ARTIN'S CRITERIA FOR ALGEBRAICITY REVISITED

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Abstract. Using notions of homogeneity we give new proofs of M. Artin's algebraicity criteria for functors and groupoids. Our methods give a more general result, unifying Artin's two theorems and clarifying their differences.

Introduction

Classically, moduli spaces in algebraic geometry are constructed using either projective methods or by forming suitable quotients. In his reshaping of the foundations of algebraic geometry half a century ago, Grothendieck shifted focus to the functor of points and the central question became whether certain functors are representable. Early on, he developed formal geometry and deformation theory, with the intent of using these as the main tools for proving representability. Grothendieck's proof of the existence of Hilbert and Picard schemes, however, is based on projective methods. It was not until ten years later that Artin completed Grothendieck's vision in a series of landmark papers. In particular, Artin vastly generalized Grothendieck's existence result and showed that the Hilbert and Picard schemes exist—as algebraic spaces—in great generality. It also became clear that the correct setting was that of algebraic spaces—not schemes—and algebraic stacks.

In his two eminent papers [Art69b, Art74], M. Artin gave precise criteria for algebraicity of functors and stacks. These criteria were later clarified by B. Conrad and J. de Jong [CJ02] using Néron–Popescu desingularization, by H. Flenner [Fle81] using Exal, and the first author [Hal17] using coherent functors. The criterion in [Hal17] is very streamlined and elegant and suffices to deal with most problems. It does not, however, supersede Artin's criteria as these are more general. Another conundrum is that Artin gives two different criteria—the first [Art69b, Thm. 5.3] is for functors and the second [Art74, Thm. 5.3] is for stacks—but neither completely generalizes the other.

The purpose of this paper is to use the ideas of Flenner and the first author to give a new criterion that supersedes all present criteria. We also introduce several new ideas that broaden the criteria and simplify the proofs of [Art69b, Art74, Fle81]. In positive characteristic, we also identify a subtle issue in Artin's algebraicity criterion for stacks. With the techniques that we develop, this problem is circumvented. We now state our criterion for algebraicity.

Date: May 27, 2018.

2010 *Mathematics Subject Classification.* Primary 14D15; Secondary 14D23.

This collaboration was supported by the Göran Gustafsson foundation. The first author was supported by the Australian Research Council DE150101799. The second author is also supported by the Swedish Research Council 2011-5599 and 2015-05554.

Main Theorem. *Let S be an excellent scheme. Then a category X, fibered in groupoids over the category of S-schemes, Sch/S, is an algebraic stack, locally of finite presentation over S, if and only if it satisfies the following conditions:*

(1) *X is a stack over $(\mathsf{Sch}/S)_{\mathrm{fppf}}$;*
(2) *X is limit preserving (Definition 1.7);*
(3) *X is weakly effective (Definition 9.1);*
(4) *X is $\mathbf{Art}^{\mathbf{triv}}$-homogeneous (Definition 1.3, also see below);*
(5a) *X has bounded automorphisms and deformations (Conditions 6.1(i)–6.1(ii));*
(5b) *X has constructible automorphisms and deformations (Conditions 6.3(i)–6.3(ii));*
(5c) *X has Zariski local automorphisms and deformations (Conditions 6.4(i)–6.4(ii));*
(6b) *X has constructible obstructions (Condition 6.3(iii), or 7.3); and*
(6c) *X has Zariski local obstructions (Condition 6.4(iii), or 7.4).*

In addition,

(α) *if S is Jacobson, then conditions* (5c) *and* (6c) *are superfluous;*
(β) *if X is $\mathbf{DVR}$-homogeneous (Notation 2.14), then conditions* (5c) *and* (6c) *are superfluous and condition* (6b) *may be replaced with Condition 8.3;*
(γ) *conditions* (1) *and* (4) *can be replaced with*
  (1′) *X is a stack over $(\mathsf{Sch}/S)_{\text{Ét}}$ and*
  (4′) *X is $\mathbf{Art}^{\mathbf{insep}}$-homogeneous; and*
(δ) *if the residue fields of S at points of finite type are perfect, then* (4) *and* (4′) *are equivalent.*

In particular, if S is a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$*, then conditions* (5c) *and* (6c) *are superfluous and* (1) *can be replaced with* (1′).

The $\mathbf{Art}^{\mathbf{triv}}$-homogeneity (resp. $\mathbf{Art}^{\mathbf{insep}}$-homogeneity) condition is the following Schlessinger–Rim condition: for every diagram of local artinian S-schemes of finite type $[\operatorname{Spec} B \leftarrow \operatorname{Spec} A \hookrightarrow \operatorname{Spec} A']$, where $A' \twoheadrightarrow A$ is surjective and the residue field extension $B/\mathfrak{m}_B \to A/\mathfrak{m}_A$ is trivial (resp. purely inseparable), the natural functor

$$X(\operatorname{Spec}(A' \times_A B)) \to X(\operatorname{Spec} A') \times_{X(\operatorname{Spec} A)} X(\operatorname{Spec} B)$$

is an equivalence of categories.

Perhaps the most striking difference between our conditions and Artin's conditions is that our homogeneity condition (4) only involves local artinian schemes and that we do not need any conditions on étale localization of deformation and obstruction theories. If S is Jacobson, e.g., of finite type over a field, then we do not even need compatibility with Zariski localization. There is also no condition on compatibility with completions for automorphisms and deformations. We will give a detailed comparison between our conditions and other versions of Artin's conditions in Section 11.

All existing algebraicity proofs, including ours, consist of the following four steps:

(i) existence of formally versal deformations;
(ii) algebraization of formally versal deformations;
(iii) openness of formal versality; and
(iv) formal versality implies formal smoothness.

Step (i) was eloquently dealt with by Schlessinger [Sch68, Thm. 2.11] for functors and by Rim [SGA7, Exp. VI] for groupoids. This step uses conditions (4) and (5a) ($\mathbf{Art}^{\mathbf{triv}}$-homogeneity and boundedness of tangent spaces). Step (ii) begins with the effectivization of formally versal deformations using condition (3). One may then algebraize this family using either Artin's results [Art69a, Art69b] or B. Conrad and J. de Jong's result [CJ02]. In the latter approach, Artin approximation is replaced with Néron–Popescu desingularization, and S is only required to be excellent. This step requires condition (2).

The last two steps are more subtle and it is here that [Art69b, Art74, Fle81, Sta06, Hal17] and our present treatment diverge—both when it comes to the criteria themselves and the techniques employed. We begin with discussing step (iv).

Formal versality implies formal smoothness. It is readily seen that our criterion is weaker than Artin's two criteria [Art69b, Art74] except that, in positive characteristic, we need X to be a stack in the fppf topology, or otherwise strengthen (4). This is similar to [Art69b, Thm. 5.3] where the functor is assumed to be an fppf-sheaf. In [Art69b, Thm. 5.3], Artin deftly uses the fppf sheaf condition to deduce that formally universal deformations are formally étale [Art69b, pp. 50–52], settling step (iv) for functors. This argument relies on the existence of universal deformations and thus does not extend to stacks with infinite or non-reduced stabilizers. Using a different approach, we extend this result to fppf stacks in Lemma 1.9.

In his second paper [Art74], Artin only assumes that the groupoid is an étale stack. His proof of step (iv) for groupoids [Art74, Prop. 4.2], however, does not treat inseparable extensions. We do not understand how this problem can be overcome without strengthening the criteria and assuming that either (1) the groupoid is a stack in the fppf topology or (4′) requiring (semi)homogeneity for inseparable extensions (see Lemmas 1.9 and 2.2). We wish to emphasize that if S is of finite type over $\operatorname{Spec}\mathbb{Z}$ or a perfect field, then the main result of [Art74] holds without change. See Remark 2.8 for further discussion. Flenner does not discuss formal smoothness, and in [Hal17] formal smoothness is obtained by strengthening the homogeneity condition (4).

Openness of formal versality. Step (iii) uses constructibility, boundedness, and Zariski localization of deformations and obstruction theories (Theorem 4.4). In our treatment, localization is only required when passing to non-closed points of finite type. Such points only exist when S is not Jacobson, e.g., if S is the spectrum of a discrete valuation ring. Our proof is very similar to Flenner's proof. It may appear that Flenner does not need Zariski localization in his criterion, but this is due to the fact that his conditions are expressed in terms of deformation and obstruction *sheaves*.

As in Flenner's proof, openness of versality becomes a matter of simple algebra. It comes down to a criterion for the openness of the *vanishing locus* of half-exact functors (Theorem 3.3) that easily follows from the Ogus–Bergman Nakayama Lemma for half-exact functors (Theorem 3.7). Flenner proves a stronger statement that implies the Ogus–Bergman result (Remark 3.8).

At first, it seems that we need more than $\mathbf{Art}^{\mathbf{triv}}$-homogeneity to even make sense of conditions (5a)–(6c). This will turn out to not be the case. Using steps (ii) and (iv), we prove that conditions (1)–(4) and (5a) at fields guarantee that we have homogeneity for arbitrary integral morphisms (Lemma 10.4). It follows that $\mathrm{Aut}_{X/S}(T,-)$, $\mathrm{Def}_{X/S}(T,-)$ and $\mathrm{Obs}_{X/S}(T,-)$ are additive functors.

Applications. We believe that a distinct advantage of the criterion in the present paper contrasted with all prior criteria is the dramatic weakening of the homogeneity. Whereas the criteria [Hal17] and [Art69b] require **Aff**-, and **DVR**-homogeneity respectively, involving knowledge of the functor over non-noetherian rings, we only need homogeneity for artinian rings. This is particularly useful for more subtle moduli problems such as Angéniol's Chow functor [Ang81, 5.2], which is difficult to define over non-noetherian rings.

The ideas in this paper have also led to a criterion for a half-exact functor to be coherent [HR12]. Although both the statement and the proof bear a close resemblance to the Main Theorem, this coherence criterion does not follow from any algebraicity criterion.

Outline. The technical results of the paper are summarized by Proposition 10.2. The Main Theorem follows from Proposition 10.2 by a bootstrapping process and the relationship between automorphisms, deformations, obstructions and extensions. A significant part of the paper (§§5–9) is devoted to making this relationship precise. Sections §§1–4 form the technical heart of the paper. We now briefly summarize the contents of the paper in more detail.

In Section 1 we recall the notions of homogeneity, limit preservation and extensions from [Hal17]. We also introduce homogeneity that only involves artinian rings and show that residue field extensions are harmless for stacks in the fppf topology. In Section 2 we then relate formal versality, formal smoothness and vanishing of Exal.

In Section 3 we study additive functors and their vanishing loci. This is applied in Section 4 where we give conditions on Exal that assure that the locus of formal versality is open. The results are then assembled in Theorem 4.4.

In Section 5 we repeat the definitions of automorphisms, deformations and minimal obstruction theories from [Hal17]. In Section 6, we give conditions on Aut, Def and Obs that imply the corresponding conditions on Exal needed in Theorem 4.4. In Section 7 we introduce n-step obstruction theories. In Section 8 we formulate the conditions on obstructions without using linear obstruction theories, as in [Art69b]. In Section 9, we discuss effectivity. Finally, in Section 10 we prove the Main Theorem. Comparisons with other criteria are given in Section 11.

Notation. We follow standard conventions and notation. In particular, we adhere to the notation of [Hal17]. Recall that if T is a scheme, then a point $t \in |T|$ is of *finite type* if $\operatorname{Spec} \kappa(t) \to T$ is of finite type. Points of finite type are locally closed. A point of a Jacobson scheme is of finite type if and only if it is closed. If $f\colon X \to Y$ is of finite type and $x \in |X|$ is of finite type, then $f(x) \in |Y|$ is of finite type.

Acknowledgment. We would like to thank M. Artin for encouraging comments and L. Moret–Bailly for answering a question on MathOverflow about Jacobson schemes. We would also especially like to thank the referees for their patience, support and a number of useful comments.

1. Homogeneity, limit preservation, and extensions

Fix a scheme S. An S-*groupoid* is a category X together with a functor $a_X : X \to \mathsf{Sch}/S$ that is fibered in groupoids. A 1-morphism of S-groupoids $\Phi\colon (Y, a_Y) \to (Z, a_Z)$ is a functor between categories Y and Z that commutes strictly over Sch/S. We will typically refer to an S-groupoid (X, a_X) as "X".

A closed immersion of schemes $j\colon V \hookrightarrow V'$ is *nilpotent* if there exists an integer $n > 0$ such that $J^n = 0$, where J is the quasi-coherent sheaf of ideals defining j. A closed immersion of schemes is *locally nilpotent* if fppf-locally it is nilpotent.

If X is an S-groupoid and $[\operatorname{Spec} B \leftarrow \operatorname{Spec} A \xrightarrow{j} \operatorname{Spec} A']$ is a diagram of S-schemes, where j is a nilpotent closed immersion, then the condition that the functor

$$X(\operatorname{Spec}(B \times_A A')) \to X(\operatorname{Spec} B) \times_{X(\operatorname{Spec} A)} X(\operatorname{Spec} A')$$

is an equivalence for a collection of diagrams has been a feature of deformation theory since Schlessinger [Sch68] and Rim [SGA7, Exp. VI]. Consequently, these are typically called *Schlessinger–Rim* conditions.

In this section, we review the concept of homogeneity—a variation of the Schlessinger–Rim conditions that we attribute to J. Wise [Wis11, §2]—in the formalism of [Hal17, §§1–2]. We will also briefly discuss limit preservation and extensions.

Let X be an S-groupoid. An X-*scheme* is a pair (T, σ_T), where T is an S-scheme and $\sigma_T\colon \mathsf{Sch}/T \to X$ is a 1-morphism of S-groupoids. A morphism of X-schemes $U \to V$ is a morphism of S-schemes $f\colon U \to V$ (which canonically determines a 1-morphism of S-groupoids $\mathsf{Sch}/f\colon \mathsf{Sch}/U \to \mathsf{Sch}/V$) together with a 2-morphism $\alpha\colon \sigma_U \Rightarrow \sigma_V \circ \mathsf{Sch}/f$. The collection of all X-schemes forms a 1-category, which we denote by Sch/X. It is readily seen that Sch/X is an S-groupoid and that there is a natural equivalence of S-groupoids $\mathsf{Sch}/X \to X$. For a 1-morphism of S-groupoids $\Phi\colon Y \to Z$ there is an induced functor $\mathsf{Sch}/\Phi\colon \mathsf{Sch}/Y \to \mathsf{Sch}/Z$.

Notation 1.1. Frequently, we will be interested in the following classes of morphisms of S-schemes:

- **Nil** – locally nilpotent closed immersions,
- **Cl** – closed immersions,
- **rNil** – morphisms $X \to Y$ such that there exists $(X_0 \to X) \in \mathbf{Nil}$ with the composition $(X_0 \to X \to Y) \in \mathbf{Nil}$,
- **rCl** – morphisms $X \to Y$ such that there exists $(X_0 \to X) \in \mathbf{Nil}$ with the composition $(X_0 \to X \to Y) \in \mathbf{Cl}$,
- $\mathbf{Art}^{\mathbf{fin}}$ – morphisms between local artinian schemes of finite type over S,
- $\mathbf{Art}^{\mathbf{sep}}$ – $\mathbf{Art}^{\mathbf{fin}}$-morphisms with separable residue field extensions,
- $\mathbf{Art}^{\mathbf{insep}}$ – $\mathbf{Art}^{\mathbf{fin}}$-morphisms with purely inseparable residue field extensions,
- $\mathbf{Art}^{\mathbf{triv}}$ – $\mathbf{Art}^{\mathbf{fin}}$-morphisms with trivial residue field extensions,
- **Fin** – finite morphisms,
- **Int** – integral morphisms and
- **Aff** – affine morphisms.

We certainly have a containment of classes of morphisms of S-schemes:

$$\begin{array}{ccccccccc}
 & & \mathbf{rNil} & \subseteq & \mathbf{rCl} & \subseteq & \mathbf{Int} & \subseteq & \mathbf{Aff}. \\
 & & \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid & & \\
 & /\!\!/ & \mathbf{Nil} & \subseteq & \mathbf{Cl} & \subseteq & \mathbf{Fin} & & \\
 & & & & & & \cup\!\shortmid & & \\
\mathbf{Art}^{\mathbf{triv}} & \subseteq & \mathbf{Art}^{\mathbf{insep}} & & \subseteq & & \mathbf{Art}^{\mathbf{fin}} & &
\end{array}$$

Note that for a morphism $X \to Y$ of locally noetherian S-schemes, the properties **rNil** and **rCl** simply mean that $X_{\mathrm{red}} \to Y$ is **Nil** and **Cl** respectively. Note that the classes of morphisms above are all closed under composition.

Let P be a class of morphisms of S-schemes. In [Hal17, §1], P-nil pairs and P-homogeneity were defined. In the present article, it will be necessary to consider some natural refinements of these notions.

Definition 1.2. Fix a scheme S, a class P of morphisms of S-schemes, an S-groupoid X and an X-scheme V. A *P-nil pair over X at V* is a pair $(V \xrightarrow{p} T, V \xrightarrow{j} V')$, where p and j are morphisms of X-schemes, $p \in P$ and $j \in \mathbf{Nil}$. A *P-nil square over X at V* is a commutative diagram of X-schemes

$$\begin{tikzcd}
V \arrow[r, "p"] \arrow[d, hook, "j"'] & T \arrow[d, "i"] \\
V' \arrow[r, "p'"] & T',
\end{tikzcd} \tag{1.1}$$

where the pair $(V \xrightarrow{p} T, V \xrightarrow{j} V')$ is P-nil over X at V. A P-nil square over X at V is *cocartesian* if it is cocartesian in the category of X-schemes. A P-nil square over X at V is *geometric* if p' is affine, i is a locally nilpotent closed immersion, and there is a natural isomorphism

$$\mathcal{O}_{T'} \to i_*\mathcal{O}_T \times_{p'_*j_*\mathcal{O}_V} p'_*\mathcal{O}_{V'}.$$

Note that every geometric P-nil square is cartesian [Fer03, Lem. 1.3c]. Moreover if $P \subseteq \mathbf{Aff}$, then every cocartesian P-nil square is geometric [Hal17, Lem. 1.5(1)].

Definition 1.3 (P-Homogeneity)**.** Fix a scheme S and a class P of morphisms of S-schemes. A 1-morphism of S-groupoids $\Phi\colon Y \to Z$ is *P-homogeneous at a Y-scheme V* if the following two conditions are satisfied:

(${}^V\mathrm{H}_1^P$) a P-nil square over Y at V is cocartesian if and only if the induced P-nil square over Z at V is cocartesian; and

(${}^V\mathrm{H}_2^P$) if a P-nil pair over Y at V can be completed to a cocartesian P-nil square over Z at V, then it can be completed to a P-nil square over Y at V.

We also say that Φ is *P-homogeneous* if it is P-homogeneous at every Y-scheme V. Similarly, Φ satisfies (H_1^P) (resp. (H_2^P)) if it satisfies (${}^V\mathrm{H}_1^P$) (resp. (${}^V\mathrm{H}_2^P$)) for every Y-scheme V. An S-groupoid X is *P-homogeneous at V* if its structure 1-morphism is P-homogeneous at V and is *P-homogeneous* if its structure morphism is P-homogeneous. If Z satisfies (H_1^P), then Y satisfies (H_1^P) if and only if Φ has P-homogeneous diagonal after pull-back to schemes, see Lemma B.2.

If we only assume (${}^V\mathrm{H}_2^P$) in the above, then we obtain the weaker notion of *P-semihomogeneity*. This notion was used in the work of Artin and Flenner.

Remark 1.4. In [Hal17], a number of results are established for 1-morphisms of P-homogeneous S-groupoids $\Phi\colon Y \to Z$. With trivial modifications, most of these results hold using the more refined notion of P-homogeneity at a Y-scheme V. We will use this observation frequently and without further comment.

By [Wis11, Prop. 2.1], every algebraic stack is **Aff**-homogeneous. Also, **rNil**-homogeneity at an artinian scheme V is equivalent to $\mathbf{Art}^{\mathbf{triv}}$-homogeneity at V.

If P is Zariski local (e.g., P is listed in Notation 1.1), then P-homogeneity of an S-groupoid X that is a stack over $(\mathsf{Sch}/S)_{\text{Ét}}$ is equivalent to the functor:

$$X(\operatorname{Spec}(B \times_A A')) \to X(\operatorname{Spec} B) \times_{X(\operatorname{Spec} A)} X(\operatorname{Spec} A') \tag{1.2}$$

being an equivalence for every P-nil pair $(\operatorname{Spec} A \to \operatorname{Spec} B, \operatorname{Spec} A \to \operatorname{Spec} A')$ over S [Hal17, Lem. 1.5(4)]. If X has representable diagonal, then the functor above is always fully faithful for all **Aff**-nil pairs over S—even if X is not necessarily **Aff**-homogeneous (Lemma B.2).

The main computational tools that P-homogeneity bring are contained in [Hal17, Lem. 1.5], an important part of which we now recall.

