{M},\mathcal{N}[l]}(I) \cong \mathbb{H}_{\mathcal{M}_0,\mathcal{N}_0[l]}(g_*I)$. Example 3.6 now gives the result. □

We can now prove Theorem C.

Proof of Theorem C. We must prove that $\mathsf{D}_{\mathrm{qc}}(X) = \mathcal{T}^{\mathcal{N}[0]}_{X/S}$. By Lemmas 1.6 and 4.3, we may immediately reduce to the case where $f\colon X \to S$ is proper and of finite presentation. Thus, by Lemma 4.2(2), it suffices to prove that $\mathbb{H}_{\mathcal{M},\mathcal{N}[n]}$ is coherent for all integers n, and all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{M}$.

Now, $\mathbb{H}_{\mathcal{M},\mathcal{N}[n]} \equiv 0$ for all $n < 0$ and all $\mathcal{M} \in \mathsf{QCoh}(X)$. We now prove, by induction on $n \geq -1$, that $\mathbb{H}_{\mathcal{M},\mathcal{N}[n]}$ is coherent. Certainly, the result is true for $n = -1$. Thus, we fix $n \geq 0$ and assume the result has been proven for all $r < n$.

By Lemma 4.4, it suffices to prove that $\mathbb{H}_{\mathcal{M},\mathcal{N}[n]}$ is coherent when $\mathcal{M}$ is of finite presentation. Thus, by standard limit methods [Ryd09, Prop. B.3], there exists an affine scheme S_0 of finite type over $\operatorname{Spec}\mathbb{Z}$, a proper morphism of algebraic stacks $f_0\colon X_0 \to S_0$, and a morphism of affine schemes $g\colon S \to S_0$ inducing an isomorphism of algebraic stacks $h\colon X \to X_0 \times_{S_0} S$. This data may be chosen so that there exists coherent $\mathcal{O}_{X_0}$-modules $\mathcal{M}_0$ and $\mathcal{N}_0$, with $\mathcal{N}_0$ flat over S, as well as isomorphisms of $\mathcal{O}_X$-modules $h^*\mathcal{N}_0 \cong \mathcal{N}$, $h^*\mathcal{M}_0 \cong \mathcal{M}$. By Theorem B and Example 3.13, the functors $\mathbb{H}_{\mathcal{M}_0,\mathcal{N}_0[l]}$ are coherent for all l. The inductive hypothesis also implies that $\mathbb{H}_{\mathcal{F},\mathcal{N}[r]}$ is coherent for all $\mathcal{F} \in \mathsf{QCoh}(X)$ and all integers $r < n$. By Lemma 4.5, $\mathbb{H}_{\mathcal{M},\mathcal{N}[n]}$ is coherent. □

5. Coherence of Hom-functors: non-flat case

In this section we prove Theorem E. For this section we also retain the notation of §4. We begin by dispatching Theorem E in the projective case.

Lemma 5.1. *Fix an affine and noetherian scheme S and a morphism of schemes $f\colon X \to S$ that is projective. If $\mathcal{N} \in \mathsf{D}^b_{\mathsf{Coh}}(X)$, then $\mathcal{T}^{\mathcal{N}}_{X/S} = \mathsf{D}_{\mathrm{qc}}(X)$.*

Proof. By Lemma 4.2(1), it is sufficient to show that $\mathsf{D}^-_{\mathrm{qc}}(X) \subseteq \mathcal{T}^{\mathcal{N}}_{X/S}$. Since f is a projective morphism and S is an affine and noetherian scheme, f has an ample family of line bundles. It now follows from [SGA6, II.2.2.9] that if $\mathcal{M} \in \mathsf{D}^-_{\mathrm{qc}}(X)$, then $\mathcal{M}$ is quasi-isomorphic to a complex $\mathcal{Q}$ whose terms are direct sums of shifts of line bundles. Thus, by Lemma 4.2, it is sufficient to prove that if $\mathcal{L} \in \mathsf{Coh}(X)$ is a line bundle, then $\mathcal{L}[0] \in \mathcal{T}^{\mathcal{N}}_{X/S}$. If n is an integer, then we have natural isomorphisms:

