

Fun With Complex Numbers

Algebra 5/Trig

Spring 2010

Instructions: There are none! This contains background information and some suggestions for your project. Feel free to make changes and ask different questions if you want, subject to your teacher's approval. Note in particular that on these pages I present several different ideas that may be separate projects. Consider just picking one or two of these ideas to explore. To make this easier, I have separated the ideas into subsections, and labeled the suggested questions by these subsections.

1 Background

Recall that the complex number system $\mathbb{C}$ is the set of all numbers of the form $z = a + bi$, where a is a real number called the **real part** of z , b is a real number called the **imaginary part** of z and $i^2 = -1$ is the **imaginary unit**.

At this point you may want to review complex numbers in section 1.5 of your textbook.

You should be accustomed to thinking of the real numbers as represented by the real number line. The correct way to picture complex numbers is with a plane. If we put Cartesian coordinates on this plane then the x -axis becomes the set of all a 's (in the numbers $a + bi$) and the y -axis becomes the set of all b 's.

1.1 Complex Numbers Exist

One advantage of using the plane is that we can justify the existence of complex numbers. Recall that a point in the plane represents a vector whose tail is at the origin and whose head is at the given point. For example, the point $(2, 3)$ represents the vector $\langle 2, 3 \rangle$, with the arrow pointing from this origin to $(2, 3)$. It may be helpful to review vectors in section 8.3 of your textbook.

Using this notion we can identify a complex number with a vector. For example, the number $2 + 3i$ becomes the point $(2, 3)$ in the plane or the vector $\langle 2, 3 \rangle$ from the origin to the point $(2, 3)$. Multiplication of complex numbers is now just a way of defining how to multiply vectors!

1.2 Polar Forms

The usual description of a complex number as $z = a + bi$ gives the plane Cartesian coordinates. You can use polar coordinates to describe complex numbers as well. Review the transformation rules

for polar coordinates. You should not be surprised that the polar coordinates for the point in the plane corresponding to $z = a + bi$ are given by

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ \theta &= \tan^{-1}(b/a). \end{aligned}$$

In this context, the number r is called the **modulus** of the complex number and the angle θ is called the **argument**.

The real and imaginary parts of a complex number can be recovered from polar coordinates by:

$$\begin{aligned} a &= r \cos \theta \\ b &= r \sin \theta. \end{aligned}$$

Therefore, given the polar coordinates for a point in the plane, the corresponding complex number can be written in standard form as

$$z = r \cos \theta + (r \sin \theta)i$$

or after factoring out r :

$$z = r(\cos \theta + i \sin \theta).$$

Finally, there is the wonderful formula:

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

The verification of this formula requires calculus, so we will take it as given. Using this formula we can now write our complex number in **polar form**:

$$z = re^{i\theta}.$$

One advantage of polar form is that it allows us to interpret the geometric meaning of complex multiplication.

1.3 DeMoivre's Theorem

DeMoivre's Theorem says that the following property of exponents still holds when the exponent is imaginary:

$$(e^{i\theta})^n = e^{in\theta}.$$

Using the identity

$$e^{i\theta} = \cos \theta + i \sin \theta$$

we can write DeMoivre's Theorem as

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta).$$

One consequence of DeMoivre's theorem is that double angle formulae (and similar formulae) can be derived quite easily.

Another consequence of DeMoivre's Theorem is that it is useful in finding all the n^{th} roots of

1 (called **roots of unity**). The best way to understand how to do this is to realize that if we let θ run from, say, 0 to 2π then the complex numbers $e^{i\theta}$ trace out the unit circle. Finding the n^{th} roots of unity amounts to cutting the unit circle into n equal arcs and finding the complex numbers corresponding to the partition points. Wikipedia has a great page on roots of unity.

Section 8.5 of your textbook is another resource for DeMoivre's Theorem. You may also want to review properties of exponents in section P.2 of your textbook.

2 Some Questions

1. Plot the following complex numbers in the plane: i , 3 , $2 + 4i$, $-1 - i$ and $3 - 4i$.
2. **Complex Numbers Exist** Write the complex numbers 1 and i as points in the plane and as vectors.
3. **Complex Numbers Exist** Recall that we add vectors by components: $\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle$. For example:

$$\langle 2, 3 \rangle + \langle 1, -2 \rangle = \langle 2 + 1, 3 - 2 \rangle = \langle 3, 1 \rangle.$$

Does addition of complex numbers work the same way?

4. **Complex Numbers Exist** Can we multiply vectors by components? In other words, can we multiply vectors using the formula $\langle a, b \rangle \times \langle c, d \rangle = \langle ac, bd \rangle$? Use this formula to multiply $\langle 1, 0 \rangle \times \langle 0, 1 \rangle$ and state what is wrong with this form of multiplication.
5. **Complex Numbers Exist** Define vector multiplication by the formula:

$$\langle a, b \rangle \times \langle c, d \rangle = \langle ac - bd, cb + ad \rangle.$$

Verify that when you write the vector for the complex number i , this definition of multiplication gives back $i^2 = -1$. This justifies the existence of complex numbers: they are simply vectors in the plane that can be multiplied together using this formula.

6. **Polar Forms** Find the modulus, argument and polar forms of the following complex numbers: 1 , i , $1 + i$, $-3 + 4i$.
7. **Polar Forms** Write the following complex numbers in standard form: $e^{\pi i/4}$, $e^{\pi i/2}$, $6e^{3\pi i/4}$, $e^{i\pi}$. The last example gives rise to "Euler's identity," considered one of the most beautiful equations in mathematics.
8. **Polar Forms** The following property of exponents holds when the exponent is imaginary:

$$e^{i\theta} e^{i\varphi} = e^{i(\theta+\varphi)}.$$

Therefore complex multiplication of two complex numbers in polar form can be written as

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

Pick some pairs of complex numbers from the previous two exercises and multiply them using the polar forms.

9. **Polar Forms** The previous exercise gives rise to an interpretation of complex multiplication. Multiplying the modulus r_1 by the modulus r_2 **dilates** the complex number - places it closer or further from the origin. The addition of arguments $\theta_1 + \theta_2$ gives rise to **rotation**. Using this, interpret what it means to multiply a complex number z by i .

10. **Polar Forms** ★ Using the identities

$$\begin{aligned} e^{i\theta} e^{i\varphi} &= e^{i(\theta+\varphi)} \\ e^{i\theta} &= \cos \theta + i \sin \theta \end{aligned}$$

verify the addition of angles formulae (for sine and cosine).

11. **DeMoivre's Theorem** Use DeMoivre's Theorem to derive the double-angle identities.

12. **DeMoivre's Theorem** Use DeMoivre's Theorem to find triple-angle identities.

13. **DeMoivre's Theorem** Find all of the 3rd, 4th, 6th and 8th roots of 1.