Lemma 1.5. *Let S be a scheme and let $P \subseteq \mathbf{Aff}$ be a class of morphisms. Let X be an S-groupoid that is P-homogeneous at an X-scheme V. If $(V \xrightarrow{p} T, V \xrightarrow{j} V')$ is a P-nil pair at V, then there exists a cocartesian and geometric P-nil square at V as in* (1.1)*. Moreover if P is listed in Notation 1.1, then p' is P.*

Proof. The main claim is [Hal17, Lem. 1.5(3)]. What remains is trivial except for $P \in \{\mathbf{Nil}, \mathbf{Cl}, \mathbf{Fin}, \mathbf{Int}\}$. In these cases, however, it is known [Fer03, 5.6 (3)]. ☐

Remark 1.6. Let S be a noetherian scheme. If $(\operatorname{Spec} A \to \operatorname{Spec} B, \operatorname{Spec} A \hookrightarrow \operatorname{Spec} A']$ is a **Fin**-nil pair, where $\operatorname{Spec} B$ is of finite type over S, then $\operatorname{Spec}(B \times_A A')$ is of finite type over S. This follows from the fact that $B \times_A A' \subseteq B \times A'$ is an integral extension [AM69, Prop. 7.8]. On the other hand, if $\operatorname{Spec} A \to \operatorname{Spec} B$ is only affine, then $\operatorname{Spec}(B \times_A A')$ is typically not of finite type over S. For example, if $B = k[x]$, $A = k[x, x^{-1}]$ and $A' = k[x, x^{-1}, y]/y^2$, then $B' = B \times_A A' = k[x, y, yx^{-1}, yx^{-2}, \dots]/(y, yx^{-1}, \dots)^2$ which is not of finite type over $S = \operatorname{Spec} k$.

We also recall the following definition (cf. [Art74, §1] and [Hal17, §3]).

Definition 1.7. Let X be a stack over $(\mathsf{Sch}/S)_{\text{Ét}}$. We say that X is *limit preserving* if for every inverse system of affine S-schemes $\{\operatorname{Spec} A_j\}_{j \in J}$ with inverse limit $\operatorname{Spec} A$, the natural functor:

$$\varinjlim_j X(\operatorname{Spec} A_j) \to X(\operatorname{Spec} A)$$

is an equivalence of categories.

If X is an algebraic stack, then X is limit preserving if and only if $X \to S$ is locally of finite presentation [LMB, Prop. 4.15].

By Lemmas B.2 and B.3, if X is a limit preserving stack over $(\mathsf{Sch}/S)_{\text{Ét}}$ with representable diagonal and S is locally noetherian, then **rCl**-homogeneity is equivalent to Artin's *semi*homogeneity condition [Art74, 2.2(S1a)] for X.

Homogeneity supplies an S-groupoid with a quantity of linear data, which we now recall from [Hal17, §2]. An X-*extension* is a square zero closed immersion of X-schemes $i\colon T \hookrightarrow T'$. The collection of X-extensions forms a category, which we denote by $\mathbf{Exal}_X$. There is a natural functor $\mathbf{Exal}_X \to \mathsf{Sch}/X$ that takes $(i\colon T \hookrightarrow T')$ to T.

We denote by $\mathbf{Exal}_X(T)$ the fiber of the category $\mathbf{Exal}_X$ over the X-scheme T—we call these the X-extensions of T. There is a natural functor:

$$\mathbf{Exal}_X(T)^\circ \to \mathsf{QCoh}(T), \quad (i\colon T \hookrightarrow T') \mapsto \ker(i^{-1}\mathcal{O}_{T'} \to \mathcal{O}_T).$$

We denote by $\mathbf{Exal}_X(T,I)$ the fiber category of $\mathbf{Exal}_X(T)$ over the quasi-coherent $\mathcal{O}_T$-module I—we refer to these as the X-extensions of T by I. Denote the set of isomorphism classes of the category $\mathbf{Exal}_X(T,I)$ by $\mathrm{Exal}_X(T,I)$.

Let W be a scheme and let J be a quasi-coherent $\mathcal{O}_W$-module. We let $W[J]$ denote the W-scheme $\underline{\mathrm{Spec}}_W(\mathcal{O}_W[J])$ with structure morphism $r_{W,J}\colon W[J] \to W$. If W is an X-scheme, we consider $W[J]$ as an X-scheme via $r_{W,J}$. The X-extension $W \hookrightarrow W[J]$ is thus *trivial* in the sense that it admits an X-retraction.

By [Hal17, Prop. 2.4], if the S-groupoid X is **Nil**-homogeneous at T, then the groupoid $\mathbf{Exal}_X(T,I)$ is a Picard category. Thus, we have additive functors

$$\begin{aligned} \mathrm{Der}_X(T,-)\colon \mathsf{QCoh}(T) \to \mathsf{Ab}, &\quad I \mapsto \mathrm{Aut}_{\mathbf{Exal}_X(T,I)}(T[I]); \quad \text{and} \\ \mathrm{Exal}_X(T,-)\colon \mathsf{QCoh}(T) \to \mathsf{Ab}, &\quad I \mapsto \mathrm{Exal}_X(T,I). \end{aligned}$$

We now record here the following easy consequences of [Hal17, 2.3–2.6 & 3.4].

Lemma 1.8. *Let S be a scheme, let X be an S-groupoid, and let T be an X-scheme.*

(1) *Let I be a quasi-coherent $\mathcal{O}_T$-module. Then $\mathrm{Exal}_X(T,I) = 0$ if and only if every X-extension $i\colon T \hookrightarrow T'$ of T by I admits an X-retraction.*

(2) *Let P be a class of a morphisms of S-schemes and let $p\colon V \to T$ be an affine morphism in P. If X is P-homogeneous at V, then for every $N \in \mathsf{QCoh}(V)$ there is a natural functor*
$$p_\#\colon \mathbf{Exal}_X(V,N) \to \mathbf{Exal}_X(T,p_*N).$$

(3) *If X is* **rNil**-*homogeneous at T, then the functor $M \mapsto \mathrm{Exal}_X(T,M)$ is half-exact.*

(4) *Suppose that X is* **Nil**-*homogeneous at T and limit preserving. If T is of finite presentation over S, then the functor $M \mapsto \mathrm{Exal}_X(T,M)$ preserves direct limits.*

(5) *Let $p\colon U \to T$ be an* affine *étale morphism and let N be a quasi-coherent $\mathcal{O}_U$-module. Then there is a natural functor $\psi\colon \mathbf{Exal}_X(T,p_*N) \to \mathbf{Exal}_X(U,N)$. If $(i\colon T \hookrightarrow T') \in \mathbf{Exal}_X(T,p_*N)$ with image $(j\colon U \hookrightarrow U') \in \mathbf{Exal}_X(U,N)$, then there is a cartesian diagram of X-schemes*
$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[d, "p"'] & U' \arrow[d, "p'"] \\ T \arrow[r, hook, "i"] & T', \end{tikzcd}$$
which is cocartesian as a diagram of S-schemes. If X is **Aff**-*homogeneous at U, then ψ is an equivalence.*

Proof. The claim (1) is [Hal17, Lem. 2.3].

For (2), if $j\colon V \hookrightarrow V'$ is an X-extension of V by N, then there is an induced P-nil pair $(V \xrightarrow{p} T, V \xrightarrow{j} V')$ over X at V. Since X is P-homogeneous at V, by Lemma 1.5, there functorially exists a cocartesian and geometric P-nil square over X at V as in (1.1) completing the P-nil pair. The resulting morphism $i\colon T \hookrightarrow T'$ is an X-extension of T by p_*N and we have thus defined the functor $p_\#$.

The claim (3) is [Hal17, Cor. 2.5]. The claim (4) is [Hal17, Prop. 3.4(2)]. The claim (5) is [Hal17, Cor. 2.6]. □

Finally, we give conditions that imply $\mathbf{Art}^{\mathbf{sep}}$- and $\mathbf{Art}^{\mathbf{fin}}$-homogeneity.

Lemma 1.9. *Let S be a scheme and let X be an S-groupoid that is $\mathbf{Art}^{\mathbf{triv}}$-homogeneous. Consider the following conditions on X.*

(1) *X is a stack in the fppf topology.*
(2) *X is a stack in the étale topology and $\mathbf{Art}^{\mathbf{insep}}$-homogeneous.*
(3) *X is a stack in the étale topology and S is a $\mathbb{Q}$-scheme.*
(4) *X is a stack in the étale topology.*

Then any of the conditions (1), (2) or (3) imply that X is $\mathbf{Art}^{\mathbf{fin}}$-homogeneous and condition (4) implies that X is $\mathbf{Art}^{\mathbf{sep}}$-homogeneous.

Proof. We begin by noting that trivially (3) implies (2). Next, let $(\operatorname{Spec} A \to \operatorname{Spec} B, \operatorname{Spec} A \hookrightarrow \operatorname{Spec} A')$ be an $\mathbf{Art}^{\mathbf{fin}}$-nil pair over S. Let $\operatorname{Spec} B' = \operatorname{Spec}(A' \times_A B)$ be the pushout of this diagram in the category of S-schemes. We have to prove that the functor

$$\varphi\colon X(\operatorname{Spec} B') \to X(\operatorname{Spec} A') \times_{X(\operatorname{Spec} A)} X(\operatorname{Spec} B)$$

is an equivalence. If X is a stack in either the fppf or étale topology, then the equivalence of φ is a local question for the respective topology on B' since fiber products of rings commute with flat base change.

Now there is a finite (resp. finite separable) field extension K/k_B such that the residue fields of $k_A \otimes_{k_B} K$ are trivial (resp. purely inseparable) extensions of K. There is then a local artinian ring $\widetilde{B}'$ and a finite flat (resp. finite étale) extension $B' \hookrightarrow \widetilde{B}'$ with $k_{\widetilde{B}'} = K$ [EGA, $\mathbf{0}_{\text{III}}$.10.3.2]. Let $\widetilde{A} = A \otimes_{B'} \widetilde{B}'$, $\widetilde{A}' = A' \otimes_{B'} \widetilde{B}'$ and $\widetilde{B} = B \otimes_{B'} \widetilde{B}'$. Then $\widetilde{A}$, $\widetilde{A}'$, $\widetilde{B}$ are artinian rings such that all residue fields equal K (resp. are purely inseparable extensions of K). However, $\widetilde{A}$ and $\widetilde{A}'$ need not be local. Now let $\widetilde{A} = \prod_{i=1}^n \widetilde{A}_i$ and $\widetilde{A}' = \prod_{i=1}^n \widetilde{A}'_i$ be decompositions such that $\widetilde{A}' \twoheadrightarrow \widetilde{A}_i$ factors through $\widetilde{A}'_i$. Then $\widetilde{B}' = (\widetilde{A}'_1 \times_{\widetilde{A}_1} \widetilde{B}) \times_{\widetilde{B}} (\widetilde{A}'_2 \times_{\widetilde{A}_2} \widetilde{B}) \times_{\widetilde{B}} \cdots \times_{\widetilde{B}} (\widetilde{A}'_n \times_{\widetilde{A}_n} \widetilde{B})$ is an iterated fiber product of local artinian rings.

If X is $\mathbf{Art}^{\mathbf{triv}}$-homogeneous (resp. $\mathbf{Art}^{\mathbf{insep}}$-homogeneous) and a stack for the fppf (resp. étale) topology, it follows that φ is an equivalence. If the $\mathbf{Art}^{\mathbf{fin}}$-nil pair that we started with was an $\mathbf{Art}^{\mathbf{sep}}$-nil pair and X is a stack for the étale topology, then it also follows that φ is an equivalence. This proves the result. □

2. Formal versality and formal smoothness

In this section we address a subtle point about the relationship between formal versality and formal smoothness. We begin by recalling and refining some results of [Hal17, §4].

Definition 2.1. Let S be a scheme, let X be an S-groupoid, and let T be an X-scheme. Consider the following lifting problem in the category of X-schemes: given a pair of morphisms of X-schemes $(V \xrightarrow{p} T, V \xrightarrow{j} V')$, where j is a locally nilpotent closed immersion, complete the following diagram so that it commutes:

$$\begin{tikzcd} V \arrow[r, "p"] \arrow[d, hook, "j"'] & T \\ V'. \arrow[ur, dashed] & \end{tikzcd} \tag{2.1}$$

The X-scheme T is:

formally smooth if the lifting problem can always be solved Zariski-locally on V';

formally smooth at $t \in |T|$ if the lifting problem can always be solved whenever the X-schemes V and V' are local artinian, with closed points v and v', respectively, such that $p(v) = t$, and the field extension $\kappa(t) \subseteq \kappa(v)$ is finite;

formally versal at $t \in |T|$ if the lifting problem can always be solved whenever the X-schemes V and V' are local artinian, with closed points v and v', respectively, such that $p(v) = t$, and the field extension $\kappa(t) \subseteq \kappa(v)$ is an isomorphism.

We certainly have the following implications:

$$\begin{aligned} \text{formally smooth} &\implies \text{formally smooth at all } t \in |T| \\ &\implies \text{formally versal at all } t \in |T|. \end{aligned}$$

Formal smoothness and formal versality at all $t \in |T|$ are not obviously equivalent. Even for morphisms of finite type between noetherian schemes, it is a non-trivial result that they are equivalent [EGA, **IV**.17.14.2] (also see [Stacks, Tag 02HX] and Corollary 2.5).

Formal smoothness at t and formal versality at t are also not obviously equivalent. Moreover without stronger assumptions, it is not obvious to the authors that formal smoothness or formal versality is smooth-local on the source. We will see, however, that these subtleties vanish whenever the S-groupoid is $\mathbf{Art}^{\mathbf{fin}}$-homogeneous. For formal versality and formal smoothness at a point, it is sufficient that liftings exist when $\kappa(v) \cong j^{-1}\ker(\mathcal{O}_{V'} \to \mathcal{O}_V)$.

The goal of this section is to give sufficient conditions for a family, formally versal at all *closed* points, to be formally *smooth*. In Artin's papers, Artin approximation is used to address this. With our formulation, excellence (or related) assumptions are irrelevant. For some further discussion on Artin's approach, see Remark 2.8.

There is a tight connection between formal smoothness (resp. formal versality) and X-extensions in the *affine* setting. Most of the next result was proved in [Hal17, Lem. 4.3], which utilized arguments similar to those of [Fle81, Satz 3.2].

Lemma 2.2. *Let S be a scheme, let X be an S-groupoid, and let T be an* affine *X-scheme. Let $t \in |T|$ be a point. Consider the following conditions.*

1. *The X-scheme T is formally smooth at t.*
2. *The X-scheme T is formally versal at t.*
3. *X is $\mathbf{Nil}$-homogeneous at T and $\mathrm{Exal}_X(T, \kappa(t)) = 0$.*

Then (1) $\Longrightarrow$ (2) and if X is $\mathbf{Art}^{\mathbf{fin}}$-semihomogeneous and t is of finite type, then (2) $\Longrightarrow$ (1). If X is $\mathbf{Cl}$-homogeneous, T is noetherian and t is a closed *point, then (2) $\Longrightarrow$ (3). If X is $\mathbf{rCl}$-homogeneous and t is a* closed *point, then (3) $\Longrightarrow$ (2).*

Thus, assuming that an S-groupoid X is $\mathbf{rCl}$-homogeneous, we can reformulate formal versality of an affine X-scheme T at a closed point $t \in |T|$ in terms of the triviality of the abelian group $\mathrm{Exal}_X(T,\kappa(t))$. Understanding the set of points $U \subseteq |T|$ where $\mathrm{Exal}_X(T,\kappa(u)) = 0$ for $u \in |U|$ will be accomplished in the next section.

Remark 2.3. If X is **Aff**-homogeneous and $\mathrm{Exal}_X(T,-) \equiv 0$, then T is formally smooth [Hal17, Lem. 4.3] but we will not use this. If Exal_X commutes with Zariski localization, that is, if for every open immersion of affine schemes $U \subseteq T$ the canonical map $\mathrm{Exal}_X(T,M) \otimes_{\Gamma(\mathcal{O}_T)} \Gamma(\mathcal{O}_U) \to \mathrm{Exal}_X(U, M|_U)$ is bijective, then the implications (2) $\implies$ (3) and (3) $\implies$ (2) also hold for non-closed points. This is essentially what Flenner proves in [Fle81, Satz 3.2] as his $\mathcal{E}x(T \to X, M)$ is the sheafification of the presheaf $U \mapsto \mathrm{Exal}_X(U, M|_U)$.

Proof of Lemma 2.2. The implication (1) $\implies$ (2) follows from the definition. The implications (2) $\implies$ (3) and (3) $\implies$ (2) are proved in [Hal17, Lem. 4.3]. The implication (2) $\implies$ (1) follows from a similar argument: assume that T is formally versal at t and fix a lifting problem as in diagram (2.1), where $j\colon V \to V'$ is a closed immersion of local artinian schemes with closed points v and v', respectively, such that $p(v) = t$ and $\kappa(v)/\kappa(t)$ is a finite extension. Let W be the image of $V \to \mathrm{Spec}(\mathcal{O}_{T,t})$. Then W is a local artinian scheme with residue field $\kappa(t)$. As X is $\mathbf{Art^{fin}}$-semihomogeneous, the $\mathbf{Art^{fin}}$-nil pair $(V \to W, V \xrightarrow{j} V')$ over X can be completed to a geometric $\mathbf{Art^{fin}}$-nil square over X:

$$\begin{tikzcd} V \arrow[r] \arrow[d, hook] & W \arrow[d, hook] \\ V' \arrow[r] & W', \end{tikzcd}$$

where $W \hookrightarrow W'$ is a closed immersion of local artinian schemes. Since the closed point of W has the same residue field as that of t, by formal versality, we obtain a lift of $W \to T$ to $W' \to T$ over X. The result follows. $\square$

Lemma 2.2 is already quite powerful. In the following Proposition, we give a simple proof of [EGA, $\mathbf{0}_{\mathrm{IV}}$.22.1.4] in the case of a finitely generated or separable extension of residue fields (also see [Stacks, Tag 02HT]).

Proposition 2.4. *Let $f\colon T \to X$ be a morphism of locally noetherian schemes and let $t \in |T|$ with image $x = f(t)$. Consider the following conditions.*

1. *The ring homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{T,t}$ is preadically formally smooth* [EGA, $\mathbf{0}_{\mathrm{IV}}$.19.3.1].
2. *f is formally smooth at t.*
3. *f is formally versal at t.*

Then (1) $\implies$ (2) $\iff$ (3)*. If $\kappa(x) \subseteq \kappa(t)$ is finitely generated or separable, then* (3) $\implies$ (1)*.*

Proof. We recall [EGA, $\mathbf{0}_{\mathrm{IV}}$.19.3.1] for our situation. The preadic topology on a noetherian local ring has as a basis of open neighborhoods the powers of the maximal ideal. A local ring homomorphism $(A,\mathfrak{m}) \to (B,\mathfrak{n})$, where A and B are noetherian and preadically topologized, is smooth for the preadic topologies if for every discrete and continuous A-algebra C and nilpotent ideal $I \subseteq C$, all continuous A-algebra homomorphisms $B \to C/I$ factor continuously as $B \to C \to C/I$. Since A and B have their preadic topologies, this means that we can choose $n \gg 0$ such that $A \to C$ factors through $A \to A/\mathfrak{m}^n$ and $B \to C$ factors through $B \to B/\mathfrak{n}^n$. Note that both $A/\mathfrak{m}^n$ and $B/\mathfrak{n}^n$ are local artinian. Hence, (1) $\implies$ (2) $\implies$ (3).

For (3) $\implies$ (2): we may assume that $X = \operatorname{Spec} \mathcal{O}_{X,x}$ and $T = \operatorname{Spec} \mathcal{O}_{T,t}$. In particular, $t \in |T|$ is a finite type point and X is $\mathbf{Art}^{\mathbf{fin}}$-homogeneous. By Lemma 2.2, the claim follows.

To prove (3) $\implies$ (1) we will take $(A, \mathfrak{m}) = (\mathcal{O}_{X,x}, \mathfrak{m}_x)$ and $(B, \mathfrak{n}) = (\mathcal{O}_{T,t}, \mathfrak{m}_t)$ and consider the lifting problem described above. Take $D = \operatorname{im}(B \to C/I)$, which is a local artinian ring with residue field $K = B/\mathfrak{n}$. Next take $E = D \times_{C/I} C$. Then $E \to D$ is surjective and $E \subseteq C$. It remains to show that there is a lifting $B \to E$. If E was artinian, then we would be done by formal versality. But E need not be noetherian and we will instead construct an A-subalgebra $E_0 \subseteq E$ which is artinian and such that $E_0 \to E \to D$ is surjective with nilpotent kernel. Then $B \to D$ factors via A-homomorphisms $B \to E_0 \to E \to D$ by formal versality.

Let $k = A/\mathfrak{m}$ and first assume that $k \to K$ is a finitely generated extension. Since $E \to D \to K$ is surjective we may choose $t_1, \ldots, t_r \in E$ such that $k(t_1, \ldots, t_r) = K$. Further choose $u_1, \ldots, u_s \in E$ such that their images in D generate the maximal ideal. Let E_0 be the total quotient ring of the A-subalgebra of E generated by $t_1, \ldots, t_r, u_1, \ldots, u_s$. Then $E_0 \subseteq E$ is local artinian, $E_0 \to D$ is surjective, and by formal versality we have the required lift.

If instead $k \to K$ is separable, then there exists a Cohen A-algebra A' such that $A' \otimes_A k = K$. Recall that A' is a complete local noetherian ring and that $A \to A'$ is preadically formally smooth [EGA, $\mathbf{0}_{\mathrm{IV}}$.19.8.2]. Since $E \to D \to K$ is surjective with nilpotent kernel, we obtain a factorization $A \to A' \to E$ such that $A' \to E$ induces an isomorphism on residue fields. We can now take E_0 as the A'-subalgebra of E generated by $u_1, \ldots, u_s$. $\square$

We now obtain the following well-known corollary (cf. [EGA, **IV**.17.14.2]).