$$\begin{aligned}\mathbb{H}_{\mathcal{L}[0],\mathcal{N}[n]} &= \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L}[0], \mathcal{N}[n] \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}(-))\\ &\cong \mathcal{H}^0(\mathsf{R}\Gamma(X, (\mathcal{L}^{-1}[0] \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N}[n] \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}f^*_{\mathrm{qc}}(-))\\ &\cong \mathcal{H}^0(\mathsf{R}\Gamma(X, \mathcal{L}^{-1}[0] \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N}[n]) \otimes^{\mathsf{L}}_{\mathcal{O}_S} (-)) \qquad \text{(Proposition 1.3(3))}.\end{aligned}$$

Since $\mathcal{L}$ is $\mathcal{O}_X$-flat and $\mathcal{N} \in \mathsf{D}^b_{\mathsf{Coh}}(X)$, it follows that $\mathcal{L}^{-1}[0] \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N}[n] \in \mathsf{D}^b_{\mathsf{Coh}}(X)$. Hence, $\mathsf{R}\Gamma(X, \mathcal{L}^{-1}[0] \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{N}[n]) \in \mathsf{D}^b_{\mathsf{Coh}}(S)$ [SGA6, III.2.2]. By Example 3.13, $\mathbb{H}_{\mathcal{L},\mathcal{N}[n]}$ is coherent, so $\mathcal{L}[0] \in \mathcal{T}^{\mathcal{N}}_{X/S}$. □

We can now prove Theorem E.

Proof of Theorem E. Arguing by induction on the length of the complex $\mathcal{N}$, it is sufficient to prove the result when $\mathcal{N}$ is only supported in cohomological degree 0. In particular, there exists a closed immersion $i\colon Y \to X$, such that the composition

$f \circ i$ is proper, together with a coherent $\mathcal{O}_Y$-module $\mathcal{N}_0$ and a quasi-isomorphism $i_*\mathcal{N}_0[0] \cong \mathcal{N}$. By Lemma 4.3, it suffices to prove that $\mathcal{T}_{Y/S}^{\mathcal{N}_0[0]} = \mathsf{D}_{\mathrm{qc}}(Y)$. Hence, we have reduced the claim to the case where the morphism f is proper and where $\mathcal{N} \simeq \mathcal{N}[0]$ for some $\mathcal{N} \in \mathsf{Coh}(X)$.

Now let $\mathcal{C}_{X/S} \subseteq \mathsf{Coh}(X)$ denote the full subcategory with objects those $\mathcal{N} \in \mathsf{Coh}(X)$ such that $\mathcal{T}_{X/S}^{\mathcal{N}[0]} = \mathsf{D}_{\mathrm{qc}}(X)$. By the 5-Lemma, it is plain to see that $\mathcal{C}_{X/S}$ is an exact subcategory (in the sense of [EGA, III.3.1]). We now prove by noetherian induction on the closed substacks of X that $\mathcal{C}_{X/S} = \mathsf{Coh}(X)$. By virtue of Lemma 4.3 and the technique of dévissage [EGA, Proof of III.3.2], it is sufficient to prove that $\mathcal{C}_{X/S} = \mathsf{Coh}(X)$ when X is integral and $\mathcal{T}_{X/S}^{\mathcal{Q}[0]} = \mathsf{D}_{\mathrm{qc}}(X)$ for all coherent $\mathcal{O}_X$-modules $\mathcal{Q}$ such that $\operatorname{supp} \mathcal{Q} \subsetneq |X|$.

So, we fix $\mathcal{N} \in \mathsf{Coh}(X)$. By Chow's Lemma [Knu71, Thm. IV.3.1], there exists a morphism $p\colon X' \to X$ that is proper, surjective, and birational such that X' is a projective S-scheme. By Lemma 5.1, we deduce that $\mathcal{T}_{X'/S}^{p^*\mathcal{N}[0]} = \mathsf{D}_{\mathrm{qc}}(X')$.

Next, by Lemma 4.3, $\mathcal{T}_{X/S}^{\mathsf{R}p_*p^*\mathcal{N}} = \mathsf{D}_{\mathrm{qc}}(X)$. Also, p is birational, thus generically affine, so the support of the cohomology sheaves of $\tau^{\geq 1}(\mathsf{R}p_*p^*\mathcal{N})$ does not contain the generic point. By noetherian induction, we deduce that $\mathcal{T}_{X/S}^{\tau^{\geq 1}(\mathsf{R}p_*p^*\mathcal{N})} = \mathsf{D}_{\mathrm{qc}}(X)$, thus $\mathcal{T}_{X/S}^{p_*p^*\mathcal{N}[0]} = \mathsf{D}_{\mathrm{qc}}(X)$. In particular, $p_*p^*\mathcal{N} \in \mathcal{C}_{X/S}$. Since p is birational and X is integral, the natural map $\theta\colon \mathcal{N} \to p_*p^*\mathcal{N}$ is an isomorphism over a dense open $U \subseteq X$. The exactness of the subcategory $\mathcal{C}_{X/S} \subseteq \mathsf{Coh}(X)$ and dévissage now prove that $\mathcal{N} \in \mathcal{C}_{X/S}$. □

6. Applications

6.1. **Representability of** Hom**-spaces.** As promised, Theorem D is now completely elementary.

Proof of Theorem D. The latter claim follows from the former by standard limit methods. Since $\underline{\mathrm{Hom}}_{\mathcal{O}_X/S}(\mathcal{M},\mathcal{N})$ is an étale sheaf, it is sufficient to prove the result when S is affine. By Theorem C and Corollary 1.4(3), the functor:

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes_{\mathcal{O}_X} f^*(-))\colon \mathsf{QCoh}(S) \to \mathsf{Ab}$$

is coherent. The flatness of $\mathcal{N}$ over S also shows that this functor is left-exact, so by Example 3.10, it is corepresentable by a quasi-coherent $\mathcal{O}_S$-module $Q_{\mathcal{M},\mathcal{N}}$. If $\tau\colon T \to S$ is affine, then there are natural isomorphisms:

$$\begin{aligned}
\underline{\mathrm{Hom}}_{\mathcal{O}_X/S}(\mathcal{M},\mathcal{N})[T \xrightarrow{\tau} S] &= \mathrm{Hom}_{\mathcal{O}_{X_T}}(\tau_X^*\mathcal{M}, \tau_X^*\mathcal{N}) \\
&\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, (\tau_X)_*\tau_X^*\mathcal{N}) \\
&\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes_{\mathcal{O}_X} f^*[\tau_*\mathcal{O}_T]) && \text{(Corollary 1.4)} \\
&\cong \mathrm{Hom}_{\mathcal{O}_S}(Q_{\mathcal{M},\mathcal{N}}, \tau_*\mathcal{O}_T) \\
&\cong \mathrm{Hom}_{\mathcal{O}_S\text{-Alg}}(\mathrm{Sym}^\bullet_{\mathcal{O}_S} Q_{\mathcal{M},\mathcal{N}}, \tau_*\mathcal{O}_T) \\
&\cong \mathrm{Hom}_{\mathsf{Sch}/S}(T \xrightarrow{\tau} S, \underline{\mathrm{Spec}}_{\mathcal{O}_S} \mathrm{Sym}^\bullet_{\mathcal{O}_S} Q_{\mathcal{M},\mathcal{N}}).
\end{aligned}$$