Corollary 2.5. *Let $f \colon T \to X$ be a locally of finite type morphism of locally noetherian schemes. Let $t \in |T|$. The following are equivalent.*

1. *f is smooth at t* [EGA, **IV**.17.3.7].
2. *f is formally smooth at $t \in |T|$.*
3. *f is formally versal at $t \in |T|$.*

Proof. Since f is locally of finite type, $\kappa(f(t)) \subseteq \kappa(t)$ is a finitely generated extension. By Proposition 2.4, it follows that conditions (2) and (3) are equivalent to $\mathcal{O}_{X,f(t)} \to \mathcal{O}_{T,t}$ being preadically formally smooth. By [EGA, **IV**.17.5.3], we have the claim. We can also argue as follows: the natural map $\operatorname{Exal}_X(T, \kappa(t)) \to \operatorname{Exal}_X(\operatorname{Spec} \mathcal{O}_{T,t}, \kappa(t))$ is an isomorphism. Indeed, the cotangent complex of the morphism $\operatorname{Spec} \mathcal{O}_{T,t} \to T$ vanishes. By Lemma 2.2, formal versality implies that $\operatorname{Exal}_X(\operatorname{Spec} \mathcal{O}_{T,t}, \kappa(t)) \cong 0$. By [Hal17, Lem. 5.4], the functor on quasi-coherent $\mathcal{O}_T$-modules $\operatorname{Exal}_X(T, -)$ is coherent and limit preserving. By [Hal14, Cor. 7.7], there is thus an affine open neighborhood $j \colon U \subseteq T$ of t such that the functor $\operatorname{Exal}_X(T, j_*(-))$ vanishes. But $\operatorname{Exal}_X(T, j_*(-)) \simeq \operatorname{Exal}_X(U, -)$, so $U \to X$ is formally smooth [Hal17, Lem. 4.3(1)]. $\square$

Corollary 2.6. *Let S be a locally noetherian scheme and let X be a limit preserving S-groupoid. Let T be an X-scheme that is locally of finite type over S and let $t \in |T|$ be a point such that:*

1. *T is formally smooth at $t \in |T|$ as an X-scheme and*

(2) *the morphism $T \to X$ is representable by algebraic spaces.*

If W is an X-scheme, then the morphism $T \times_X W \to W$ is smooth in a neighborhood of every point over t. In particular, if $T \to X$ is formally smooth at every point of finite type*, then $T \to X$ is formally smooth.*

Proof. By a standard limit argument, we can assume that $W \to S$ is of finite type. It is then enough to verify that $T \times_X W \to W$ is smooth at closed points in the fiber of t. Let $u\colon U \to T \times_X W$ be an étale and surjective morphism, where U is a scheme. Then $U \to W$ is formally smooth at closed points in the fiber of t. By Corollary 2.5, the composition $U \to W$ is smooth at every point over t, and we deduce the claim. The last statement follows from the fact that every closed point of $T \times_X W$ maps to a point of finite type of T. □

Combining Lemma 2.2 and Corollary 2.6 we obtain the following key result.

Corollary 2.7. *Let S be a locally noetherian scheme and let X be a limit preserving and $\mathbf{Art}^{\mathbf{fin}}$-semihomogeneous S-groupoid. If T is an X-scheme such that*

(1) *$T \to S$ is locally of finite type,*
(2) *$T \to X$ is formally versal at all points of finite type, and*
(3) *$T \to X$ is representable by algebraic spaces,*

then $T \to X$ is formally smooth.

Remark 2.8. To establish algebraicity of a functor or groupoid in the spirit of Artin's criteria, one must provide conditions for an algebraic family that is formally versal at all points of finite type to be formally smooth. In the present paper, this is Corollary 2.7, where we use $\mathbf{Art}^{\mathbf{fin}}$-semihomogeneity. This result was known to several experts. In Artin's paper for functors, this is [Art69b, Lem. 5.4], where the functor is assumed to be an fppf sheaf and $\mathbf{Art}^{\mathbf{triv}}$-homogeneous. By Lemma 1.9, the fppf stack condition together with $\mathbf{Art}^{\mathbf{triv}}$-homogeneity imply $\mathbf{Art}^{\mathbf{fin}}$-homogeneity, so the results of our paper recover Artin's. As discussed in the Introduction, Artin's arguments for functors do not extend to groupoids.

In Artin's paper for groupoids, the relationship between formal versality and smoothness is established in [Art74, Prop. 4.2]. The relevant standing assumption is **rCl**-semihomogeneity. Assuming **rCl**-homogeneity makes no difference to our discussion below. We feel that it is worthwhile to digress into some of the technicalities that arise here. We wish to assure the reader that, as mentioned in the Introduction, if S is of finite type over $\operatorname{Spec} \mathbb{Z}$ or a perfect field, then the proof of the main result of [Art74] is essentially correct, with only minor modifications to the arguments necessary.

Our interpretation of Artin's definition of formal smoothness [Art74, p. 173] is that it coincides with ours given in Definition 2.1. In particular, in the notation of [Art74, p. 173], to verify formal smoothness the residue fields of A are unconstrained. But the proof of [Art74, Prop. 4.2] relies on [Art74, Thm. 3.3], which requires that the residue field of A is equal to the residue field of R (here both A and R are henselian local rings). If the residue field extension is separable, then it is possible to conclude using [Art74, Prop. 4.3], which uses étale localization of obstruction theories (also see Proposition 2.9). We do not know how to complete the argument if the residue field extension is inseparable. The essential problem is the verification that formal versality is smooth-local.

It was suggested by a referee that Artin's definition of formal smoothness can be interpreted as follows. In the notation of [Art74, p. 173], the morphism $\operatorname{Spec} A \to \operatorname{Spec} R$ should induce an isomorphism of residue fields at every point of finite type over S. With this definition of formal smoothness, Artin's proof of [Art74, Prop. 4.2] is correct. This definition of formal smoothness seems too limited to prove his main result [Art74, Cor. 5.2] without further assumptions, however. Indeed, it is essential in [Art74, Cor. 5.2] that formal smoothness is stable under base change. Artin omits the proof of this stability under base change and we were unable to prove it ourselves. Again, it is the presence of inseparable field extensions that complicates matters. Note that our definition of formal smoothness is obviously stable under base change.

2.1. **Étale localization.** We also obtain the following result showing that, under mild hypotheses, formal versality is stable under étale-localization. This improves [Art74, Prop. 4.3], which requires the existence of an obstruction theory that is compatible with étale localization.

Proposition 2.9. *Let S be a scheme and let X be an $\mathbf{Art}^{\mathbf{sep}}$-semihomogeneous S-groupoid (cf. Lemma 1.9). Let T be an X-scheme. If $(U,u) \to (T,t)$ is a pointed étale morphism of S-schemes, then formal versality at $t \in |T|$ implies formal versality at $u \in |U|$.*

Proof. To see that formal versality at $t \in |T|$ implies formal versality at $u \in |U|$, it is enough to show that the lifting property holds for T and a square-zero extension of local artinian schemes $V \hookrightarrow V'$ such that $\kappa(v) = \kappa(u)$. This follows from an identical argument as in the proof of Lemma 2.2(2) $\Longrightarrow$ (1). ☐

Using Lemma 2.2, one can show that Proposition 2.9 admits a partial converse. Indeed, if $u \in |U|$ and $t \in |T|$ are closed, X is $\mathbf{rCl}$-homogeneous, U and T are affine and noetherian, and $T \to X$ is representable by algebraic spaces, then formal versality at $u \in |U|$ implies formal versality at $t \in |T|$. This will not be used, however.

Remark 2.10. The conditions on obstruction theories in the criteria for algebraicity are used to prove that formal versality is an open condition. Proposition 2.9 proves that it is enough to find suitable obstruction theories étale-locally. This idea is present in [Art74, 4.9–4.11]. We do not understand the given arguments, however, as they rely on [Art74, Prop. 4.3], which requires the existence of a global obstruction theory. But these are isolated remarks, having no bearing on the main results of the article.

2.2. **Zariski localization.** Next, we give a condition that ensures that if an X-scheme T is formally versal at all *closed* points, then it is formally versal at all points of *finite type*.

Condition 2.11. Let X be $\mathbf{Nil}$-homogeneous and let T be an affine X-scheme. The extensions of X are *Zariski local at T* if for every open immersion $p\colon U \to T$ of affine X-schemes and every point $u \in |U|$ of finite type, the natural map:

$$\operatorname{Exal}_X(T, \kappa(u)) \to \operatorname{Exal}_X(U, \kappa(u))$$

is surjective. The extensions of X are *Zariski local* if they are Zariski local at every affine X-scheme that is locally of finite type over S.

Note that Lemma 1.8(5) implies that if an S-groupoid X is **Aff**-homogeneous, then its extensions are Zariski local. As the following lemma shows, it is also satisfied whenever S is Jacobson.

Lemma 2.12. *Let X be a* **Nil**-*homogeneous Zariski S-stack and let $p\colon U \to T$ be an open immersion of affine X-schemes. If $u \in |U|$ is a point that is closed in T, then the natural map:*

$$\mathrm{Exal}_X(T, \kappa(u)) \to \mathrm{Exal}_X(U, \kappa(u))$$

is an isomorphism. In particular, if S is Jacobson, then extensions of X are Zariski local (Condition 2.11).

Proof. We construct an inverse by taking an X-extension $U \hookrightarrow U'$ of U by $\kappa(u)$ to the gluing of U' and $T \setminus \{u\}$ along $U' \setminus \{u\} \cong U \setminus \{u\}$. If S is Jacobson and $T \to S$ is locally of finite type, then T is Jacobson and every point of finite type $u \in |U|$ is closed in T so Condition 2.11 holds. □

We now extend the implication (3) $\implies$ (2) of Lemma 2.2 to points of finite type.

Proposition 2.13. *Fix a scheme S, an* **rCl**-*homogeneous S-groupoid X and an affine X-scheme T, locally of finite type over S. Assume that extensions of X are Zariski local at T (Condition 2.11). If $t \in |T|$ is a point of* finite type *and* $\mathrm{Exal}_X(T, \kappa(t)) = 0$*, then the X-scheme T is formally versal at t.*

Proof. Finite type points are locally closed so there exists an open affine neighborhood $U \subseteq T$ of t such that $t \in |U|$ is closed. By Condition 2.11, $0 = \mathrm{Exal}_X(T, \kappa(t)) \twoheadrightarrow \mathrm{Exal}_X(U, \kappa(t))$, so the X-scheme U is formally versal at t by Lemma 2.2. It then follows, from the definition, that the X-scheme T also is formally versal at t. □

2.3. DVR-homogeneity.

In this subsection, we will increase our homogeneity assumption instead of assuming that Exal commutes with localization.

Recall that a *geometric discrete valuation ring* is a discrete valuation ring D such that $\mathrm{Spec}\, D \to S$ is essentially of finite type and the residue field is of finite type over S [Art69b, p. 38].

Notation 2.14. Let **DVR** $\subseteq$ **Aff** be the class of morphisms $(\mathrm{Spec}\, K \to \mathrm{Spec}\, D)$ such that D is a geometric discrete valuation ring with fraction field K.

Artin's condition [4a] of [Art69b, Thm. 3.7] implies **DVR**-semihomogeneity and Artin's conditions [5′](b) and [4′](a,b) of [Art69b, Thm. 5.3] imply **DVR**-homogeneity. We conclude this section by showing that **DVR**-homogeneity implies that formal smoothness is stable under generizations. This is accomplished by the following lemma, which is a generalization of [Art69b, Lem. 3.10] from functors to categories fibered in groupoids. To guarantee sufficiently many geometric discrete valuation rings, we assume that we are over an excellent base.

Lemma 2.15. *Let S be an excellent scheme and let X be a limit preserving* **DVR**-*homogeneous S-groupoid. If T is an X-scheme such that*

1. *$T \to S$ is locally of finite type,*
2. *$T \to X$ is representable by algebraic spaces, and*
3. *$T \to X$ is formally smooth at a point $t \in |T|$ of finite type,*

then $T \to X$ is formally smooth at every generization $t' \in |T|$ of t.

Proof. Consider a diagram of X-schemes

$$\begin{tikzcd} Z_0 \arrow[r, "g"] \arrow[d, hook] & T \arrow[d] \\ Z \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

where $Z_0 \hookrightarrow Z$ is a closed immersion of local artinian schemes and the image $t' = g(z_0)$ of the closed point $z_0 \in |Z_0|$ is a generization of $t \in T$ and $\kappa(z_0)/\kappa(t')$ is finite. We have to prove that every such diagram admits a lifting as indicated by the dashed arrow.

As X is limit preserving, we can factor $Z \to X$ as $Z \to W \to X$ where W is an S-scheme of finite type. Let $h\colon T \times_X W \to T$ denote the first projection. The pullback $T \times_X W \to W$ is smooth at every point of the fiber $h^{-1}(t)$ by Corollary 2.6. Let T_t denote the local scheme $\mathrm{Spec}(\mathcal{O}_{T,t})$. It is enough to prove that $T \times_X W \to W$ is smooth at every point of $h^{-1}(T_t)$.

Let $y \in |T \times_X W|$ be a point of $h^{-1}(T_t)$. It is enough to prove that $Y = \overline{\{y\}}$ contains a point at which $T \times_X W \to W$ is smooth. By Chevalley's theorem, $h(Y)$ contains a constructible subset. Thus, there is a point $t_1 \in h(Y) \cap T_t$ such that the closure $T_1 = \overline{\{t_1\}}$ in the local scheme T_t is of dimension 1. By Corollary 2.6, it is enough to show that $T \to X$ is formally smooth at t_1. Thus, consider a diagram

$$\begin{tikzcd} \mathrm{Spec}\, K' \arrow[r, "g"] \arrow[d, hook] & T \arrow[d] \\ \mathrm{Spec}\, K'' \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

of X-schemes where $K'' \twoheadrightarrow K'$ is a surjection of local artinian rings such that $g(\eta) = t_1$ and $\kappa(\eta)/\kappa(t_1)$ is finite. Let $D \subseteq K = \kappa(\eta)$ be a geometric DVR dominating $\mathcal{O}_{T_1,t}$ (which exists since $\mathcal{O}_{T_1,t}$ is excellent). We may then, using **DVR**-homogeneity, extend the situation to a diagram

$$\begin{tikzcd} \mathrm{Spec}\, K' \arrow[r] \arrow[d, hook] & \mathrm{Spec}\, D' \arrow[r] \arrow[d, hook] & T \arrow[d] \\ \mathrm{Spec}\, K'' \arrow[r] & \mathrm{Spec}\, D'' \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

where $D' = D \times_K K'$ and $D'' = D \times_K K''$ so $D' \twoheadrightarrow D$ and $D'' \twoheadrightarrow D$ have nilpotent kernels. Now, by Corollary 2.6, the pullback $T \times_X \mathrm{Spec}\, D'' \to \mathrm{Spec}\, D''$ is smooth at the image of $\mathrm{Spec}\, D'$ so there is a lifting as indicated by the dashed arrow. Thus $T \to X$ is formally smooth at t_1 and hence also at t'. $\square$

In Lemma 10.4 we will show that under mild hypotheses, **DVR**-homogeneity actually implies **Aff**-homogeneity and thus also Condition 2.11.

Remark 2.16. If we replace geometric DVRs with all DVRs in **DVR**-homogeneity, then it is enough that S is noetherian instead of excellent and t need not be of finite type.

3. Vanishing loci for additive functors

Let T be a scheme. In this section we will be interested in additive functors $F\colon \mathsf{QCoh}(T) \to \mathsf{Ab}$. It is readily seen that the collection of all such functors forms an abelian category, with all limits and colimits computed "pointwise". For example, given additive functors F, $G\colon \mathsf{QCoh}(T) \to \mathsf{Ab}$ as well as a natural transformation $\varphi\colon F \to G$, then $\ker\varphi\colon \mathsf{QCoh}(T) \to \mathsf{Ab}$ is the functor

$$(\ker\varphi)(M) = \ker(F(M) \xrightarrow{\varphi(M)} G(M)).$$

Next, we set $A = \Gamma(\mathcal{O}_T)$. Note that the natural action of A on the abelian category $\mathsf{QCoh}(T)$ induces for every $M \in \mathsf{QCoh}(T)$ an action of A on the abelian group $F(M)$. Thus we see that the functor F is canonically valued in the category $\mathsf{Mod}(A)$. It will be convenient to introduce the following notation: for a quasi-compact and quasi-separated morphism of schemes $g\colon W \to T$ and a functor $F\colon \mathsf{QCoh}(T) \to \mathsf{Ab}$, define $F_W\colon \mathsf{QCoh}(W) \to \mathsf{Ab}$ to be the functor $F_W(N) = F(g_*N)$. If F is additive (resp. preserves direct limits), then the same is true of F_W. The *vanishing locus of* F is the following subset [Hal14, §7.2]:

$$\begin{aligned}\mathbb{V}(F) &= \{t \in |T| \,:\, F(M) = 0 \quad \forall M \in \mathsf{QCoh}(T),\ \mathrm{supp}(M) \subseteq \mathrm{Spec}(\mathcal{O}_{T,t})\}\\ &= \{t \in |T| \,:\, F_{\mathrm{Spec}(\mathcal{O}_{T,t})} \equiv 0\} \quad \text{(if } T \text{ is quasi-separated).}\end{aligned}$$

The main result of this section, Theorem 3.3, which gives a criterion for the set $\mathbb{V}(F)$ to be Zariski open, is essentially due to H. Flenner. In [Fle81, Lem. 4.1], for an S-groupoid X and an affine X-scheme V, locally of finite type over S, a specific result about the vanishing locus of the functor $M \mapsto \mathrm{Exal}_X(V, M)$ is proved. In [Fle81], a standing assumption is that the S-groupoid X is *semi*homogeneous, thus the functor $M \mapsto \mathrm{Exal}_X(T, M)$ is only set-valued, which complicates matters. Since we are assuming **Nil**-homogeneity of X, the functor $M \mapsto \mathrm{Exal}_X(T, M)$ takes values in abelian groups. As we will see, this simplifies matters considerably.

We now make the following trivial observation.

Lemma 3.1. *Let T be a scheme and let $F\colon \mathsf{QCoh}(T) \to \mathsf{Ab}$ be an additive functor. Then the subset $\mathbb{V}(F) \subseteq |T|$ is stable under generization.*

By Lemma 3.1, we thus see that the subset $\mathbb{V}(F) \subseteq |T|$ will be Zariski open if we can determine sufficient conditions on the functor F and the scheme T such that the subset $\mathbb{V}(F)$ is (ind)constructible. We make the following definitions.

Definition 3.2. Let $T = \operatorname{Spec} A$ be an affine scheme and let $F\colon \mathsf{QCoh}(T) \to \mathsf{Ab}$ be an additive functor.

- The functor F is *bounded* if the scheme T is noetherian and $F(M)$ is finitely generated for every finitely generated A-module M.
- The functor F is *weakly bounded* if the scheme T is noetherian and for every integral closed subscheme $T_0 \hookrightarrow T$, the $\Gamma(\mathcal{O}_{T_0})$-module $F(\mathcal{O}_{T_0})$ is finitely generated.
- The functor F is *GI* (resp. *GS*, resp. *GB*) if there exists a dense open subset $U \subseteq |T|$ such that for all points $u \in |U|$ of finite type, the natural map
$$F(\mathcal{O}_T) \otimes_A \kappa(u) \to F(\kappa(u))$$
is injective (resp. surjective, resp. bijective).
- The functor F is *CI* (resp. *CS*, resp. *CB*) if for every integral closed subscheme $T_0 \hookrightarrow T$, the functor F_{T_0} is GI (resp. GS, resp. GB).

In the above definition, GI (resp. GS, resp. GB) is an acronym for *generically* injective (resp. surjective. resp. bijective). Similarly, CI (resp. CS, resp. CB) is an acronym for *constructibly* injective (resp. surjective, resp. bijective).

We can now state the main result of this section.

Theorem 3.3 (Flenner). *Let T be an affine noetherian scheme and let $F\colon \mathsf{QCoh}(T) \to \mathsf{Ab}$ be a half-exact, additive, and bounded functor that commutes with direct limits. If the functor F is CS, then the subset $\mathbb{V}(F) \subseteq |T|$ is Zariski open.*

Functors of the above type occur frequently in algebraic geometry.

Example 3.4. Let T be an affine noetherian scheme and let $Q \in \mathsf{D}^-_{\mathsf{Coh}}(T)$. Then, for all $i \in \mathbb{Z}$, the functors on quasi-coherent $\mathcal{O}_T$-modules given by $M \mapsto \operatorname{Ext}^i_{\mathcal{O}_T}(Q, M)$ and $M \mapsto \operatorname{Tor}_i^{\mathcal{O}_T}(Q, M)$ are additive, bounded, half-exact, commute with direct limits, and CB.

Example 3.5. Let T be an affine noetherian scheme and let $p\colon X \to T$ be a morphism that is projective and flat. Then the functor $M \mapsto \Gamma(X, p^*M)$ is CB. Indeed, one interpretation of the Cohomology and Base Change Theorem asserts that the functor $M \mapsto \Gamma(X, p^*M)$ is of the form given in Example 3.4.