The natural isomorphism above extends to all S-schemes as $\underline{\mathrm{Hom}}_{\mathcal{O}_X/S}(\mathcal{M},\mathcal{N})$ is an étale sheaf. The result follows. □

6.2. **Cohomology and base change.** To prove Theorem A, we will consider some refinements of the vanishing results of A. Ogus and G. Bergman [OB72] that occur in the setting of finitely generated functors. So, we fix a ring A and an A-linear functor Q. Define

$$\mathbb{V}(Q) = \{\mathfrak{p} \in \operatorname{Spec} A \;:\; Q(N) = 0 \quad \forall N \in \mathsf{Mod}(A_{\mathfrak{p}})\}.$$

Finitely generated functors immediately demonstrate their utility.

Proposition 6.1. *Fix a ring A and an A-linear functor F that preserves direct limits. If F is finitely generated, then the subset $\mathbb{V}(F) \subseteq \operatorname{Spec} A$ is Zariski open.*

Proof. Fix a generator (I,η) for F and a prime ideal $\mathfrak{p} \lhd A$ such that $F_{A_{\mathfrak{p}}} \equiv 0$. For $a \in A$ there is the localisation morphism $l_a \colon I \to I_a$ (resp. $l_{\mathfrak{p}} \colon I \to I_{\mathfrak{p}}$) and we set $\eta_a = (l_a)_*\eta$ (resp. $\eta_{\mathfrak{p}} = (l_{\mathfrak{p}})_*\eta$). Since $F(I_{\mathfrak{p}}) = 0$, it follows that $\eta_{\mathfrak{p}} = 0$. However, as $I_{\mathfrak{p}} = \varinjlim_{a\notin\mathfrak{p}} I_a$ and F preserves direct limits of A-modules, there exists $a \notin \mathfrak{p}$ such that $\eta_a = 0$ in $F(I_a)$. Since the pair (I_a,η_a) generates F_{A_a}, we have that $F_{A_a} \equiv 0$. □

We now record for future reference a result that is likely well-known, though we are unaware of a reference. For an A-module N, define $Q^N \colon \mathsf{Mod}(A) \to \mathsf{Sets}$ to be the functor $Q(- \otimes_A N) \colon \mathsf{Mod}(A) \to \mathsf{Sets}$. For another ring C, a C-module K, and a functor $G \colon \mathsf{Mod}(A) \to \mathsf{Mod}(C)$, there is the functor $G \otimes_C K \colon \mathsf{Mod}(A) \to \mathsf{Mod}(C) \colon M \mapsto G(M) \otimes_C K$. Given an A-linear functor H, and an A-module N, there is a natural transformation of functors

$$\delta_{H,N} \colon H \otimes_A N \Rightarrow H^N.$$

Indeed, it is sufficient to describe a natural morphism $N \to \operatorname{Hom}_A(H(M), H(M \otimes_A N)$ for each A-module M. So, let $n \in N$, which we may view as a morphism $e_n \colon A \to N$. Applying the functor $H(M \otimes_A -)$ to this morphism gives a natural morphism $H(M) \to H(M \otimes_A N)$, as required.

Proposition 6.2. *Fix a ring A and an A-linear functor F. If F preserves direct limits, then for every flat A-module M the natural transformation*

$$\delta_{F,M} \colon F \otimes_A M \Rightarrow F^M$$

is an isomorphism of functors.

Proof. First, assume that the A-module M is finite free. The functor F commutes with finite products, thus the natural transformation $\delta_{F,M}$ is an isomorphism. For the general case, by Lazard's Theorem [Laz64], we may write $M = \varinjlim_i P_i$, where each P_i is a finite free A-module. Since tensor products commute with direct limits, as does the functor F, the Proposition follows from the case already considered. □

Let R be a noetherian ring. An R-linear functor G is *bounded* if $M \in \mathsf{Coh}(R)$ implies $G(M) \in \mathsf{Coh}(R)$.

Corollary 6.3. *Fix a noetherian ring R and a bounded R-linear functor G that preserves direct limits.*

1. *For every quasi-finite R-algebra R', the functor $G_{R'}$ is bounded.*
2. *For every $\mathfrak{p} \in \operatorname{Spec} R$, the functor $G_{R_{\mathfrak{p}}}$ is bounded.*

Proof. For (1), by Zariski's Main Theorem [EGA, IV.18.12.13], the homomorphism $R \to R'$ factors as $R \to \widetilde{R} \to R'$ where $R \to \widetilde{R}$ is finite and $\operatorname{Spec} R' \to \operatorname{Spec} \widetilde{R}$ is an open immersion. Since the functor G is bounded, the functor $G_{\widetilde{R}}$ is bounded. Thus, it suffices to consider the case where $\operatorname{Spec} R' \to \operatorname{Spec} R$ is an open immersion. For every coherent R'-module M', there exists a coherent R-module M together with an isomorphism of R'-modules $M \otimes_R R' \cong M'$. Since the homomorphism $R \to R'$ is flat, Proposition 6.2 implies that $G(M) \otimes_R R' \cong G(M \otimes_R R') \cong G(M')$. The functor G is bounded, thus the R'-module $G(M) \otimes_R R'$ is coherent, giving the claim.

For (2), fix a coherent $R_{\mathfrak{p}}$-module N, then for some $f \in R - \mathfrak{p}$ there exists an R_f-module L, together with an $R_{\mathfrak{p}}$-module isomorphism $L \otimes_{R_f} R_{\mathfrak{p}} \cong N$. By Proposition

6.2, we have $G(N) \cong G(L) \otimes_{R_f} R_{\mathfrak{p}}$. By (1), since $R \to R_f$ is quasi-finite, it follows that G_{R_f} is bounded. Thus, $G(N)$ is a coherent $R_{\mathfrak{p}}$-module. □

Remark 6.4. Corollary 6.3(2) also holds for the henselization and strict henselization of $R_{\mathfrak{p}}$.

Fix a noetherian ring R and a half-exact, bounded, R-linear functor F. A. Ogus and G. Bergman show in [OB72, Thm. 2.1] that if $F(\kappa(\mathfrak{q})) = 0$ for all closed points $\mathfrak{q} \in \operatorname{Spec} R$, then F is the zero functor. We have the following amplification.

Corollary 6.5. *Fix a noetherian ring R and a bounded, half-exact, R-linear functor G. If G preserves direct limits, then*

$$\mathbb{V}(G) = \{\mathfrak{q} \in \operatorname{Spec} R \,:\, G(\kappa(\mathfrak{q})) = 0\}.$$

Proof. Clearly, if $\mathfrak{q} \in \mathbb{V}(G)$, then $G(\kappa(\mathfrak{q})) = 0$. For the other inclusion, let $\mathfrak{q} \in \operatorname{Spec} R$ satisfy $G(\kappa(\mathfrak{q})) = 0$. By Corollary 6.3(2), the functor $G_{R_{\mathfrak{q}}}$ is bounded. Thus, [OB72, Thm. 2.1] applies, giving $G_{R_{\mathfrak{q}}} \equiv 0$, and so $\mathfrak{q} \in \mathbb{V}(G)$. □

An R-linear functor G is *universally bounded* if for every noetherian R-algebra R', the functor $G_{R'}$ is bounded. To combine Proposition 6.1 and Corollary 6.5, it is useful to have the following easily proven Lemma at hand.

Lemma 6.6. *Fix a ring A and an A-linear functor F that preserves direct limits. If F is finitely generated, then there exists a generator (I, η) with I a finitely presented A-module. In particular, if A is noetherian, then F is universally bounded.*

Combining Proposition 6.1, Corollary 6.5, and Lemma 6.6, we obtain the vanishing result we desire.