Example 3.6. Let T be an affine noetherian scheme. An additive functor $F\colon \mathsf{QCoh}(T) \to \mathsf{Ab}$, commuting with direct limits, is *coherent* [Aus66] if there exists a homomorphism $M \to N$ of coherent $\mathcal{O}_T$-modules such that

$$F(-) = \operatorname{coker}\Big(\operatorname{Hom}_{\mathcal{O}_T}(N, -) \longrightarrow \operatorname{Hom}_{\mathcal{O}_T}(M, -)\Big).$$

It is easily seen that a coherent functor is CB and bounded. Indeed, boundedness is obvious and if $i\colon T_0 \hookrightarrow T$ is an integral closed subscheme, then $F|_{T_0} = \operatorname{coker}(\operatorname{Hom}_{\mathcal{O}_{T_0}}(i^*N, -) \to \operatorname{Hom}_{\mathcal{O}_{T_0}}(i^*M, -))$ and after passing to a dense open subscheme, we may assume that i^*N and i^*M are flat. Then $F|_{T_0}(-) = \operatorname{coker}((i^*N)^\vee \to (i^*M)^\vee) \otimes_{\mathcal{O}_{T_0}} (-)$ commutes with all tensor products. It is well-known, and easily seen, that the functors of the previous two examples are coherent.

Conversely, let $F\colon \mathsf{QCoh}(T) \to \mathsf{Ab}$ be a half-exact bounded additive functor that commutes with direct limits and is CS. Then for every integral closed subscheme $T_0 \hookrightarrow T$, there is an open dense subscheme $U_0 \subseteq T_0$ such that $F|_{U_0}$ is coherent. In particular, for half-exact bounded additive functors that commute with direct limits, CS implies CB.

The main ingredient in the proof of Theorem 3.3 is a remarkable Nakayama Lemma for half-exact functors, due to A. Ogus and G. Bergman [OB72, Thm. 2.1]. We state the following amplification, which follows from the mild strengthening given in [Hal14, Cor. 7.5] and Lemma 3.1.

Theorem 3.7 (Ogus–Bergman). *Let T be an affine noetherian scheme and let $F\colon \mathsf{QCoh}(T) \to \mathsf{Ab}$ be a half-exact, additive, and bounded functor that commutes with direct limits. Then*

$$\mathbb{V}(F) = \{t \in |T| \,:\, F(\kappa(t)) = 0\}.$$

In particular, if $F(\kappa(t)) = 0$ for all closed points $t \in |T|$, then $F \equiv 0$.

Remark 3.8. Let F be as in Theorem 3.7 and let $I \subseteq A$ be an ideal. Then Flenner proves that the natural map $F(M) \otimes_A \hat{A}_{/I} \to \varprojlim_n F(M/I^nM)$ is injective for every finitely generated A-module M. In fact, this is the special case $X = Y =$

Spec A of [Fle81, Kor. 6.3]. The Ogus–Bergman Nakayama lemma is an immediate consequence of the injectivity of this map.

Before we address vanishing loci of functors, the following simple application of Lazard's Theorem [Laz64], which appeared in [Hal14, Prop. 7.2], will be a convenient tool to have at our disposal.

Proposition 3.9. *Let $T = \operatorname{Spec} A$ be an* affine *scheme and let $F\colon \mathsf{QCoh}(T) \to \mathsf{Ab}$ be an additive functor that commutes with direct limits. Let M and L be A-modules. If L is* flat*, then the natural map:*

$$F(M) \otimes_A L \to F(M \otimes_A L)$$

is an isomorphism. In particular, for every A-algebra B and every flat B-module L, the natural map:

$$F(B) \otimes_B L \to F(L)$$

is an isomorphism.

We may now prove Flenner's theorem.

Proof of Theorem 3.3. The subset $\mathbb{V}(F) \subseteq |T|$ is open if and only if it is closed under generization and its intersection with any irreducible closed subset $T_0 \subseteq |T|$ contains a non-empty open subset of T_0 or is empty [EGA, **IV**.1.10.1]. By Lemma 3.1, we have witnessed the stability under generization. Thus it remains to address the latter claim.

Let $T_0 \hookrightarrow T$ be an integral closed subscheme. If $|T_0| \cap \mathbb{V}(F) \neq \emptyset$, then the generic point $\eta \in |T_0|$ belongs to $\mathbb{V}(F)$ (Lemma 3.1), thus $F(\kappa(\eta)) = 0$. Since by assumption the functor F is CS, there exists a dense open subset $U_0 \subseteq |T_0|$ such that the map $F_{T_0}(\mathcal{O}_{T_0}) \otimes_{\Gamma(\mathcal{O}_{T_0})} \kappa(u) \to F(\kappa(u))$ is surjective for all $u \in U_0$ of finite type.

As $\kappa(\eta)$ is a quasi-coherent and flat $\mathcal{O}_{T_0}$-module, the natural map $F_{T_0}(\mathcal{O}_{T_0}) \otimes_{\Gamma(\mathcal{O}_{T_0})} \kappa(\eta) \to F(\kappa(\eta))$ is an isomorphism by Proposition 3.9. But $\eta \in \mathbb{V}(F)$, thus the finitely generated $\Gamma(\mathcal{O}_{T_0})$-module $F_{T_0}(\mathcal{O}_{T_0})$ is torsion. Hence there is a dense open subset $U_0 \subseteq |T_0|$ with the property that if $u \in U_0$ is of finite type, then $F(\kappa(u)) = 0$. Using Theorem 3.7 we infer that $U_0 \subseteq \mathbb{V}(F) \cap |T_0|$. ☐

We record for future reference a useful lemma.

Lemma 3.10. *Let $T = \operatorname{Spec} A$ be an affine noetherian scheme and let $F\colon \mathsf{QCoh}(T) \to \mathsf{Ab}$ be an additive functor.*

1. *If the functor F is* half-exact*, then F is bounded if and only if F is weakly bounded.*
2. *If the functor F is (weakly) bounded, then every additive sub-quotient functor of F is (weakly) bounded.*
3. *If F is GS (resp. CS), then so is every additive quotient functor of F.*
4. *If F is weakly bounded and CI, then so is every additive subfunctor of F.*
5. *Consider an exact sequence of additive functors $\mathsf{QCoh}(T) \to \mathsf{Ab}$:*

$$H_1 \longrightarrow H_2 \longrightarrow H_3 \longrightarrow H_4.$$

   (a) *If H_1 and H_3 are CS and H_4 is CI and weakly bounded, then H_2 is CS.*
   (b) *If H_1 is CS, H_2 and H_4 are CI, and H_4 is weakly bounded, then H_3 is CI.*

If T is reduced, then (4), (5a)*, and* (5b) *hold with GI and GS instead of CI and CS.*

Proof. For claim (1), note that every coherent $\mathcal{O}_T$-module M admits a finite filtration whose successive quotients are of the form $i_*\mathcal{O}_{T_0}$, where $i\colon T_0 \hookrightarrow T$ is an integral closed subscheme. Induction on the length of the filtration, combined with the half-exactness of the functor F, proves the claim. Claims (2) and (3) are trivial. For (4), it is sufficient to prove the claim about GI and we can assume that T is a disjoint union of integral schemes. Fix an additive subfunctor $K \subseteq F$, then there is an exact sequence of additive functors: $0 \to K \to F \to H \to 0$. By (2) we see that H is weakly bounded and so $H(\mathcal{O}_T)$ is a finitely generated A-module. As A is reduced, generic flatness implies that there is a dense open subset $U \subseteq |T|$ such that $H(\mathcal{O}_T)_u$ is a flat A-module $\forall u \in U$. Thus, for all $u \in U$ the sequence:

$$0 \longrightarrow K(\mathcal{O}_T) \otimes_A \kappa(u) \longrightarrow F(\mathcal{O}_T) \otimes_A \kappa(u) \longrightarrow H(\mathcal{O}_T) \otimes_A \kappa(u) \longrightarrow 0$$

is exact. Since F is GI, we may further assume that the map $F(\mathcal{O}_T) \otimes_A \kappa(u) \to F(\kappa(u))$ is injective for all points $u \in U$ of finite type after shrinking U. We then conclude that K is GI from the commutative diagram:

$$\begin{tikzcd} K(\mathcal{O}_T) \otimes_A \kappa(u) \arrow[r, hook] \arrow[d] & F(\mathcal{O}_T) \otimes_A \kappa(u) \arrow[d, hook] \\ K(\kappa(u)) \arrow[r, hook] & F(\kappa(u)). \end{tikzcd}$$

Claims (5a) and (5b) follow from a similar argument and the 4-Lemmas. □

We conclude this section with a criterion for a functor to be GI (and consequently a criterion for a functor to be CI). This will be of use when we express Artin's criteria for algebraicity without obstruction theories in Section 8.

Proposition 3.11. *Let $T = \operatorname{Spec} A$ be an affine and integral (i.e., reduced and irreducible) noetherian scheme with function field K. Let $F\colon \mathsf{QCoh}(T) \to \mathsf{Ab}$ be an additive functor that commutes with direct limits. If $F(\mathcal{O}_T)$ is a finitely generated A-module, then F is GI if and only if the following condition is satisfied:*

- (†) *for every $f \in A$, every free A_f-module M of finite rank, and $\omega \in F(M)$ such that for all non-zero A-module maps $\epsilon\colon M \to K$ we have $\epsilon_*\omega \neq 0$ in $F(K)$, there exists a dense open subset $V_\omega \subseteq D(f) \subseteq |T|$ such that for every non-zero A-module map $\gamma\colon M \to \kappa(v)$, where $v \in V_\omega$ is of finite type, we have $\gamma_*\omega \neq 0$ in $F(\kappa(v))$.*

Proof. Let M be a free A_f-module of finite rank and let $M^\vee = \operatorname{Hom}_{A_f}(M, A_f)$. Then the canonical homomorphism $F(A)_f \otimes_{A_f} M \to F(M)$ is an isomorphism (Proposition 3.9) so there is a one-to-one correspondence between elements $\omega \in F(M)$ and homomorphisms $\overline{\omega}\colon M^\vee \to F(A)_f$. Moreover, $\overline{\omega}$ is injective if and only if $\overline{\omega} \otimes_A K\colon M^\vee \otimes_A K \to F(A) \otimes_A K = F(K)$ is injective and this happens exactly when $\epsilon_*\omega \neq 0$ in $F(K)$ for every non-zero map $\epsilon\colon M \to K$.

Let $t \in |T|$ and let $\delta_t\colon F(A) \otimes_A \kappa(t) \to F(\kappa(t))$ denote the natural map. Then condition (†) can be reformulated as: for every free A_f-module M of finite rank and every injective homomorphism $\overline{\omega}\colon M^\vee \to F(A)_f$, there exists a dense open subset $V_\omega \subseteq D(f)$ such that $\delta_t \circ \big(\overline{\omega} \otimes_A \kappa(t)\big)$ is injective for all points $t \in V_\omega$ of finite type.

To show that (†) implies that F is GI, choose $f \in A \setminus 0$ such that $F(A)_f$ is free, let $M = F(A)_f^\vee$ and let $\omega \in F(M)$ correspond to the inverse of the canonical isomorphism $F(A)_f \to M^\vee$. If (†) holds, then there exists an open subset V_ω such that δ_t is injective for all $t \in V_\omega$, i.e., F is GI.

Conversely, if F is GI, then there is an open subset V such that δ_t is injective for all $t \in V$ of finite type. Given a finite free A_f-module M and $\omega \in F(M)$, we let $V_\omega = V \cap W$ where $W \subseteq D(f)$ is an open dense subset over which the cokernel of $\overline{\omega}$ is flat. If $\overline{\omega}$ is injective, it then follows that $\delta_t \circ \big(\overline{\omega} \otimes_A \kappa(t)\big)$ is injective for all $t \in V_\omega$ of finite type, that is, condition (†) holds. □

4. Openness of formal versality

As the title suggests, we now address the openness of the formally versal locus. Let S be a scheme. We isolate the following conditions for an S-groupoid X.

Condition 4.1. Let T be an affine X-scheme. The extensions of X are *bounded at* T if X is **Nil**-homogeneous at T and the functor $M \mapsto \mathrm{Exal}_X(T, M)$ is bounded. The extensions of X are *bounded* if X has bounded extensions at every affine X-scheme T, locally of finite type over S.

Condition 4.2. Let T be an affine X-scheme. The extensions of X are *constructible at* T if X is **Nil**-homogeneous at T and the functor $M \mapsto \mathrm{Exal}_X(T, M)$ is CS. The extensions of X are *constructible* if X has constructible extensions at every affine X-scheme T, locally of finite type over S.

That these conditions are plausible is implied by the following lemma.

Lemma 4.3. *Let S be a locally noetherian scheme, let X be an algebraic S-stack, and let T be an affine X-scheme. If both X and T are locally of finite type over S, then the functors $M \mapsto \mathrm{Der}_X(T, M)$ and $M \mapsto \mathrm{Exal}_X(T, M)$ are bounded and CB.*

Proof. By [Ols06, Thm. 1.1] there is a complex $L_{T/X} \in \mathsf{D}^-_{\mathsf{Coh}}(T)$ such that for all quasi-coherent $\mathcal{O}_T$-modules M, there are natural isomorphisms $\mathrm{Der}_X(T, M) \cong \mathrm{Ext}^0_{\mathcal{O}_T}(L_{T/X}, M)$ and $\mathrm{Exal}_X(T, M) \cong \mathrm{Ext}^1_{\mathcal{O}_T}(L_{T/X}, M)$. The result now follows from a consideration of Example 3.4. □

In their current form, Conditions 4.1 and 4.2 are difficult to verify. In §6, this will be rectified. Nonetheless, we can now prove the following.

Theorem 4.4. *Let S be a locally noetherian scheme, let X be an S-groupoid and let T be an affine X-scheme, locally of finite type over S. Assume, in addition, that*

1. *X is limit preserving,*
2. *X is* **rCl**-*homogeneous,*
3. *X has bounded extensions at T (Condition 4.1),*
4. *X has constructible extensions at T (Condition 4.2) and*
5. *X has Zariski local extensions at T (Condition 2.11).*

Let $t \in |T|$ be a closed *point. If T is formally versal at $t \in |T|$, then T is formally versal at every point of finite type in a Zariski open neighborhood of t. In particular, if X is also* $\mathbf{Art}^{\mathbf{fin}}$-*homogeneous and $T \to X$ is representable by algebraic spaces, then T is formally smooth in a Zariski open neighborhood of t.*

Proof. By Condition 4.1 and Lemma 1.8, the functor $M \mapsto \mathrm{Exal}_X(T, M)$ is bounded, half-exact, and preserves direct limits. Condition 4.2 now implies that the functor $M \mapsto \mathrm{Exal}_X(T, M)$ satisfies the criteria of Theorem 3.3. Thus, $\mathbb{V}(\mathrm{Exal}_X(T, -)) \subseteq$

$|T|$ is a Zariski open subset. By Lemma 2.2(2) $\implies$ (3) and Theorem 3.7, we have that $t \in \mathbb{V}(\mathrm{Exal}_X(T,-))$. So, there exists an open neighborhood $t \in U \subseteq |T|$ with $\mathrm{Exal}_X(T,\kappa(u)) = 0$ for all $u \in U$. By Proposition 2.13, every point $u \in U$ of finite type is formally versal. The last assertion follows from Corollary 2.7. □

5. Automorphisms, deformations, and obstructions

In this section, we introduce a deformation-theoretic framework that makes it possible to verify Conditions 2.11, 4.1 and 4.2. To do this, we recall the formulation of deformations and obstructions given in [Hal17, §6].

Let S be a scheme and let $\Phi\colon Y \to Z$ be a 1-morphism of S-groupoids. Define the category $\mathbf{Def}_\Phi$ to have objects the pairs $(i\colon T \hookrightarrow T', r\colon T' \to T)$, where i is a Y-extension and r is a Z-retraction of i, with the obvious morphisms. Graphically, it is the category of completions of the following diagram:

$$\begin{tikzcd} T \arrow[r] \arrow[d, hook] & Y \arrow[d, "\Phi"] \\ T[J] \arrow[r] \arrow[ur, dashed, "\eta"] & Z. \end{tikzcd}$$

Forgetting the retraction, there is a natural functor $\mathbf{Def}_\Phi \to \mathbf{Exal}_Y$. If T is a Y-scheme, then we denote the fiber of this functor over $\mathbf{Exal}_Y(T) \subseteq \mathbf{Exal}_Y$ by $\mathbf{Def}_\Phi(T)$. It follows that there is an induced functor $\mathbf{Def}_\Phi(T) \to \mathsf{QCoh}(T)^\circ$, whose fiber over a quasi-coherent $\mathcal{O}_T$-module I we denote by $\mathbf{Def}_\Phi(T,I)$. Note that the category $\mathbf{Def}_\Phi(T,I)$ is naturally pointed by the trivial Y-extension $i_{T,J}$ of T by J. Denote the set of isomorphism classes of $\mathbf{Def}_\Phi(T,J)$ by $\mathrm{Def}_\Phi(T,J)$ and let $\mathrm{Aut}_\Phi(T,J)$ denote the set $\mathrm{Aut}_{\mathbf{Def}_\Phi(T,J)}(i_{T,J})$.

If Y and Z are $\mathbf{Nil}$-homogeneous at T, then the groupoid $\mathbf{Def}_\Phi(T,J)$ is a Picard category [Hal17, Prop. 6.5]. Thus we obtain $\Gamma(T,\mathcal{O}_T)$-linear functors

$$\begin{aligned} \mathrm{Def}_\Phi(T,-)\colon \mathsf{QCoh}(T) \to \mathsf{Ab}, &\quad J \mapsto \mathrm{Def}_\Phi(T,J); \quad \text{and} \\ \mathrm{Aut}_\Phi(T,-)\colon \mathsf{QCoh}(T) \to \mathsf{Ab}, &\quad J \mapsto \mathrm{Aut}_{\mathbf{Def}_\Phi(T,J)}(i_{T,J}). \end{aligned}$$

The lemma that follows is an easy consequence of [Hal17, Lem. 6.4].

Lemma 5.1. *Let S be a scheme and let $\Phi\colon Y \to Z$ be a 1-morphism S-groupoids. Let $i\colon W \hookrightarrow T$ be a closed immersion of Y-schemes and let N be a quasi-coherent $\mathcal{O}_W$-module. If Y and Z are* $\mathbf{Cl}$*-homogeneous at W, then the natural maps:*

$$\mathrm{Aut}_\Phi(T, i_*N) \to \mathrm{Aut}_\Phi(W,N) \quad \textit{and} \quad \mathrm{Def}_\Phi(T,i_*N) \to \mathrm{Def}_\Phi(W,N)$$

are bijective.

We recall the exact sequence of [Hal17, Prop. 6.7], which is our fundamental computational tool.

Proposition 5.2. *Let S be a scheme and let $\Phi\colon Y \to Z$ be a 1-morphism of S-groupoids. Let T be a Y-scheme and let J be a quasi-coherent $\mathcal{O}_T$-module. If Y and Z are* $\mathbf{Nil}$*-homogeneous at T, then there is a natural 6-term exact sequence of* *abelian groups:*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Aut}_\Phi(T,J) \arrow[r] & \mathrm{Der}_Y(T,J) \arrow[r] & \mathrm{Der}_Z(T,J) \arrow[dll] \\ & \mathrm{Def}_\Phi(T,J) \arrow[r] & \mathrm{Exal}_Y(T,J) \arrow[r] & \mathrm{Exal}_Z(T,J). \end{tikzcd}$$

If Y and Z are **Nil**-homogeneous at T and J is a quasi-coherent $\mathcal{O}_T$-module, then we let

$$\mathrm{Obs}_\Phi(T,J) = \mathrm{coker}\big(\mathrm{Exal}_Y(T,J) \to \mathrm{Exal}_Z(T,J)\big).$$

This defines a $\Gamma(T,\mathcal{O}_T)$-linear functor

$$\mathrm{Obs}_\Phi(T,-)\colon \mathsf{QCoh}(T) \to \mathsf{Ab}, \quad J \mapsto \mathrm{Obs}_\Phi(T,J),$$

the *minimal obstruction theory* of Φ at T (see §7). If Y and Z are **rNil**-homogeneous at T, then $\mathrm{Aut}_\Phi(T,-)$ and $\mathrm{Def}_\Phi(T,-)$ are half-exact [Hal17, Cor. 6.6]. There is no reason to expect that $\mathrm{Obs}_\Phi(T,-)$ is half-exact, however. We have the following analogues of Lemmas 1.8(2) and 5.1 for obstructions.

Lemma 5.3. *Let S be a scheme and let P be a class of morphisms of S-schemes. Let $\Phi\colon Y \to Z$ be a 1-morphism of S-groupoids. Let $p\colon V \to T$ be an affine morphism of Y-schemes that is P. If Y and Z are P-homogeneous at V and* **Nil**-*homogeneous at T, then there is a natural map $p_\#\colon \mathrm{Obs}_\Phi(V,N) \to \mathrm{Obs}_\Phi(T,p_*N)$, which is injective and functorial in N.*

Proof. The existence of $p_\#$ follows immediately from Lemma 1.8(2). That $p_\#$ is injective is obvious. □

Lemma 5.4. *Let S be a scheme, and let $\Phi\colon Y \to Z$ be a 1-morphism of* **Cl**-*homogeneous S-groupoids. Let $i\colon W \hookrightarrow T$ be a closed immersion of affine noetherian Y-schemes and let N be a quasi-coherent $\mathcal{O}_W$-module. If $\mathrm{Obs}_\Phi(T,i_*N)$ is a finitely generated $\Gamma(T,\mathcal{O}_T)$-module, then there exists an infinitesimal neighborhood $i_n\colon W_n \to T$ of W in T, i.e., a factorization of i as $W \xrightarrow{j} W_n \xrightarrow{i_n} T$, where j is a locally nilpotent closed immersion, such that*

$$(i_n)_\#\colon \mathrm{Obs}_\Phi(W_n, j_*N) \to \mathrm{Obs}_\Phi(T,i_*N)$$

is an isomorphism.