Corollary 6.7. *Fix a noetherian ring R and an R-linear and half-exact functor F that is finitely generated and preserves direct limits. If $\mathfrak{q} \in \operatorname{Spec} R$ and $F(\kappa(\mathfrak{q})) = 0$, then there exists $r \in R - \mathfrak{q}$ such that $F_{R_r} \equiv 0$.*

Proof. By Corollary 6.5, $\mathfrak{q} \in \mathbb{V}(F)$. By Proposition 6.1, the set $\mathbb{V}(F)$ is Zariski open, thus there exists $r \in R - \mathfrak{q}$ such that $\mathfrak{p} \in \operatorname{Spec} R_r \subseteq \mathbb{V}(F)$. Let $N \in \mathsf{Mod}(R_r)$ and $\mathfrak{p} \in \operatorname{Spec} R_r$, then by Proposition 6.2 it follows that $F(N)_{\mathfrak{p}} = F(N_{\mathfrak{p}})$. But $\mathfrak{p} \in \mathbb{V}(F)$ and so $F(N)_{\mathfrak{p}} = 0$. Since $F(N)$ is an R_r-module, the result follows. □

Corollary 6.8. *Fix a noetherian ring R and an R-linear and half-exact functor F that is finitely generated and preserves direct limits. If $\mathfrak{q} \in \operatorname{Spec} R$ and the natural map $F(R) \otimes_R \kappa(\mathfrak{q}) \to F(\kappa(\mathfrak{q}))$ is surjective, then there exists $r \in R - \mathfrak{q}$ such that $F_{R_r}(-) \cong F(R) \otimes_R -$. In particular, F_{R_r} is right-exact.*

Proof. Let F_0 be the R-linear functor that assigns to each R-module M the R-module $\operatorname{coker}(F(R) \otimes_R M \to F(M))$. Clearly, F_0 is finitely generated, preserves direct limits, and satisfies $F_0(\kappa(\mathfrak{q})) = 0$. By [Har98, Prop. 3.1], F_0 is half exact. The result now follows from Corollary 6.7. □

We now combine Corollary 6.7 with the exchange property proved by A. Ogus and G. Bergman [OB72, Cor. 5.1]. Some notation: for a ring A, a pair of A-linear functors F^0 and F^1 is *cohomological* if for every exact sequence of A-modules $0 \to M' \to M \to M'' \to 0$, there is a functorially induced exact sequence of A-modules:

$$F^0(M') \longrightarrow F^0(M) \longrightarrow F^0(M'') \longrightarrow F^1(M') \longrightarrow F^1(M) \longrightarrow F^1(M'').$$

Corollary 6.9 (Property of exchange). *Fix a noetherian ring R and a cohomological pair of R-linear functors F^0 and F^1 that preserve direct limits. For $i = 0$ and 1 and for every $M \in \mathsf{Mod}(R)$, there is a natural map:*

$$\phi^i(M)\colon F^i(R) \otimes_R M \to F^i(M).$$

Let $\mathfrak{q} \in \operatorname{Spec} R$ and let $\kappa(\mathfrak{q}) \subseteq k$ be a field extension. Assume that F^0 and F^1 are finitely generated.

(1) *If F^1 is coherent, then the following are equivalent.*
   (a) *$\phi^0(k)$ is surjective.*
   (b) *There exists $r \in R - \mathfrak{q}$ such that for all $M \in \mathsf{QCoh}(R_r)$, the map $\phi^0(M)$ is an isomorphism.*
   (c) *There exists an $r \in R - \mathfrak{q}$, a coherent R_r-module N, and a natural isomorphism*
   $$F^1(M) \cong \operatorname{Hom}_{R_r}(N, M)$$
   for all R_r-modules M.

(2) *If $\phi^1(k)$ is surjective, then the following are equivalent:*
   (a) *$\phi^0(k)$ is surjective;*
   (b) *there exists an $l \in R - \mathfrak{q}$ such that the R_l-module $F^1(R)_l$ is free.*

Proof. First note that $\phi^i(k)$ is surjective if and only if $\phi^i(\kappa(\mathfrak{q}))$ is surjective. To see this, we note since $\kappa(\mathfrak{q})$ is field, k is a flat $\kappa(\mathfrak{q})$-module. By Proposition 6.2, applied to the functor $(F^i)_{\kappa(\mathfrak{q})}$ and the module k, it follows that $F^i(\kappa(\mathfrak{q})) \otimes_R k \cong F^i(k)$. The claim follows and we may henceforth assume that $\kappa(\mathfrak{q}) = k$.

We first treat (1). The implication (a)⇒(b) is Corollary 6.8. For (b)⇒(c), we see that $F^0_{R_r}$ is right-exact and so $F^1_{R_r}$ is left-exact. Since F^1 is coherent, it follows that $F^1_{R_r}$ is coherent (Example 3.6). By Example 3.10, we deduce the claim. For the implication (c)⇒(a), we see that $F^1_{R_r}$ is left-exact, so $F^0_{R_r}$ is right-exact, and so $\phi^0(\kappa(\mathfrak{q}))$ is surjective.

For (2), by Lemma 6.6, the functors F_i are bounded. For the implication (a)⇒(b), by Corollary 6.8, there exists an $l \in R - \mathfrak{q}$ such that $F^1_{R_l} \cong F^1(R) \otimes_R -$ and $F^0_{R_l}$ is right-exact. In particular, $F^1_{R_l}$ is also left-exact, so $F^1(R)_l$ is a finitely generated and flat R_l-module. Shrinking $\operatorname{Spec} R_l$ around $\mathfrak{q}$ gives the claim. For (b)⇒(a), by Corollary 6.8, $F^1_{R_\mathfrak{q}} \cong F^1(R)_\mathfrak{q} \otimes_R -$. Since $F^1(R)_\mathfrak{q}$ is free, it follows that $F^1_{R_\mathfrak{q}}$ is also left-exact. Hence, $F^0_{R_\mathfrak{q}}$ is right-exact and so $\phi^0(\kappa(\mathfrak{q}))$ is surjective. □

Proof of Theorem A. Throughout we fix a 2-cartesian diagram of noetherian algebraic stacks:

$$\begin{array}{ccc} X_T & \xrightarrow{\tau_X} & X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\tau} & S. \end{array}$$

The results are all smooth local on S and T, thus we may assume that $S = \operatorname{Spec} A$ and $T = \operatorname{Spec} B$ and τ is induced by a ring homomorphism $A \to B$. We may also work with the global Ext-groups instead of the relative $\mathcal{E}xt$-sheaves.

For an A-module I set $E^q(I) = \operatorname{Ext}^q_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes_{\mathcal{O}_X} f^*I)$, which gives an A-linear functor. By Theorem C, the functor E^q is coherent and by Lemma 1.2 the functor preserves direct limits. By Corollary 1.4, if J is a B-module, there is a natural isomorphism:

$$E^q(J) = \operatorname{Ext}^q_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes_{\mathcal{O}_X} f^*\tau_*J) \cong \operatorname{Ext}^q_{\mathcal{O}_{X_T}}(\mathsf{L}(\tau_X)^*_{\mathrm{qc}}\mathcal{M}, \tau_X^*\mathcal{N} \otimes_{\mathcal{O}_{X_T}} f_T^*J).$$

In particular, taking $J = B$ we obtain a natural map:

$$E^q(A) \otimes_A B = \operatorname{Ext}^q_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \otimes_A B \to \operatorname{Ext}^q_{\mathcal{O}_{X_T}}(\mathsf{L}(\tau_X)^*_{\mathrm{qc}}\mathcal{M}, \tau_X^*\mathcal{N}) = E^q(B).$$

The result now follows from Theorem C and Corollary 6.9. □

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