Proof. Given an obstruction $\omega \in \mathrm{Obs}_\Phi(T,i_*N)$, we can realize it as a Z-extension $k\colon T \hookrightarrow T'$ of T by i_*N. The ideal sheaf $k_*i_*N \subseteq \mathcal{O}_{T'}$ is then annihilated by the ideal sheaf I defining the closed immersion $k \circ i\colon W \hookrightarrow T'$. Thus, by the Artin–Rees lemma, we have that $(k_*i_*N) \cap I^n = 0$ for some n. Let W_1' and W_1 be the closed subschemes of T' defined by I^n and $I^n + k_*i_*N$. Then the morphisms in the diagram:

$$\begin{tikzcd} W \arrow[r, hook, "j_1"] & W_1 \arrow[r, hook, "i_1"] \arrow[d, hook] & T \arrow[d, hook] \\ & W_1' \arrow[r, hook] & T' \end{tikzcd}$$

are closed immersions and the square is cartesian and cocartesian in the category of Z-schemes (because Z is **Cl**-homogeneous at W_1). If we let $\omega_1 = [W_1 \hookrightarrow W_1'] \in \mathrm{Obs}_\Phi(W_1,(j_1)_*N)$ denote the obstruction to lifting W_1' to a Y-scheme; then $\omega = (i_1)_\#(\omega_1)$.

We have thus shown that every element $\omega \in \mathrm{Obs}_\Phi(T, i_*N)$ is in the image of $\mathrm{Obs}_\Phi(W_l, (j_l)_*N)$ for some infinitesimal neighborhood $j_l \colon W \hookrightarrow W_l$, depending on ω. Since $\mathrm{Obs}_\Phi(T, i_*N)$ is a finitely generated $\Gamma(T, \mathcal{O}_T)$-module and T is affine and noetherian, it follows that there exists an infinitesimal neighborhood $j \colon W \hookrightarrow W_n$ such that $\mathrm{Obs}_\Phi(W_n, j_*N) \to \mathrm{Obs}_\Phi(T, i_*N)$ is an isomorphism. □

6. Relative conditions

Let S be a locally noetherian scheme. In this section, we introduce a number of conditions for a 1-morphism of S-groupoids $\Phi \colon Y \to Z$. These are the relative versions of the conditions that appear in (5a), (5b), (5c), (6b) and (6c) of the Main Theorem. For any of the conditions given in this section, an S-groupoid X is said to have that condition if the structure 1-morphism $X \to \mathsf{Sch}/S$ has the condition. These conditions are stated "relatively" for two reasons. The first reason is to make it clear that this paper subsumes the results of [Sta06] on the stability of Artin's criteria under composition. This follows immediately from the exact sequence of [Hal17, Prop. 6.13] and Lemma 3.10. Secondly, and of most importance, is that the relative formulation permits a process of bootstrapping the diagonal. This is an important and subtle point of this paper, which we will discuss in more detail when we prove the Main Theorem in Section 10.

Condition 6.1. Let T be an affine Y-scheme. Assume that Y and Z are **Nil**-homogeneous at every closed subscheme of T. Automorphisms (resp. deformations, resp. obstructions) of Φ are *bounded at* T if for every integral closed subscheme $i \colon T_0 \hookrightarrow T$, condition (i) (resp. (ii), resp. (iii)) below holds:

(i) $\mathrm{Aut}_\Phi(T_0, \mathcal{O}_{T_0})$ is a finitely generated $\Gamma(\mathcal{O}_{T_0})$-module;
(ii) $\mathrm{Def}_\Phi(T_0, \mathcal{O}_{T_0})$ is a finitely generated $\Gamma(\mathcal{O}_{T_0})$-module;
(iii) $\mathrm{Obs}_\Phi(T, i_*\mathcal{O}_{T_0})$ is a finitely generated $\Gamma(\mathcal{O}_{T_0})$-module.

Automorphisms (resp. deformations, resp. obstructions) of Φ are *bounded* if they are bounded at every affine Y-scheme T, locally of finite type over S.

Morphisms of S-groupoids typically have bounded obstructions (Condition 6.1(iii)). For example, if Y is **Nil**-homogeneous and Z is algebraic, then Z has bounded extensions (Condition 4.1) and Φ has bounded obstructions.

Lemma 6.2. *Let S be a locally noetherian scheme and let $\Phi \colon Y \to Z$ be a 1-morphism of* **rCl**-*homogeneous S-groupoids with bounded deformations (Condition 6.1(ii)) at an affine Y-scheme T, locally of finite type over S. If Z has bounded extensions at T (Condition 4.1), then so does Y.*

Proof. By Lemma 1.8(3) the functor $M \mapsto \mathrm{Exal}_Y(T, M)$ is half-exact. Thus, by Lemma 3.10(1), it is sufficient to prove that for every integral closed subscheme $i \colon T_0 \hookrightarrow T$, the $\Gamma(\mathcal{O}_{T_0})$-module $\mathrm{Exal}_Y(T, i_*\mathcal{O}_{T_0})$ is finitely generated. Now, by Proposition 5.2, there is an exact sequence:

$$\mathrm{Def}_\Phi(T, i_*\mathcal{O}_{T_0}) \longrightarrow \mathrm{Exal}_Y(T, i_*\mathcal{O}_{T_0}) \longrightarrow \mathrm{Exal}_Z(T, i_*\mathcal{O}_{T_0}).$$

By Condition 4.1, the $\Gamma(\mathcal{O}_{T_0})$-module $\mathrm{Exal}_Z(T, i_*\mathcal{O}_{T_0})$ is finitely generated. By Lemma 5.1, $\mathrm{Def}_\Phi(T, i_*\mathcal{O}_{T_0}) \cong \mathrm{Def}_\Phi(T_0, \mathcal{O}_{T_0})$, which is also a finitely generated $\Gamma(\mathcal{O}_{T_0})$-module by Condition 6.1(ii). The result now follows from the exact sequence above. □

Similarly, to enable the verification that an S-groupoid has constructible extensions (Condition 4.2), we introduce the following conditions.

Condition 6.3. Let T be an affine Y-scheme. Assume that Y and Z are **Nil**-homogeneous at every closed subscheme of T. Automorphisms (resp. deformations, resp. obstructions) of Φ are *constructible at* T if for every closed subscheme $T_1 \subseteq T$, such that T_1 is irreducible and $i\colon T_0 \hookrightarrow T_1$ denotes the reduction, condition (i) (resp. (ii), resp. (iii)) below holds:

(i) $\mathrm{Aut}_\Phi(T_0, -)\colon \mathsf{QCoh}(T_0) \to \mathsf{Ab}$ is GB;
(ii) $\mathrm{Def}_\Phi(T_0, -)\colon \mathsf{QCoh}(T_0) \to \mathsf{Ab}$ is GB;
(iii) $\mathrm{Obs}_\Phi(T_1, i_*-)\colon \mathsf{QCoh}(T_0) \to \mathsf{Ab}$ is GI.

Automorphisms (resp. deformations, resp. obstructions) of Φ are *constructible* if they are constructible at every affine Y-scheme T, locally of finite type over S.

We now proceed to Zariski local extensions (Condition 2.11). Note that the following condition trivially holds when S is Jacobson. Indeed, in that case, $U_1 = T_1 = \{\eta\}$.

Condition 6.4. Let T be an affine Y-scheme. Assume that Y and Z are **Nil**-homogeneous at every closed subscheme of T. Automorphisms (resp. deformations, resp. obstructions) of Φ are *Zariski local at* T if for every closed subscheme $T_1 \subseteq T$ and non-empty open subscheme $U_1 \subseteq T_1$, such that T_1 is irreducible and the generic point $\eta \in |T_1|$ is of finite type over S, and $U_0 \subseteq T_0$ denotes the reductions, condition (i) (resp. (ii), resp. (iii)) below holds:

(i) the natural map $\mathrm{Aut}_\Phi(T_0, \kappa(\eta)) \to \mathrm{Aut}_\Phi(U_0, \kappa(\eta))$ is bijective;
(ii) the natural map $\mathrm{Def}_\Phi(T_0, \kappa(\eta)) \to \mathrm{Def}_\Phi(U_0, \kappa(\eta))$ is bijective;
(iii) the natural map $\mathrm{Obs}_\Phi(T_1, \kappa(\eta)) \to \mathrm{Obs}_\Phi(U_1, \kappa(\eta))$ is injective.

Automorphisms (resp. deformations, resp. obstructions) of Φ are *Zariski local* if they are Zariski local at every affine Y-scheme T, locally of finite type over S.

The following proposition is one of the major results of the article.

Proposition 6.5. *Let S be a locally noetherian scheme. Let $\Phi\colon Y \to Z$ be a 1-morphism of* **Cl**-*homogeneous S-groupoids with bounded obstructions at an affine Y-scheme T, locally of finite type over S (Condition 6.1(iii)).*

1. *Assume, in addition, that Φ has constructible deformations and obstructions at T (Conditions 6.3(ii)–6.3(iii)). If Z has constructible extensions at T (Condition 4.2), then so does Y.*
2. *Assume, in addition, that Φ has Zariski local deformations and obstructions at T (Conditions 6.4(ii)–6.4(iii)). If Z has Zariski local extensions at T (Condition 2.11), then so does Y.*

Proof. We prove (1). By Proposition 5.2 there is an exact sequence of additive functors $\mathsf{QCoh}(T) \to \mathsf{Ab}$:

$$\mathrm{Def}_\Phi(T, -) \longrightarrow \mathrm{Exal}_Y(T, -) \longrightarrow \mathrm{Exal}_Z(T, -) \longrightarrow \mathrm{Obs}_\Phi(T, -) \longrightarrow 0.$$

Let $i\colon T_0 \hookrightarrow T$ be an integral closed subscheme. By Lemma 5.1 we have that $\mathrm{Def}_\Phi(T_0, -) = \mathrm{Def}_\Phi(T, i_*(-))$. Condition 6.3(ii) gives that $\mathrm{Def}_\Phi(T_0, -)$ is GS, so the functor $\mathrm{Def}_\Phi(T, -)$ is CS. Condition 4.2 says that $\mathrm{Exal}_Z(T, -)$ is CS. The remaining two conditions together with Lemma 5.4 imply that $\mathrm{Obs}_\Phi(T, -)$ is CI and weakly bounded. In fact, for every integral closed subscheme $i\colon T_0 \hookrightarrow T$, there is an infinitesimal neighborhood $j\colon T_0 \hookrightarrow T_1$ such that $\mathrm{Obs}_\Phi(T_1, j_*\mathcal{O}_{T_0}) \cong \mathrm{Obs}_\Phi(T, i_*\mathcal{O}_{T_0})$ and $\mathrm{Obs}_\Phi(T_1,\kappa(t)) \hookrightarrow \mathrm{Obs}_\Phi(T,\kappa(t))$ is injective for all $t \in |T_0|$. It now follows from Lemma 3.10(5a) that the functor $\mathrm{Exal}_Y(T,-)$ is CS.

The proof of (2) is similar: let $u \in U \subseteq T$ be as in Condition 2.11, use the exact sequence above, take $T_0 = \overline{\{u\}}$, and apply Lemmas 5.1 and 5.4 as before. $\square$

7. Obstruction theories

Throughout this section, we let S be a locally noetherian scheme and let $\Phi\colon Y \to Z$ be a 1-morphism of **Nil**-homogeneous S-groupoids. In this section, we will expand the conditions on obstructions given in the previous sections to obtain more readily verifiable conditions. We begin with recalling the definition of an n-step relative obstruction theory given in [Hal17, Defn. 6.8].

An *n-step relative obstruction theory for* Φ, denoted $\{\mathrm{o}^l(-,-), \mathrm{O}^l(-,-)\}_{l=1}^n$, is for each Y-scheme T, a sequence of additive functors (the obstruction spaces):

$$\mathrm{O}^l(T,-)\colon \mathsf{QCoh}(T) \to \mathsf{Ab}, \quad J \mapsto \mathrm{O}^l(T,J), \quad l = 1,\dots,n$$

as well as natural transformations of functors (the obstruction maps):

$$\begin{aligned} \mathrm{o}^1(T,-)\colon\ & \mathrm{Exal}_Z(T,-) \Rightarrow \mathrm{O}^1(T,-)\\ \mathrm{o}^l(T,-)\colon\ & \ker \mathrm{o}^{l-1}(T,-) \Rightarrow \mathrm{O}^l(T,-) \quad \text{for } l = 2,\ \dots,\ n, \end{aligned}$$

such that the natural transformation of functors:

$$\mathrm{Exal}_Y(T,-) \Rightarrow \mathrm{Exal}_Z(T,-)$$

has image $\ker \mathrm{o}^n(T,-)$. Furthermore, we say that the obstruction theory is

- *(weakly) bounded*, if for every affine Y-scheme T, locally of finite type over S, the obstruction spaces $M \mapsto \mathrm{O}^l(T,M)$ are (weakly) bounded functors;
- *Zariski- (resp. étale-) functorial* if for every open immersion (resp. étale morphism) of affine Y-schemes $g\colon V \to T$, and $l = 1,\ \dots,\ n$, there is a natural transformation of functors:

  $$C^l_g\colon \mathrm{O}^l(T, g_*(-)) \Rightarrow \mathrm{O}^l(V,-),$$

  which for every quasi-coherent $\mathcal{O}_V$-module N, make the following diagrams commute:

  $$\begin{tikzcd} \mathrm{Exal}_X(T,g_*N) \arrow[r] \arrow[d] & \mathrm{O}^1(T,g_*N) \arrow[d] \\ \mathrm{Exal}_X(V,N) \arrow[r] & \mathrm{O}^1(V,N) \end{tikzcd} \qquad \begin{tikzcd} \ker \mathrm{o}^{l-1}(T,g_*N) \arrow[r] \arrow[d] & \mathrm{O}^l(T,g_*N) \arrow[d] \\ \ker \mathrm{o}^{l-1}(V,N) \arrow[r] & \mathrm{O}^l(V,N). \end{tikzcd}$$

  Here the leftmost map is the map ψ of Lemma 1.8 (5). We also require for every open immersion (resp. étale morphism) of affine schemes $h\colon W \to V$, an isomorphism of functors:

  $$\alpha^l_{g,h}\colon C^l_h \circ C^l_g \Rightarrow C^l_{gh}.$$

Remark 7.1 (Comparison with Artin's obstruction theories). An obstruction theory in the sense of [Art74, 2.6] is a 1-step bounded obstruction theory "that is functorial in the obvious sense". We take this to mean étale-functorial in the above sense. Obstruction theories are usually half-exact and functorial for every morphism, but Exal is only contravariantly functorial for étale morphisms so the condition above does not make sense for arbitrary morphisms. On the other hand, for **Aff**-homogeneous stacks, Exal is *covariantly* functorial for every affine morphism (Lemma 1.8(2)) and the minimal obstruction theory Obs_Φ is étale-functorial (Lemma 1.8(5)).

We have the following simple lemma.

Lemma 7.2. *Let S be a locally noetherian scheme and let $\Phi\colon Y \to Z$ be a 1-morphism of* **Nil**-*homogeneous S-groupoids. Let $\{\mathrm{o}^l, \mathrm{O}^l\}_{l=1}^n$ be an n-step relative obstruction theory for Φ. Let $\widetilde{\mathrm{O}}^l(T,M) \subseteq \mathrm{O}^l(T,M)$ be the image of $\mathrm{o}^l(T,M)$ for $l = 1,\dots,n$. Then $\{\mathrm{o}^l, \widetilde{\mathrm{O}}^l\}_{l=1}^n$ is an n-step relative obstruction theory for Φ. Moreover, let $\mathrm{Obs}^l(T,-) = \mathrm{Exal}_Z(T,-)/\ker \mathrm{o}^l$ and $\mathrm{Obs}^0(T,-) = 0$. Then $\mathrm{Obs}^n(T,-) = \mathrm{Obs}_\Phi(T,-)$ and we have exact sequences*

$$0 \longrightarrow \widetilde{\mathrm{O}}^l(T,-) \longrightarrow \mathrm{Obs}^l(T,-) \longrightarrow \mathrm{Obs}^{l-1}(T,-) \longrightarrow 0$$

for $l = 1,2,\dots,n$. In particular, if the obstruction theory is (weakly) bounded, then so is the minimal obstruction theory $\mathrm{Obs}_\Phi(T,-)$.

We now introduce variations of Conditions 6.3(iii) and 6.4(iii) (constructible and Zariski local obstructions) in terms of an n-step relative obstruction theory.

Condition 7.3 (Constructible obstructions II)**.** There exists a weakly bounded n-step relative obstruction theory for Φ, $\{\mathrm{o}^l(-,-), \mathrm{O}^l(-,-)\}_{l=1}^n$, such that for every affine irreducible Y-scheme T that is locally of finite type over S, the obstruction spaces $\mathrm{O}^l(T,-)|_{T_0}\colon \mathsf{QCoh}(T_0) \to \mathsf{Ab}$, are GI for $l = 1,\dots,n$ where $T_0 = T_{\mathrm{red}}$.

Condition 7.4 (Zariski local obstructions II)**.** There exists a functorial, n-step relative obstruction theory for Φ, $\{\mathrm{o}^l(-,-), \mathrm{O}^l(-,-)\}_{l=1}^n$, such that for every affine irreducible Y-scheme T that is locally of finite type over S and whose generic point $\eta \in |T|$ is of finite type, and for every open subscheme $U \subseteq T$, the canonical maps $\mathrm{O}^l(T,\kappa(\eta)) \to \mathrm{O}^l(U,\kappa(\eta))$ are injective for $l = 1,\dots,n$.

Lemma 7.5. *Let S be a locally noetherian scheme and let $\Phi\colon Y \to Z$ be a 1-morphism of* **Nil**-*homogeneous S-groupoids.*

(1) *(Constructibility) Φ has bounded and constructible obstructions (Conditions 6.1(iii) and 6.3(iii)) if and only if Φ satisfies Condition 7.3.*
(2) *(Zariski localization) Φ has Zariski local obstructions (Condition 6.4(iii)) if and only if Φ satisfies Condition 7.4.*

Proof. If Φ has bounded deformations and obstructions (Conditions 6.1(iii) and 6.3(iii)), then the minimal obstruction theory satisfies Condition 7.3. Conversely, assume that we are given an obstruction theory $\mathrm{O}^l(-,-)$ as in Condition 7.3. Let T be an affine irreducible Y-scheme that is locally of finite type over S. Then the subfunctors $\widetilde{\mathrm{O}}^l(T,-)|_{T_0} \subseteq \mathrm{O}^l(T,-)|_{T_0}$ of Lemma 7.2 are also GI and weakly bounded by Lemma 3.10(4). Since $\mathrm{Obs}_\Phi(T,-)$ is an iterated extension of the $\widetilde{\mathrm{O}}^l(T,-)$'s, it follows that $\mathrm{Obs}_\Phi(T,-)|_{T_0}$ is GI and weakly bounded by Lemma 3.10(5b)—thus Φ has bounded and constructible obstructions (Conditions 6.3(iii) and 6.1(iii)).

If Condition 6.4(iii) holds then the minimal obstruction theory satisfies 7.4. That Condition 7.4 implies Condition 6.4(iii) follows from Lemma 7.2. $\square$

8. Conditions on obstructions without an obstruction theory

In this section we give conditions without reference to linear obstruction theories, just as in [Art69b, Thm. 5.3 [5′c]] and [Sta06]. In the comparison we provide between our conditions on obstructions we use **Aff**-homogeneity, while Artin uses **DVR**-homogeneity and Starr uses homogeneity along localization morphisms (not just Zariski localizations). Starr's localization-homogeneity is stronger than **DVR**-homogeneity, but weaker than **Aff**-homogeneity. However, we make our assumption because shortly we will prove that **DVR**-homogeneity—in all cases of relevance to the proof of the Main Theorem—implies **Aff**-homogeneity (Lemma 10.4).

Definition 8.1 ([Art69b, 5.1], [Sta06, Defn. 2.1])**.** By a *deformation situation* for $\Phi\colon Y \to Z$, we will mean data $(T \hookrightarrow T', M)$, where T is an irreducible affine Y-scheme that is locally of finite type over S, where M is a quasi-coherent $\mathcal{O}_{T_{\mathrm{red}}}$-module, and where $T \hookrightarrow T'$ is an Z-extension of T by M. We say that the deformation situation is *obstructed* if the Z-extension $T \hookrightarrow T'$ cannot be lifted to a Y-extension $T \hookrightarrow T'$.

Notation 8.2. For a deformation situation $(T \hookrightarrow T', M)$, let $T_0 = T_{\mathrm{red}}$, let $\eta_0 = \operatorname{Spec} K_0$ denote the generic point of T_0, let $\eta = \operatorname{Spec}(\mathcal{O}_{T,\eta_0})$, and let $\eta' = \operatorname{Spec}(\mathcal{O}_{T',\eta_0})$. Thus $\eta \hookrightarrow \eta'$ is a Z-extension of η by $M_\eta = M \otimes_{\mathcal{O}_{T_0}} K_0$.

Condition 8.3 (Constructible obstructions III)**.** Given a deformation situation such that M is a free $\mathcal{O}_{T_0}$-module of finite rank and such that for every non-zero $\mathcal{O}_{T_0}$-module map $\epsilon\colon M_\eta \to K_0$, the resulting Z-extension $\eta \hookrightarrow \eta'_\epsilon$ of η by K_0 is obstructed, then there exists a dense open subset $U_0 \subseteq |T_0|$ such that for all points $u \in U_0$ of finite type, and all non-zero $\mathcal{O}_{T_0}$-module maps $\gamma\colon M \to \kappa(u)$, the resulting Z-extension $T \hookrightarrow T'_\gamma$ of T by $\kappa(u)$ is obstructed.

Lemma 8.4. *Let S be a locally noetherian scheme and let $\Phi\colon Y \to Z$ be a 1-morphism of limit preserving,* **Aff***-homogeneous S-groupoids. If Φ has bounded obstructions (Condition 6.1(iii)), then Φ has constructible obstructions (Condition 6.3(iii)) if and only if Φ satisfies Condition 8.3.*

Proof. Fix an irreducible affine Y-scheme T and let T_0 be its reduction. To see that Conditions 6.3(iii) and 8.3 are equivalent we will use condition (†) of Proposition 3.11 for $F(-) = \operatorname{Obs}_\Phi(T,-)|_{T_0}$. Some care is needed, though, as these two conditions are not quite equivalent for a fixed T.

Consider a deformation situation $(T \hookrightarrow T', M)$ as in Condition 8.3 and let $\omega \in F(M) = \operatorname{Obs}_\Phi(T, M)$ be the obstruction of the deformation situation. Then for every non-zero $\epsilon\colon M \to K_0$, the element $\epsilon_*\omega \in F(K_0)$ is non-zero since its image under $F(K_0) = \operatorname{Obs}_\Phi(T, K_0) \to \operatorname{Obs}_\Phi(T_\eta, K_0)$ is non-zero. If F is GI, then condition (†) is satisfied for F, M and ω. Thus, there is an open dense subset $U_0 \subseteq |T_0|$ such that $\gamma_*\omega \in F(\kappa(u))$ is non-zero for all $u \in U_0$ of finite type and non-zero maps $\gamma\colon M \to \kappa(u)$, that is, Condition 8.3 holds.

Conversely, let f, M and ω be as in condition (†) for $F(-)$. Let $V_0 = \operatorname{Spec}(A_f) \subseteq T_0 = \operatorname{Spec} A$ and let $V \subseteq T$ denote the corresponding open subscheme. Since Y and Z are **Aff**-homogeneous, the natural morphism $F(-)|_{A_f} = \operatorname{Obs}_\Phi(T,-)|_{V_0} \to \operatorname{Obs}_\Phi(V,-)|_{V_0}$ is an isomorphism (Lemma 1.8(5)). Since M is an A_f-module, we may thus consider $\omega \in F(M)$ as an obstruction class in $\operatorname{Obs}_\Phi(V, M)$. This class can be realized by a deformation situation $(V \hookrightarrow V', M)$. We assume that Condition 8.3 holds for this deformation situation.

Since Y and Z are **Aff**-homogeneous, we also have an isomorphism $\mathrm{Obs}_\Phi(T,-)|_{\eta_0} \to \mathrm{Obs}_\Phi(\eta,-)|_{\eta_0}$. In particular, for all $\epsilon\colon M_\eta \to K_0$, the resulting Z-extension $\eta \hookrightarrow \eta'_\epsilon$ of η by K_0 is obstructed. Thus, there exists a dense open subset $U_0 \subseteq |V_0|$ such that for all points $u \in U_0$ of finite type and maps $\gamma\colon M \to \kappa(u)$, the induced Z-extension $(V \hookrightarrow V'_\gamma, \kappa(u))$ is obstructed. In particular, $\gamma_*\omega \in F(\kappa(u)) = \mathrm{Obs}_\Phi(T,\kappa(u)) = \mathrm{Obs}_\Phi(V,\kappa(u))$ is non-zero. Thus, Condition (†) holds for the given f, M and ω with $V_\omega = U_0$.

Thus, if for a given T, Condition 8.3 holds for all deformation situations $(V_0 \hookrightarrow V, M)$ where $V \subseteq T$ is an open subscheme, then F is GI. □

Remark 8.5. If S is of finite type over a Dedekind domain as in [Art69b] (or Jacobson), then in Condition 8.3 it is enough to consider closed points $u \in U$. Indeed, in the proof of the lemma above, we are free to pass to open dense subsets and every S-scheme of finite type has a dense open subscheme which is Jacobson.

9. Effectivity

We begin with the following definition.

Definition 9.1. Let X be a category fibered in groupoids over the category of S-schemes. We say that X is *weakly effective* (resp. *effective*) if for every local noetherian ring $(B,\mathfrak{m})$, such that B is $\mathfrak{m}$-adically complete, with an S-scheme structure $\operatorname{Spec} B \to S$ such that the induced morphism $\operatorname{Spec}(B/\mathfrak{m}) \to S$ is locally of finite type, the natural functor:

$$X(\operatorname{Spec} B) \to \varprojlim_n X(\operatorname{Spec}(B/\mathfrak{m}^{n+1}))$$

is dense and fully faithful (resp. an equivalence). Here dense means that for every object $(\xi_n)_{n\geq 0}$ in the limit and for every $k \geq 0$, there exists an object $\xi \in X(\operatorname{Spec} B)$ such that its image in $X(\operatorname{Spec}(B/\mathfrak{m}^{k+1}))$ is isomorphic to ξ_k.

If X is an algebraic stack, then the functor $X(\operatorname{Spec} B) \to \varprojlim_n X(\operatorname{Spec}(B/\mathfrak{m}^{n+1}))$ is an equivalence of categories—thus every algebraic stack is effective. Also, it is clear that effectivity implies weak effectivity. We will see in Proposition 9.3 that the converse holds under mild hypotheses.

The following lemma is well-known, with the difficult parts attributed to Schlessinger [Sch68] and Rim [SGA7, Exp. VI].

Lemma 9.2. *Let S be a noetherian scheme and let X be an S-groupoid. Let* $\operatorname{Spec} \Bbbk$ *be an X-scheme, locally of finite type over S, such that $\Bbbk$ is a field. If X is*

(1) $\mathbf{Art}^{\mathbf{triv}}$*-homogeneous,*
(2) *weakly effective and*
(3) *has bounded deformations at* $\operatorname{Spec}\Bbbk$ *(Condition 6.1(ii)),*

then there exists a pointed and affine X-scheme (T,t) such that:

(a) *the point $t \in |T|$ is closed and the X-schemes* $\operatorname{Spec}\Bbbk$ *and* $\operatorname{Spec}\kappa(t)$ *are isomorphic;*
(b) *the X-scheme T is formally versal at $t \in |T|$; and*
(c) *T is affine, local, noetherian and complete.*

Proof. By Schlessinger–Rim (e.g., [Stacks, Tag 06IW]), there exists an affine, local, noetherian and complete scheme ($T = \operatorname{Spec} R, \mathfrak{m}$) and an object $(\eta_n)_{n\geq 0} \in \varprojlim_n X(T_n)$, where $T_n = \operatorname{Spec}(R/\mathfrak{m}^{n+1})$, which is a formally versal deformation (in the sense of Schlessinger–Rim) of the X-scheme structure on $\operatorname{Spec} \Bbbk$. Since X is weakly effective, there exists $\xi \in X(T)$ such that $\xi|_{T_1} \equiv \eta_1$ in $X(T_1)$. By formal versality, there exists a map of X-schemes $\phi\colon T \to T$ which restricts to the identity map on T_1. It is well-known that such maps are isomorphisms, hence ξ is formally versal. □

We now have the main result of this section.

Proposition 9.3. *Let S be a noetherian scheme. Let X be an S-groupoid that is*

(1) $\mathbf{Art}^{\mathbf{triv}}$*-homogeneous,*
(2) *weakly effective and*
(3) *has bounded deformations at every X-scheme* $\operatorname{Spec} \Bbbk$*, locally of finite type over S, such that $\Bbbk$ is a field (Condition 6.1(ii)).*

Let $(B, \mathfrak{m})$ be a local noetherian ring, complete with respect to its $\mathfrak{m}$-adic topology, such that $\operatorname{Spec}(B/\mathfrak{m}) \to S$ *is locally of finite type. If $\{J_n\}_{n\geq 0}$ is an $\mathfrak{m}$-stable filtration of B (e.g., $J_n = \mathfrak{m}^{n+1}$), then the natural functor*

$$X(\operatorname{Spec} B) \to \varprojlim_n X(\operatorname{Spec}(B/J_n))$$

is an equivalence. In particular, X is effective.

Proof. Since $\mathfrak{m}$-stable filtrations of B have bounded difference [AM69, Lem. 10.6] (in particular, there exists an n_0 such that $J_{n+n_0} \subseteq \mathfrak{m}^{n+1}$ for all $n \geq 0$), it is sufficient to prove the result when $J_n = \mathfrak{m}^{n+1}$. In this case, the functor above is already assumed to be fully faithful; thus, it remains to establish that it is essentially surjective. To see this, let $(\xi_n)_{n\geq 0} \in \varprojlim_n X(\operatorname{Spec}(B/\mathfrak{m}^{n+1}))$. Now apply Lemma 9.2 to the X-scheme structure on $\operatorname{Spec}(B/\mathfrak{m})$ determined by ξ_0. This produces an affine, local, noetherian and complete X-scheme T, formally versal at its closed point t, such that the X-schemes $\operatorname{Spec} \kappa(t)$ and ξ_0 are isomorphic. By formal versality, there exists a compatible system of maps $b_n\colon \operatorname{Spec}(B/\mathfrak{m}^{n+1}) \to T$ lifting the X-scheme structures ξ_n. It follows that there is an induced map of schemes $\operatorname{Spec} B \to T$ which, by construction, defines an object $\xi \in X(\operatorname{Spec} B)$ with image $(\xi_n)_{n\geq 0} \in \varprojlim_n X(\operatorname{Spec} B/\mathfrak{m}^{n+1})$. The result follows. □

10. Proof of Main Theorem

In this section, we prove the Main Theorem. Before we do this, however, there are several preliminary results that we must prove. Conrad and de Jong [CJ02, Thm. 1.5] extended Artin's algebraization theorem [Art69b, Thm. 1.6] to excellent rings. The following lemma summarizes their result in the language of this paper.

Theorem 10.1. *Let S be an excellent scheme and let X be an S-groupoid. Let* $\operatorname{Spec} \Bbbk$ *be an X-scheme, locally of finite type over S, such that $\Bbbk$ is a field. If X is*

(1) *limit preserving,*
(2) *weakly effective,*
(3) $\mathbf{Art}^{\mathbf{triv}}$*-homogeneous and*
(4) *has bounded deformations at* $\operatorname{Spec} \Bbbk$ *(Condition 6.1(ii)),*

then there exists a pointed and affine X-scheme (T, t) such that:

(a) *T is locally of finite type over S;*
(b) *the point $t \in |T|$ is closed and the X-schemes $\operatorname{Spec} \Bbbk$ and $\operatorname{Spec} \kappa(t)$ are isomorphic; and*
(c) *the X-scheme T is formally versal at $t \in |T|$.*

We now obtain the following algebraicity criterion for groupoids.

Proposition 10.2. *Let S be an excellent scheme. An S-groupoid X is an algebraic S-stack, locally of finite presentation over S, if and only if*

(1) *X is a stack over $(\mathsf{Sch}/S)_{\text{Ét}}$;*
(2) *X is limit preserving;*
(3) *X is weakly effective;*
(4) *X is $\mathbf{Art}^{\mathbf{insep}}$-homogeneous;*
(5) *X is $\mathbf{rCl}$-homogeneous;*

(6a) *X has bounded deformations (Condition 6.1(ii));*
(6b) *X has constructible extensions (Condition 4.2);*
(6c) *X has Zariski local extensions (Condition 2.11); and*

(7) *the diagonal morphism $\Delta_{X/S} \colon X \to X \times_S X$ is representable by algebraic spaces.*

Proof. The hypotheses imply that for every pair $(\operatorname{Spec} \Bbbk \xrightarrow{x} S, \xi)$, where $\Bbbk$ is a field, x is a morphism locally of finite type, and $\xi \in X(x)$, there exists a pointed and affine X-scheme (T_ξ, t) as in Theorem 10.1. Condition (7) implies that $T_\xi \to X$ is representable by algebraic spaces.

As X is $\mathbf{rCl}$-homogeneous and has bounded deformations (Condition 6.1(ii)), Lemma 6.2 implies that X has bounded extensions (Condition 4.1). Also by Lemma 1.9, X is $\mathbf{Art}^{\mathbf{fin}}$-homogeneous. Since X has Zariski local, bounded and constructible extensions (Conditions 2.11, 4.1 and 4.2), it follows from Theorem 4.4 that we are free to assume—by passing to an affine open neighborhood of t—that the X-scheme T_ξ is formally smooth.

We finish the proof in the same manner as the proof of [Hal17, Thm. 7.1]: define K to be the set of all morphisms $x \colon \operatorname{Spec} \Bbbk \to S$ that are locally of finite type, where $\Bbbk$ is a field. Set $T = \coprod_{x \in K, \xi \in X(x)} T_\xi$. Then the X-scheme T is representable by smooth morphisms of algebraic spaces. We will be done if we can prove that it is representable by surjective morphisms of algebraic spaces. Since X is limit preserving, this assertion may be verified on affine X-schemes V of finite type over S. By construction, the image of the morphism $T \times_X V \to V$ contains all points of finite type; since the morphism is smooth, this image is also open. The result follows. $\square$

The following bootstrap result will be applied several times in this section.

Lemma 10.3. *Let S be a scheme and let X be an S-groupoid. Let W be an $X \times_S X$-scheme. Let $(\Delta_{X/S})_W \colon D_{X/S,W} \to W$ be the W-groupoid obtained as the pull-back of $\Delta_{X/S} \colon X \to X \times_S X$ along W. This is equivalent to a presheaf on Sch/W.*

(1) *Let $P \subseteq \mathbf{Aff}$ be a class of morphisms and let T be a $D_{X/S,W}$-scheme. If $X \to S$ is P-homogeneous at T, then $D_{X/S,W} \to W$ is P-homogeneous at T. In particular, if $X \to S$ is P-homogeneous, then $D_{X/S,W} \to W$ is P-homogeneous.*

(2) *Let T be a $D_{X/S,W}$-scheme. If $X \to S$ is* **Nil***-homogeneous at T, then $D_{X/S,W} \to W$ is* **Nil***-homogeneous at T and there are natural isomorphisms for every quasi-coherent $\mathcal{O}_T$-module M:*

$$\begin{aligned}\operatorname{Aut}_{(\Delta_{X/S})_W}(T,M) &\cong 0,\\ \operatorname{Def}_{(\Delta_{X/S})_W}(T,M) &\cong \operatorname{Aut}_{X/S}(T,M),\\ \operatorname{Obs}_{(\Delta_{X/S})_W}(T,M) &\subseteq \operatorname{Def}_{X/S}(T,M).\end{aligned}$$

(3) *If X is a stack over $(\mathsf{Sch}/S)_{\text{Ét}}$ (resp. $(\mathsf{Sch}/S)_{\text{fppf}}$), then $D_{X/S,W}$ is a sheaf over $(\mathsf{Sch}/W)_{\text{Ét}}$ (resp. $(\mathsf{Sch}/W)_{\text{fppf}}$).*
(4) *If X is limit preserving over S, then $D_{X/S,W}$ is limit preserving over W.*
(5) *If S is noetherian, W is locally of finite type over S and X is effective over S, then $D_{X/S,W}$ is effective over W.*

Proof. For (1), if $X \to S$ is P-homogeneous at T, then so is $X \times_S X$ and $\Delta_{X/S}$ [Hal17, Lem. 1.5(5,7,8)]. Thus, $D_{X/S,W} \to W$ is P-homogeneous at T [Hal17, Lem. 1.5(6)]. The assertion (2) follows from (1) and [Hal17, Cor. 6.14]. The assertions (3) and (5) are straightforward. Finally, (4) follows from [Hal17, Lem. 3.2(5,6)]. □

In following lemma, we establish that under very weak boundedness hypotheses, homogeneity at artinian schemes is sufficient to imply many other forms of homogeneity.

Lemma 10.4. *Let S be an excellent scheme. Let X be an S-groupoid that is*

(1) *a stack over $(\mathsf{Sch}/S)_{\text{Ét}}$,*
(2) *limit preserving,*
(3) *weakly effective,*
(4) $\mathbf{Art}^{\mathbf{triv}}$*-homogeneous, and*
(5) *has bounded automorphisms and deformations at every X-scheme* $\operatorname{Spec} \Bbbk$*, locally of finite type over S, such that $\Bbbk$ is a field (Conditions 6.1(i),6.1(ii)).*

The following assertions hold.

(a) *X is effective,*
(b) *X is* **rCl***-homogeneous.*
(c) *If X is* $\mathbf{Art}^{\mathbf{fin}}$*-homogeneous, then X is* **Int***-homogeneous.*
(d) *If X is* $\mathbf{Art}^{\mathbf{fin}}$*-homogeneous and* **DVR***-homogeneous and $\Delta_{X/S} \colon X \to X \times_S X$ is representable by algebraic spaces, then X is* **Aff***-homogeneous.*

Proof. That X is effective is Proposition 9.3. We first establish that if X satisfies the conditions (1)–(5) and ($\mathrm{H}_1^{\mathbf{rCl}}$) (resp. ($\mathrm{H}_1^{\mathbf{Int}}$)), then assertion (b) (resp. (c)) holds. Fix an **rCl**-nil (resp. **Fin**-nil) pair $(\operatorname{Spec} A \to \operatorname{Spec} B, \operatorname{Spec} A \to \operatorname{Spec} A')$ such that B is the completion of an $\mathcal{O}_S$-algebra B_0 of finite type at a maximal ideal $\mathfrak{m}_0$ and $A' \to A$ and $B \to A$ are of finite type. By Lemma B.3(5), it is sufficient to prove that the functor:

$$X(\operatorname{Spec} B') \to X(\operatorname{Spec} B) \times_{X(\operatorname{Spec} A)} X(\operatorname{Spec} A')$$

is essentially surjective, where $B' = B \times_A A'$. Since A is complete and $B \to A$ is finite, $A = \prod_{i=1}^n A_i$ in the category of B-algebras, where each A_i is a finite and local B-algebra. Arguing as in the proof of Lemma 1.9, we may thus reduce to the situation where A and A' are local.

Since $B' \to B$ is surjective with nilpotent kernel and B is local, B' is local with maximal ideal $\mathfrak{m}'$. For each integer $n \geq 0$ let $B'_n = B'/\mathfrak{m}'^{n+1}$, $B_n = B \otimes_{B'} B'_n$, $A_n =$

$A\otimes_{B'} B'_n$ and $A'_n = A'\otimes_{B'} B'_n$. The pair $(\operatorname{Spec} A_n \to \operatorname{Spec} B_n, \operatorname{Spec} A_n \to \operatorname{Spec} A'_n)$ is $\mathbf{Art^{triv}}$-nil (resp. $\mathbf{Art^{fin}}$-nil). Let $C_n = B_n \times_{A_n} A'_n$. Note that $\varprojlim_n C_n = B \times_A A' = B'$ and that for every $n \geq \ell$, the induced map $C_n/\mathfrak{m}'^{\ell+1}C_n \to C_\ell$ is surjective but not necessarily injective. Now $\mathbf{Art^{triv}}$-homogeneity (resp. $\mathbf{Art^{fin}}$-homogeneity) implies that

$$X(\operatorname{Spec} C_n) \to X(\operatorname{Spec} B_n) \times_{X(\operatorname{Spec} A_n)} X(\operatorname{Spec} A'_n)$$

is an equivalence. By Proposition 9.3, it follows that there is an equivalence

$$X(\operatorname{Spec} B) \times_{X(\operatorname{Spec} A)} X(\operatorname{Spec} A') \simeq \varprojlim_n (X(\operatorname{Spec} B_n) \times_{X(\operatorname{Spec} A_n)} X(\operatorname{Spec} A'_n)).$$

It remains to prove that the natural functor $X(\operatorname{Spec} B') \to \varprojlim_n X(\operatorname{Spec} C_n)$ is essentially surjective. To see this, we note that the map $B' \to C_n$ is surjective with kernel $K_n = B' \cap \mathfrak{m}^n(B \oplus A')$. By the Artin–Rees Lemma [AM69, Prop. 10.9], the filtration $\{K_n\}_{n\geq 0}$ on B' is $\mathfrak{m}$-stable. By Proposition 9.3, the claim follows.

To deduce (b) (resp. (c)) in general, we apply a bootstrapping procedure. By Lemma B.2(4), to prove that X satisfies $(\mathrm{H}_1^{\mathbf{rCl}})$ (resp. $(\mathrm{H}_1^{\mathbf{Int}})$), it is sufficient to prove that $D_{X/S,W}$ is $\mathbf{rCl}$-homogeneous (resp. $\mathbf{Int}$-homogeneous) for every affine scheme W of finite type over S. Fix an affine scheme W of finite type over S. First observe that W is excellent. By Lemma 10.3, $D_{X/S,W}$ satisfies the hypotheses (1)–(5) and the hypothesis in (b) (resp. (c)). Indeed, $\mathbf{Nil}$-homogeneity at $\operatorname{Spec} \Bbbk$ is equivalent to $\mathbf{Art^{triv}}$-homogeneity at $\operatorname{Spec} \Bbbk$. Thus it is sufficient to prove the Lemma under the additional assumption that the diagonal of $X \to S$ is a monomorphism. Repeating this process, we see that it is sufficient to prove the Lemma when $X \to S$ is a monomorphism. In this case, however, the diagonal of $X \to S$ is an isomorphism, thus is representable and consequently satisfies $(\mathrm{H}_1^{\mathbf{Aff}})$. The claim follows.

To establish (d), we note that since X has diagonal representable by algebraic spaces, X satisfies $(\mathrm{H}_1^{\mathbf{Aff}})$. By Lemma B.3(5), it is thus sufficient to prove that

$$X(\operatorname{Spec} A_3) \to X(\operatorname{Spec} A_2) \times_{X(\operatorname{Spec} A_0)} X(\operatorname{Spec} A_1)$$

is essentially surjective for every $\mathbf{Aff}$-nil pair $(\operatorname{Spec} A_0 \to \operatorname{Spec} A_2, \operatorname{Spec} A_0 \to \operatorname{Spec} A_1)$, where A_2 is the henselization of a finite type $\mathcal{O}_S$-algebra B at a maximal ideal $\mathfrak{m}$ and $A_2 \to A_0$ and $A_1 \to A_0$ are of finite type and $A_3 = A_2 \times_{A_0} A_1$.

Fix $(a_2, a_1, \alpha) \in X(\operatorname{Spec} A_2) \times_{X(\operatorname{Spec} A_0)} X(\operatorname{Spec} A_1)$, which we may regard as a diagram of X-schemes

$$\begin{tikzcd}
 & W_1 \arrow[dr] \arrow[drr, bend left] & & \\
W_0 \arrow[ur, hook, "j"] \arrow[dr, "p"'] & & W_3 \arrow[r, dashed] & X, \\
 & W_2 \arrow[ur, hook] \arrow[urr, bend right] & &
\end{tikzcd}$$

where $W_i = \operatorname{Spec} A_i$, that we must complete. Let $\Bbbk = A_2/\mathfrak{m}$; then $\operatorname{Spec} \Bbbk$ inherits an X-scheme structure from $\operatorname{Spec} A_2$. Now apply Theorem 10.1 to the X-scheme $\operatorname{Spec} \Bbbk$, which produces a pointed affine X-scheme (T, t), locally of finite type over S, which is formally versal at the closed point t. Let $W'_i = W_i \times_X T$ for $i = 0$, 1, 2 and let $p' \colon W'_0 \to W'_2$ be the pullback of $p \colon W_0 \to W_2$. Since X has diagonal representable by algebraic spaces, W'_i is an algebraic space, locally of finite type over W_i, for each i. By construction, the morphism $W'_2 \to W_2$ even admits a section $s_2 \colon W_2 \to W'_2$.

For $i = 0,\, 1,\, 2$ let $W_i'^{\mathrm{sm}} \subseteq W_i'$ denote the smooth locus of $W_i' \to W_i$, which is an open subset. By Lemma 2.2, T is formally smooth at t. Since X is **DVR**-homogeneous, T is formally smooth at every generization $t' \in |T|$ of t (Lemma 2.15). Thus $W_i'^{\mathrm{sm}}$ contains the preimage of $\mathrm{Spec}(\mathcal{O}_{T,t})$ under $W_i' \to T$. Let $Z_2 = p'(W_0' \setminus j'^{-1}(W_1'^{\mathrm{sm}}))$, $W_2'' = W_2'^{\mathrm{sm}} \setminus \overline{Z_2}$, $W_0'' = p'^{-1}(W_2'')$ and $W_1'' = j'(W_0'')$, which we regard as open subsets of $W_i'^{\mathrm{sm}}$. We claim that the section $s_2 \colon W_2 \to W_2'$ factors through W_2''. To see this, it is sufficient to check that $\overline{Z_2}$ does not contain any points above t. But Z_2 does not contain any points above $\mathrm{Spec}(\mathcal{O}_{T,t})$ and since every point of $\overline{Z_2}$ is a specialization of a point in Z_2, the claim follows.

By restriction, there is an induced section $s_0 \colon W_0 \to W_0''$. Since $W_1'' \to W_1$ is smooth and W_0 is affine, the section s_0 lifts to a section $s_1 \colon W_1 \to W_1''$ of $W_1'' \to W_1$. By [Hal17, Lem. A.4], there is a commutative diagram of S-schemes:

$$\begin{tikzcd}
& W_0'' \arrow[rr, hook] \arrow[dl] \arrow[dd] & & W_1'' \arrow[dl] \arrow[dd] \\
W_2'' \arrow[rr, hook, crossing over] \arrow[dd] & & W_3'' & \\
& W_0 \arrow[rr, hook] \arrow[dl] & & W_1 \arrow[dl] \\
W_2 \arrow[rr, hook] & & W_3 \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where all faces of the cube are cartesian, the top and bottom faces are cocartesian, and the map $W_3'' \to W_3$ is flat. Since the top square is cocartesian, and there are compatible maps $W_i'' \to T$ for $i \neq 3$, there is a uniquely induced map $W_3'' \to T$. The sections s_i for $i = 0,\, 1,\, 2$ glue to a section $s_3 \colon W_3 \to W_3''$ of $W_3'' \to W_3$. Taking the composition $W_3 \to W_3'' \to T \to X$ proves the result. $\square$

We now prove a version of the Main Theorem where we assume that the diagonal is representable.

Theorem 10.5. *Let S be an excellent scheme. Then a category X, fibered in groupoids over the category of S-schemes, Sch/S, is an algebraic stack, locally of finite presentation over S, if and only if it satisfies the conditions of the Main Theorem and*

(7) *the diagonal $\Delta_{X/S} \colon X \to X \times_S X$ is representable by algebraic spaces.*

Proof. We will use the criteria of Proposition 10.2. Clearly the conditions of limit preservation (2), weak effectivity (3), bounded deformations (6a) and diagonal representable by algebraic spaces (7) of Proposition 10.2 are satisfied. Either of the stack hypotheses—(1) or (1′)—imply the étale stack condition (1) of Proposition 10.2.

Either the $\mathbf{Art}^{\mathbf{insep}}$-homogeneity hypothesis (4′), or (1) and $\mathbf{Art}^{\mathbf{triv}}$-homogeneity (4) and Lemma 1.9, imply that X is $\mathbf{Art}^{\mathbf{fin}}$-homogeneous. By Lemma 10.4, (1) or (1′), combined with (2) and (3) and bounded automorphisms and deformations (5a), implies that X is **Int**-homogeneous. In particular, (4)–(5) of Proposition 10.2 are satisfied.

Now X has constructible obstructions, by (6b) and Lemma 7.5(1). Since X also has constructible deformations (5b), it has constructible extensions (Proposition 6.5(1)). Thus, X satisfies (6b) of Proposition 10.2. Similarly by Lemma 7.5(2) and (6c), X has Zariski local obstructions. Since X also has Zariski local deformations (5c), Proposition 6.5(2) implies that X satisfies (6c) of Proposition 10.2.

If S is Jacobson (α), then X satisfies (6c) of Proposition 10.2 (Lemma 2.12), without assuming (5c) and (6c).

If X is **DVR**-homogeneous (β), then Lemma 10.4(d) implies that X is **Aff**-homogeneous; thus, X satisfies (6c) of Proposition 10.2 (Lemma 1.8(5)), without assuming (5c) and (6c). Moreover, Lemma 8.4 implies that (6b) may be substituted for Condition 8.3. The result follows. □

We are now ready to prove the Main Theorem.

Proof of Main Theorem. We will do a bootstrapping process, similar to the proof of [Hal17, Thm. A]. In this instance, however, we must be more careful because we are working with a weaker homogeneity assumption.

The hypotheses (1) and (4), or (γ), imply that X is $\mathbf{Art}^{\mathbf{fin}}$-homogeneous (Lemma 1.9). By Lemma 10.4, X is effective and **Int**-homogeneous.

Let W be an $X \times_S X$-scheme, affine and locally of finite type over S. By Lemma 10.3, the W-groupoid $(\Delta_{X/S})_W : D_{X/S,W} \to W$ satisfies the conditions of the Main Theorem. Let V be a $D_{X/S,W} \times_W D_{X/S,W}$-scheme, affine and locally of finite type over W. By Lemma 10.3, the V-groupoid $(\Delta_{D_{X/S,W}/W})_V : D_{D_{X/S,W},V} \to V$ satisfies the conditions of the Main Theorem. Note, however, that $(\Delta_{D_{X/S,W}/W})_V$ is a monomorphism, so has representable diagonal. By Theorem 10.5, $(\Delta_{D_{X/S,W}/W})_V$ is algebraic and locally of finite presentation over V, so $(\Delta_{X/S})_W$ has diagonal representable by algebraic spaces. By Theorem 10.5 again, $(\Delta_{X/S})_W$ is algebraic and locally of finite presentation over W; so X has diagonal representable by algebraic spaces. A final application of Theorem 10.5 informs us that X is algebraic and locally of finite presentation over S. □

11. Comparison with other criteria

In this section we compare our algebraicity criterion with Artin's criteria [Art69b, Art74], Starr's criterion [Sta06], the criterion of the first author [Hal17], the criterion in the stacks project [Stacks], and Flenner's criterion for openness of versality [Fle81].

11.1. **Artin's algebraicity criterion for functors.** In [Art69b, Thm. 5.3] Artin assumes $[0']$=(1) (fppf stack), $[1']$=(2) (limit preserving) and $[2']$=(3) (effectivity). Further $[4']$(b)+$[5']$(a) is **Nil**-homogeneity for irreducible schemes, which implies (4). His $[4']$(a)+(c) is boundedness, Zariski-localization and constructibility of deformations (Conditions 6.1(ii), 6.4(ii) and 6.3(ii)). His $[5']$(c) is Condition 8.3 (constructibility of obstructions). Finally, $[5']$(b) together with $[4']$(a) and $[4']$(b) implies **DVR**-homogeneity so we are in the setting of (β). Conditions on automorphisms are of course redundant for functors. Condition $[3']$(a) is only used to assure that the resulting algebraic space is locally separated (resp. separated) and condition $[3']$(b) guarantees that it is quasi-separated. If one is willing to accept non quasi-separated algebraic spaces, no separation assumptions are necessary.

11.2. **Artin's algebraicity criterion for stacks.** Let us begin with correcting two typos in the statement of [Art74, Thm. 5.3]. In (1) the condition should be that (S1$'$,2) holds for F, not merely (S1,2), and in (2) the canonical map should be fully faithful with dense image, not merely faithful with dense image. Otherwise it is not possible to bootstrap and deduce algebraicity of the diagonal.

Artin assumes that X is a stack for the étale topology, and that X is limit preserving. He assumes (1) that the Schlessinger conditions (S1′,2) hold and boundedness of automorphisms. In our terminology, (S1′) is **rCl**-homogeneity, which implies $\mathbf{Art}^{\mathbf{triv}}$-homogeneity, our (4). The other two conditions are exactly boundedness of automorphisms and deformations (5a). Artin's condition (2) is our (3) (effectivity). Artin's condition (3) is étale localization and constructibility of automorphisms, deformations and obstructions, and compatibility with completions for automorphisms and deformations. The constructibility condition is slightly stronger than our (5b)+(6b) and the étale localization condition implies the much weaker (5c)+(6c). We do not use compatibility with completions. Finally, Artin's condition (4) implies that the double diagonal of the stack is quasi-compact and this condition can be omitted if we work with stacks without separation conditions. Thus [Art74, Thm. 5.3] follows from our main theorem, except that Artin only assumes that the groupoid is a stack in the étale topology. This is related to the issue when comparing formal versality to formal smoothness mentioned in the introduction and discussed in Remark 2.8.

Remark 11.1. That automorphisms and deformations are sufficiently compatible with completions for Artin's proof to go through actually follows from the other conditions. In fact, let A be a noetherian local ring with maximal ideal $\mathfrak{m}$, let $T = \operatorname{Spec} A$ and let $T \to X$ be given. Then the injectivity of the comparison map

$$\varphi\colon \operatorname{Def}_{X/S}(T, M) \otimes_A \hat{A} \to \varprojlim_n \operatorname{Def}_{X/S}(T, M/\mathfrak{m}^n M)$$

for a finitely generated A-module M follows from the boundedness of $\operatorname{Def}_{X/S}(T, -)$, see Remark 3.8. If $T \to X$ is formally versal, then φ is also surjective. Indeed, from (S1) it follows that $\operatorname{Der}_S(T, M/\mathfrak{m}^n M) \to \operatorname{Def}_{X/S}(T, M/\mathfrak{m}^n M)$ is surjective for all n, so the composition $\operatorname{Der}_S(T, M) \otimes_A \hat{A} \cong \varprojlim_n \operatorname{Der}_S(T, M/\mathfrak{m}^n M) \to \varprojlim_n \operatorname{Def}_{X/S}(T, M/\mathfrak{m}^n M)$, which factors through φ, is surjective.

The variant [Sta06, Prop. 1.1], due to Starr, has the same conditions as [Art74, Thm. 5.3] except that it is phrased in a relative setting. From Section 6, it is clear that our conditions can be composed. The salient point is that with **rCl**-homogeneity (or even with just (S1), i.e., **rCl**-semihomogeneity, as in [Fle81]), there is always a linear minimal obstruction theory. There is further an exact sequence relating the minimal obstruction theories for the composition of two morphisms [Hal17, Prop. 6.13]. Thus [Sta06, Prop. 1.1] also follows from our main theorem.

We wish to point out that Starr proves openness of versality [Sta06, Thm. 2.15] using his formalism of generic extenders [Sta06, Defn. 2.7]. This is similar to our Condition 8.3 (and Artin's analogous condition in his algebraicity criterion for functors). The main difference is that he also assumes homogeneity along localizations (not just Zariski localizations), as opposed to **DVR**-homogeneity.

11.3. **The criterion** [Hal17] **using coherence.** There are two differences between [Hal17, Thm. A] and our main theorem. The first is that Condition (4) is strengthened to **Aff**-homogeneity. As this includes **DVR**-homogeneity, (5c) and (6c) become redundant. Zariski localization also follows immediately from **Aff**-homogeneity without involving **DVR**-homogeneity, see the discussion after Condition 2.11. We thus have the following version of our main theorem.

Theorem 11.2. *Let S be an excellent scheme. Then a category X that is fibered in groupoids over the category of S-schemes, Sch/S, is an algebraic stack that is locally of finite presentation over S, if and only if it satisfies the following conditions.*

- (1′) *X is a stack over $(\mathsf{Sch}/S)_{\text{Ét}}$.*
- (2) *X is limit preserving.*
- (3) *X is effective.*
- (4″) *X is* **Aff***-homogeneous.*
- (5a) *Automorphisms and deformations are bounded (Conditions 6.1(i)–6.1(ii)).*
- (5b) *Automorphisms and deformations are constructible (Conditions 6.3(i)–6.3(ii)).*
- (6b) *Obstructions are constructible (Condition 6.3(iii), or 7.3, or 8.3).*

The second difference is that (5a), (5b) and (6b) are replaced with the condition that $\mathrm{Aut}_{X/S}(T,-)$, $\mathrm{Def}_{X/S}(T,-)$, $\mathrm{Obs}_{X/S}(T,-)$ are *coherent* functors. This implies that the functors are bounded and CB (Example 3.6), hence satisfy (5a), (5b) and (6b).

11.4. **The criterion in the Stacks project.** In the Stacks project, the basic version of Artin's axiom [Stacks, 07XJ,07Y5] requires that

- [0] X is a stack in the étale topology,
- [1] X is limit preserving,
- [2] X is $\mathbf{Art^{fin}}$-homogeneous (this is the Rim–Schlessinger condition RS),
- [3] $\mathrm{Aut}_{X/S}(\mathrm{Spec}\, k, k)$ and $\mathrm{Def}_{X/S}(\mathrm{Spec}\, k, k)$ are finite dimensional,
- [4] X is effective, and
- [5] X, Δ_X and Δ_{Δ_X} satisfy openness of versality.

There is also a criterion for when X satisfies openness of versality [Stacks, 07YU] using naive obstruction theories with finitely generated cohomology groups. This uses the (RS*)-condition which is our **Aff**-homogeneity [Stacks, 07Y8]. The existence of the naive obstruction theory implies that $\mathrm{Aut}_{X/S}(T,-)$, $\mathrm{Def}_{X/S}(T,-)$, $\mathrm{Obs}_{X/S}(T,-)$ are bounded and CB (Example 3.4), hence satisfy (5a), (5b) and (6b) when T is an affine X-scheme that is locally of finite type over S.

In [Stacks], the condition that the base scheme S is excellent is replaced with the condition that its local rings are G-rings. In our treatment, excellency enters at two places: in the application of Néron–Popescu desingularization in Proposition 10.2 via [CJ02] and in the context of **DVR**-homogeneity in Lemma 2.15. In both cases, excellency can be replaced with the condition that the local rings are G-rings without modifying the proofs.

11.5. **Flenner's criterion for openness of versality.** Flenner does not give a precise analogue of our main theorem, but his main result [Fle81, Satz 4.3] is a criterion for the openness of versality. In his criterion he has a limit preserving S-groupoid which satisfies (S1)–(S4). The first condition (S1) is identical to Artin's condition (S1), i.e., **rCl**-semihomogeneity. The second condition (S2) is boundedness and Zariski localization of deformations. The third condition (S3) is boundedness and Zariski localization of the minimal obstruction theory. Finally (S4) is constructibility of deformations and obstructions. The Zariski localization condition is incorporated in the formulation of (S3) and (S4) which deals with *sheaves* of deformation and obstructions modules. His (S2)–(S4) are marginally stronger than our conditions, for example, treating arbitrary schemes instead of irreducible schemes. Theorem [Fle81, Satz 4.3] thus becomes the first part of Theorem 4.4, in view of Section 6, except that we assume **rCl**-homogeneity instead of **rCl**-semihomogeneity. This is a pragmatic choice that simplifies matters since $\mathrm{Exal}_X(T, M)$ becomes a module instead of a pointed set. Also, in any algebraicity criterion, we would need homogeneity to deduce that the diagonal is algebraic and, conversely, if the diagonal is algebraic, then semihomogeneity implies homogeneity.

11.6. **Criterion for local constructibility.** There is a useful criterion for when a sheaf (or a stack) is locally constructible, that is, when it corresponds to an étale algebraic space (or algebraic stack) [Art73, VII.7.2]:

Theorem 11.3. *Let S be an excellent scheme. Then a category X that is fibered in groupoids over Sch/S, is an algebraic stack that is étale over S, if and only if it satisfies the following conditions.*

(1) *X is a stack over $(\mathsf{Sch}/S)_{\text{Ét}}$.*
(2) *X is limit preserving.*
(3) *$X(B) \to X(B/\mathfrak{m})$ is an equivalence of categories for every local noetherian ring $(B, \mathfrak{m})$, such that B is $\mathfrak{m}$-adically complete, with an S-scheme structure $\operatorname{Spec} B \to S$ such that the induced morphism $\operatorname{Spec}(B/\mathfrak{m}) \to S$ is of finite type.*

The necessity of the conditions is clear. That the conditions are sufficient can be proven directly as follows. Let $j\colon (\mathsf{Sch}/S)_{\text{Ét}} \to S_{\text{ét}}$ denote the morphism of topoi corresponding to the inclusion of the small étale site into the big étale site. It is enough to prove that $j^{-1}j_*X \to X$ is an equivalence. As X is limit preserving, it is enough to verify that $f^*(X|_{S_{\text{ét}}}) \to X|_{T_{\text{ét}}}$ is an equivalence for every morphism $f\colon T \to S$ locally of finite type, and this can be checked on stalks at points of finite type. Therefore, it suffices to prove that $X(B) \to X(B/\mathfrak{m})$ is an equivalence when B is the henselization of $\mathcal{O}_{T,t}$, for every $t \in |T|$ of finite type. This follows from general Néron–Popescu desingularization and the three conditions.

A proof more in the lines of this paper goes as follows: from (3) it follows that: X is $\mathbf{Art^{fin}}$-homogeneous; X is effective; and $X \to S$ is formally étale at every point of finite type. In particular, $\mathrm{Aut}_{X/S}(T, N) = \mathrm{Def}_{X/S}(T, N) = \mathrm{Obs}_{X/S}(T, N) = 0$ for every X-scheme T that is of finite type over S and every quasi-coherent $\mathcal{O}_T$-module N with support that is artinian (use Lemmas 5.1 and 5.4). Thus, $\mathrm{Aut}_{X/S}(T, -) = \mathrm{Def}_{X/S}(T, -) = 0$ by Theorem 3.7. Theorem 11.3 would follow from the main theorem if we also can show that $\mathrm{Obs}_{X/S}(T, -) = 0$. As we do not yet know that $\mathrm{Obs}_{X/S}(T, -)$ is half-exact, it is apparently difficult to deduce that $\mathrm{Obs}_{X/S}(T, -) = 0$ without invoking Popescu desingularization. A more elementary approach, that does not rely on the main theorem, is to note that given an X-scheme T that is locally of finite presentation over S, and a point $t \in |T|$ of finite type, then $T \to X$ is formally smooth at t if and only $T \to S$ is formally smooth at t. Thus, openness of formal smoothness for $T \to X$ follows.

Appendix A. Approximation of integral morphisms

In this appendix, we give an approximation result for integral homomorphisms of rings.

Lemma A.1. *Let A be a ring, let B be an A-algebra and let C be a B-algebra. Assume that B and C are integral A-algebras. Then there exists a filtered system*

$(B_\lambda \to C_\lambda)_\lambda$ of finite and finitely presented A-algebras, with direct limit $B \to C$. In addition, if $A \to B$ (resp. $B \to C$, resp. $A \to C$) has one of the properties:

(1) *surjective,*
(2) *surjective with nilpotent kernel,*

then the system can be chosen such that the morphisms $A \to B_\lambda$ (resp. $B_\lambda \to C_\lambda$, resp. $A \to C_\lambda$) all have the corresponding property.

If we start with a system satisfying the first part of the lemma, then it is not always the case that the second part holds after increasing λ. Therefore, the approximation $B_\lambda \to C_\lambda$ has to be built with the second part in mind.

Proof of Lemma A.1. Let Λ be the set of finite subsets of $B \amalg C$, or, if $B \to C$ is surjective, only those of B. For $\lambda = \lambda_B \cup \lambda_C \in \Lambda$, let $B^\circ_\lambda \subseteq B$ be the A-subalgebra generated by λ_B and let $C^\circ_\lambda \subseteq C$ be the A-subalgebra generated by λ_C and the image of λ_B in C.

Then $B = \varinjlim_{\lambda\in\Lambda} B^\circ_\lambda$ and $C = \varinjlim_{\lambda\in\Lambda} C^\circ_\lambda$ and we have homomorphisms $B^\circ_\lambda \to C^\circ_\lambda$ for all λ. Moreover, if $A \to B$ (resp. $B \to C$, resp. $A \to C$) is surjective or surjective with nilpotent kernel then so is $A \to B^\circ_\lambda$ (resp. $B^\circ_\lambda \to C^\circ_\lambda$, resp. $A \to C^\circ_\lambda$) for every λ.

For every λ, let $P_\lambda = A[x_i \ : \ i \in \lambda_B]$ and $Q_\lambda = A[y_j \ : \ j \in \lambda]$ be polynomial rings and let $P_\lambda \to B^\circ_\lambda$ and $Q_\lambda \to C^\circ_\lambda$ be the natural surjections. We have homomorphisms $P_\lambda \to Q_\lambda$ compatible with $B^\circ_\lambda \to C^\circ_\lambda$ and if $B \to C$ is surjective, then $P_\lambda = Q_\lambda$. For a finite subset $L \subseteq \Lambda$, let $P_L = \bigotimes_{\lambda\in L} P_\lambda$ and $Q_L = \bigotimes_{\lambda\in L} Q_\lambda$, where the tensor products are over A.

For fixed $L \subseteq \Lambda$ choose finitely generated ideals $I_L \subseteq \ker(P_L \to B)$ and $I_L Q_L \subseteq J_L \subseteq \ker(Q_L \to C)$ and let $B_L = P_L/I_L$ and $C_L = Q_L/J_L$. If $A \to B$ (resp. $A \to C$) is surjective, then for sufficiently large I_L (resp. J_L), we have that $A \to B_L$ (resp. $A \to C_L$) is surjective. If $B \to C$ is surjective, then by construction $P_L = Q_L$ so $B_L \to C_L$ is surjective. If, in addition, $B \to C$ has nilpotent kernel with nilpotency index n, then we replace I_L with $I_L + J_L^n$ so that $B_L \to C_L$ has nilpotent kernel.

Consider the set Ξ of pairs $\xi = (L, I_L, J_L)$ where $L \subseteq \Lambda$ is a finite subset, and $I_L \subseteq P_L$ and $J_L \subseteq Q_L$ are finitely generated ideals as in the previous paragraph. Then $(B_L \to C_L)_\xi$ is a filtered system of finite and finitely presented A-algebras with direct limit $(B \to C)$ which satisfies the conditions of the lemma. $\square$

Lemma A.2. *Let $f\colon X \to Y$ be a morphism of affine schemes. Let P be one of the properties* **Nil**, **Cl**, **rNil**, **rCl**, **Int** *or* **Aff** *(cf. Section 1). If f has property P, then there exists a filtered system $(f_\lambda\colon X_\lambda \to Y)_\lambda$ with inverse limit $f\colon X \to Y$ such that every f_λ is of finite presentation with property P.*

Proof. The result is standard when $P \in \{\mathbf{Cl}, \mathbf{Nil}, \mathbf{Int}, \mathbf{Aff}\}$. For $P = \mathbf{rNil}$ (resp. $P = \mathbf{rCl}$), choose a nilpotent immersion $X_0 \to X$ such that $X_0 \to X \to Y$ is **Nil** (resp. **Cl**). The lemma then follows from Lemma A.1 with $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$ and $X_0 = \operatorname{Spec} C$. $\square$

Fix a scheme S and consider the category of diagrams $[Y \xleftarrow{f} X \xrightarrow{i} X']$ of S-schemes. A morphism of diagrams $\Phi\colon [Y_1 \xleftarrow{f_1} X_1 \xrightarrow{i_1} X'_1] \to [Y_2 \xleftarrow{f_2} X_2 \xrightarrow{i_2} X'_2]$ consists of morphisms $\Phi_Y\colon Y_1 \to Y_2$, $\Phi_X\colon X_1 \to X_2$ and $\Phi_{X'}\colon X'_1 \to X'_2$ such that the natural diagram is commutative but not necessarily cartesian. We say that Φ is affine if Φ_Y, Φ_X and $\Phi_{X'}$ are affine. Given an inverse system of diagrams with affine bonding maps, the inverse limit exists and is calculated component by component.

Proposition A.3. *Let S be an affine scheme and let P be one of the properties* **Nil**, **Cl**, **rNil**, **rCl**, **Int** *or* **Aff**. *Let $\mathbf{W} = [Y \xleftarrow{f} X \xrightarrow{i} X']$ be a diagram of affine S-schemes where i is* **Nil**, *and f is P. Then $\mathbf{W}$ is an inverse limit of diagrams $\mathbf{W}_\lambda = [Y_\lambda \xleftarrow{f_\lambda} X_\lambda \xrightarrow{i_\lambda} X'_\lambda]$ of affine finitely presented S-schemes where i_λ is* **Nil**, *and f_λ is P. Moreover, if we let $Y' = Y \amalg_X X'$ and $Y'_\lambda = Y_\lambda \amalg_{X_\lambda} X'_\lambda$ denote the push-outs, then $Y' = \varprojlim_{\lambda\in\Lambda} Y'_\lambda$.*

Proof. We begin by looking at the induced diagram $[Y \xrightarrow{j} Y' \xleftarrow{g} X']$. As j is a nilpotent closed immersion it follows that g has property P. We will write this diagram as an inverse limit of diagrams $[Y_\lambda \xrightarrow{j_\lambda} \overline{Y}'_\lambda \xleftarrow{g_\lambda} X'_\lambda]$ of finite presentation over S where j_λ is **Nil** and g_λ has property P. To this end, we begin by writing (using Lemma A.2):

(1) $Y' = \varprojlim \overline{Y}'_\alpha$ where $\overline{Y}'_\alpha \to S$ are affine and of finite presentation;
(2) $X' = \varprojlim X'_\beta$ where $X'_\beta \to Y'$ are P and of finite presentation; and
(3) $Y = \varprojlim Y_\gamma$ where $Y_\gamma \to Y'$ are **Nil** and of finite presentation.

For every pair (β,γ) there is ([EGA, **IV**.8.10.5]) an index $\alpha_0(\beta,\gamma)$, and a cartesian diagram

$$\begin{tikzcd} Y_\gamma \arrow[r, hook] \arrow[d] & Y' \arrow[d] & X'_\beta \arrow[l] \arrow[d] \\ Y_{\alpha_0(\beta,\gamma)\beta\gamma} \arrow[r, hook] & \overline{Y}'_{\alpha_0(\beta,\gamma)} & X'_{\alpha_0(\beta,\gamma)\beta\gamma}. \arrow[l] \end{tikzcd}$$

where $X'_{\alpha_0(\beta,\gamma)\beta\gamma} \to \overline{Y}'_{\alpha_0(\beta,\gamma)}$ and $Y_{\alpha_0(\beta,\gamma)\beta\gamma} \to \overline{Y}'_{\alpha_0(\beta,\gamma)}$ are morphisms of finite presentation that are P and **Nil** respectively.

For every $\alpha \geq \alpha_0(\beta,\gamma)$ we also let $[Y_{\alpha\beta\gamma} \to \overline{Y}'_\alpha \leftarrow X'_{\alpha\beta\gamma}]$ denote the pull-back along $\overline{Y}'_\alpha \to \overline{Y}'_{\alpha_0(\beta,\gamma)}$. Let $I = \{(\beta,\gamma,\alpha)\}$ be the set of indices such that $\alpha \geq \alpha_0(\beta,\gamma)$. For every finite subset $J \subseteq I$, we let

$$\overline{Y}'_J = \prod_{(\beta,\gamma,\alpha)\in J} \overline{Y}'_\alpha, \quad Y_J = \prod_{(\beta,\gamma,\alpha)\in J} Y_{\alpha\beta\gamma}, \quad \text{and} \quad X'_J = \prod_{(\beta,\gamma,\alpha)\in J} X'_{\alpha\beta\gamma}$$

where the products are taken over S. The finite subsets $J \subseteq I$ form a partially ordered set under inclusion and the induced morphisms:

$$\overline{Y}' \to \varprojlim_J \overline{Y}'_J, \quad Y \to \varprojlim_J Y_J, \quad \text{and} \quad X' \to \varprojlim_J X'_J$$

are closed immersions. Now, let $K_{Y_J} = \ker(\mathcal{O}_{Y_J} \to (g_J)_*\mathcal{O}_Y)$ and similarly for $K_{\overline{Y}'_J}$ and $K_{X'_J}$. Note that $K_{\overline{Y}'_J}\mathcal{O}_{Y_J} \subseteq K_{Y_J}$ and $K_{\overline{Y}'_J}\mathcal{O}_{X'_J} \subseteq K_{X'_J}$. We then let $\Lambda = \{(J, R_{Y_J}, R_{\overline{Y}'_J}, R_{X'_J})\}$ where $J \subseteq I$ is a finite subset and $R_{Y_J} \subseteq K_{Y_J}$, $R_{\overline{Y}'_J} \subseteq K_{\overline{Y}'_J}$ and $R_{X'_J} \subseteq K_{X'_J}$ are finitely generated ideals such that $R_{\overline{Y}'_J}\mathcal{O}_{Y_J} \subseteq R_{Y_J}$ and $R_{\overline{Y}'_J}\mathcal{O}_{X'_J} \subseteq R_{X'_J}$. For every $\lambda \in \Lambda$ we put

$$\overline{Y}'_\lambda = \operatorname{Spec}(\mathcal{O}_{\overline{Y}'_J}/R_{\overline{Y}'_J}), \quad Y_\lambda = \operatorname{Spec}(\mathcal{O}_{Y_J}/R_{Y_J}), \quad \text{and} \quad X'_\lambda = \operatorname{Spec}(\mathcal{O}_{X'_J}/R_{X'_J})$$

Then $[Y \to Y' \leftarrow X'] = \varprojlim_\lambda [Y_\lambda \to \overline{Y}'_\lambda \leftarrow X'_\lambda]$. Finally, we take $X_\lambda = X'_\lambda \times_{\overline{Y}'_\lambda} Y_\lambda$ so that $[Y \xleftarrow{f} X \xrightarrow{i} X'] = \varprojlim_\lambda [Y_\lambda \xleftarrow{f_\lambda} X_\lambda \xrightarrow{i_\lambda} X']$. Indeed, $X = X' \times_{Y'} Y$ and inverse limits commute with fiber products.

For the last assertion, we note that all schemes are affine and that there are exact sequences

$$0 \to \Gamma(\mathcal{O}_{Y'}) \to \Gamma(\mathcal{O}_Y) \times \Gamma(\mathcal{O}_{X'}) \to \Gamma(\mathcal{O}_X) \to 0$$
$$0 \to \Gamma(\mathcal{O}_{Y'_\lambda}) \to \Gamma(\mathcal{O}_{Y_\lambda}) \times \Gamma(\mathcal{O}_{X'_\lambda}) \to \Gamma(\mathcal{O}_{X_\lambda}) \to 0, \quad \forall \lambda \in \Lambda.$$

Note that Y'_λ can be different from $\overline{Y}'_\lambda$. As direct limits of rings are exact it follows that $Y' = \varprojlim Y'_\lambda$. □

Appendix B. Bootstrapping homogeneity

The following notation will be useful.

Notation B.1. Fix a scheme S and a 1-morphism of S-groupoids $\Phi\colon Y \to Z$.

- If W is a $Y \times_Z Y$-scheme, let $(\Delta_\Phi)_W\colon D_{\Phi,W} \to W$ denote the W-groupoid obtained by pulling back $\Delta_\Phi\colon Y \to Y \times_Z Y$ along $W \to Y \times_Z Y$.
- Fix a class P of morphisms of S-schemes. For a P-nil square over S as in (1.1), let
$$\Lambda_{Y,T'}\colon Y(T') \to Y(V') \times_{Y(V)} Y(T)$$
denote the natural functor.

The following bootstrapping lemma provides a powerful technique to verify condition (H_1^P) of Definition 1.3.

Lemma B.2. *Fix a scheme S, a class $P \subseteq$ **Aff** of morphisms of S-schemes and a 1-morphism of S-groupoids $\Phi\colon Y \to Z$. If Z satisfies (H_1^P), then the following conditions are equivalent:*

1. *Y satisfies (H_1^P);*
2. *for every geometric P-nil square over S as in* (1.1), *$\Lambda_{Y,T'}$ is fully faithful;*
3. *for every $Y \times_Z Y$-scheme W, the W-groupoid $D_{\Phi,W}$ is P-homogeneous.*

In addition, if Y and Z are limit preserving Zariski stacks and P is Zariski local, then these conditions are equivalent to the following:

4. *Φ satisfies condition* (3) *for all W affine and of finite presentation over S.*

In particular, if $\Delta_{Y/S}$ is representable by algebraic spaces, then Y satisfies $(\mathrm{H}_1^{\mathbf{Aff}})$.

Condition (3) is not equivalent to P-homogeneity of Δ_Φ unless we a priori know that $Y \times_Z Y$ is P-homogeneous—an uninteresting situation.

Proof. For (1) $\implies$ (2), fix a geometric P-nil square over S as in (1.1). We must prove that the functor $\Lambda_{Y,T'}$ is fully faithful, that is, if y_1 and y_2 are two Y-scheme structures on T' such that $\Lambda_{Y,T'}(y_1) \cong \Lambda_{Y,T'}(y_2)$, then there is a unique isomorphism of Y-schemes $y_1 \cong y_2$. Since Y satisfies (H_1^P), any Y-scheme structure on T' makes the resulting P-nil square cocartesian (because geometric P-nil squares over S are cocartesian). The claim follows.

For (2) $\implies$ (3), we fix a $Y \times_Z Y$-scheme W. To establish (H_1^P) for $D_{\Phi,W}$, it is sufficient to prove that a geometric P-nil square over $D_{\Phi,W}$ as in (1.1) is cocartesian. There is a canonical map $T' \to W$ and this corresponds to two maps $y_1, y_2\colon T' \to Y$ and a 2-isomorphism τ between $\Phi \circ y_1$ and $\Phi \circ y_2$. If Q is a $D_{\Phi,W}$-scheme with compatible maps from T and V', we obtain a map $T' \to Q$ over W and hence two maps $T' \to D_{\Phi,W}$. These two maps correspond to 2-isomorphisms α, β between y_1 and y_2 compatible with τ and such that $\Lambda_{Y,T'}(\alpha) = \Lambda_{Y,T'}(\beta)$. Since $\Lambda_{Y,T'}$ is faithful, we conclude that $\alpha = \beta$ and hence that the square is cocartesian over $D_{\Phi,W}$.

To establish (H_2^P) for $D_{\Phi,W}$, it is sufficient to prove that every P-nil pair over $D_{\Phi,W}$ may be completed to a P-nil square. Clearly, we can complete such a P-nil pair to a geometric P-nil square over W as in (1.1). It remains to promote T' to a $D_{\Phi,W}$-scheme. However, $T' \to W \to Y \times_Z Y$ factors through Y because $\Lambda_{Y,T'}$ is full, $\Lambda_{Z,T'}$ is faithful, and T' comes from a P-nil pair over Y. Thus, T' lifts to a $D_{\Phi,W}$-scheme and the claim follows.

For (3) $\implies$ (1), we have to prove that a geometric P-nil square over Y as in (1.1) is cocartesian. Thus, we must prove that if Q is a Y-scheme that fits into the following P-nil square

$$\begin{tikzcd} V \arrow[r, "p"] \arrow[d, hook, "j"'] & T \arrow[d, "a"] \\ V' \arrow[r, "b"] & Q, \end{tikzcd}$$

then there is a unique compatible map of Y-schemes $T' \to Q$. Note that since Z satisfies (H_1^P), there is a unique Z-morphism $T' \to Q$. Thus, it is sufficient to prove that the two induced Y-scheme structures on T' coincide. So we may regard T' as a $(Y \times_Z Y)$-scheme and let $D_{\Phi,T'} \to T'$ be the pullback of $\Delta_{Y/Z}$ to T'. Since $D_{\Phi,T'}$ is P-homogeneous and $(V \to T, V \to V')$ is a P-nil pair over $D_{\Phi,T'}$, it follows that the geometric P-nil square over Y is uniquely a cocartesian P-nil square over $D_{\Phi,T'}$. The claim follows.

Noting [Hal17, Lem. 1.5(7)], the equivalence (4) $\iff$ (3) is routine. $\square$

The following lemma (cf. [Hal17, Lem. 1.5(4)]) is particularly useful when combined with Lemma B.2.

Lemma B.3. *Fix a scheme S and a limit preserving étale S-stack X. Let P be one of the properties* **Nil**, **Cl**, **rNil**, **rCl**, **Int** *or* **Aff**. *If X satisfies* (H_1^P), *then the following conditions are equivalent.*

1. *X is P-homogeneous;*
2. *$\Lambda_{X,T'}$ is essentially surjective for every geometric P-nil square over S as in* (1.1) *where T, V and V' are affine;*
3. *Condition* (2) *holds when T, V and V' are of finite presentation over S; or*
4. *Condition* (2) *holds when T is the henselization of an affine scheme of finite presentation over S at a closed point, and $V \to T$, $V \to V'$ are of finite presentation;*

If in addition $P \subseteq$ **Int** *and S is excellent, then these conditions are equivalent to the following:*

5. *Condition* (2) *holds when T' is the completion of an affine scheme of finite type over S at a closed point, and $V \to T$ is finite.*

In particular, if S is locally noetherian then condition (S1′) *of* [Art74, 2.3] *is equivalent to* **rCl**-*homogeneity for X.*

Proof. Note that $\Lambda_{X,T'}$ is fully faithful (Lemma B.2) so (1) $\iff$ (2) by [Hal17, Lem. 1.5(4)]. Obviously, (2) $\implies$ (3), (4) and (5). To see (3) $\implies$ (2), as X is a Zariski stack we may assume that $S = \operatorname{Spec} R$ is affine. By Proposition A.3, every P-nil pair $(V \xrightarrow{p} T, V \xrightarrow{j} V')$, where T is affine, may be written as an inverse limit of P-nil pairs $(V_\lambda \xrightarrow{p_\lambda} T_\lambda, V_\lambda \xrightarrow{j_\lambda} V'_\lambda)$ of finite presentation over S such that T_λ is affine. Furthermore, T' is the inverse limit of the T'_λ, where $T'_\lambda = T_\lambda \amalg_{V_\lambda} V'_\lambda$. The assertion then follows from our assumption that X is limit preserving and [Hal17, Lem. 1.5(4)].

To see (4) $\implies$ (3), we fix a geometric P-nil square over S as in (1.1) with the properties prescribed by (3). On the small flat site T'_{fl}, we can consider two fibered categories that are stacks for étale covers. The first, F_1, is just the restriction of X. The second, F_2, over a flat morphism $U' \to T'$ has fiber $X(V' \times_{T'} U') \times_{X(V \times_{T'} U')} X(T \times_{T'} U')$. The functor $F_1 \to F_2$ is fully faithful (Lemma B.2); it remains to prove that it is locally surjective. Let $t \in T$ be a closed point and let T^h_t denote the henselization of T at t. This uniquely lifts to a henselization T'^h_t of T'. By assumption, $F_1(T'^h_t) \simeq F_2(T'^h_t)$. Fix $\eta \in F_2(T)$ and let η^h_t denote its image in $F_2(T'^h_t)$. It follows that there exists $\tilde{\eta}^h_t \in F_1(T'^h_t)$ inducing η^h_t. Since F_1 is limit preserving, $\tilde{\eta}^h_t$ is induced by some $\tilde{\eta}^{U'}_t \in F_1(U')$, where $(U', u) \to (T, t)$ is étale. Since $F_1 \to F_2$ is fully faithful and F_2 is limit preserving, we can arrange so that $\tilde{\eta}^{U'}_t$ agrees $\eta_t|_{U'}$. The claim follows.

Finally, to see (5) $\implies$ (4), we will argue similarly to (4) $\implies$ (3). So we fix a geometric P-nil square over S as in (1.1) with the properties prescribed by (4). Since $P \subseteq \mathbf{Int}$, this implies that T' is also the henselization of an affine scheme of finite type over S at a closed point; in particular, T' is excellent. Defining F_1 and F_2 analogously, we obtain a fully faithful morphism of groupoids $\phi\colon F_1 \to F_2$ over T'_{fl} which are stacks for étale covers. Let $\hat{T}'$ be the completion of T' at its unique closed point, by hypothesis we have that $F_1(\hat{T}') \simeq F_2(\hat{T}')$. Since T' is excellent, Néron–Popescu desingularization [Pop86] implies that $\hat{T}'$ is an inverse limit of affine and smooth T'-schemes. Now argue just as before to deduce the claim. □

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