

Éveriste Galois and Polynomial Equations

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Warm Up

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Solution Let x represent the side of the square. Then the area is x^2 , and the problem translates to

$$x^2 - x = 870.$$

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Taking the side length to be positive, we find that the side length is 30.

Warm Up

- This problem and solution was found on a Babylonian clay tablet from 1600 BC.
- Its solution involves finding the **roots** of a quadratic equation.
- Finding the roots of **polynomials** has been a quest of mankind for thousands of years.

A **polynomial** is a function of the form

$$ax^n + \dots + bx^3 + cx^2 + dx + e$$

with a finite degree.

Examples:

- $x - 5$
- $x^2 - 1$
- $x^3 - x + 2$
- $5x^{16} - 4x^{13} + \dots + 2x - 7$

The **root** of the polynomial, $p(x)$ is the value of x at which the polynomial is 0; the solution to the equation $p(x) = 0$.

Examples:

- $x - 5$ has root 5.
- $x^2 - 1$ has roots $+1$ and -1 .
- $x^3 - x + 2$ has roots 0, 2, and -1 .

In the 15 hundreds in Europe, mathematicians would engage in public math contests in which they would demonstrate their computational prowess. Computing roots of polynomials was one such tasks on which they competed.

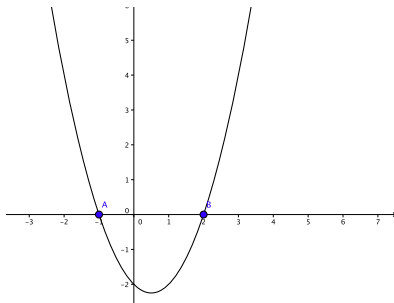
For some polynomials there are formulas for finding the roots. Such formulas gave mathematicians a competitive advantage if their opponents did not know the formula.

The Quadratic Equation

For polynomials of the form $ax^2 + bx + c$ we have the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Babylonians knew this method by 1600 BC.



The Cubic and Quartic

Similarly, there exists a formula (known by 1535) that gives the roots of the cubic polynomial,

$$ax^3 + bx^2 + cx + d,$$

and the quartic (1545),

$$ax^4 + bx^3 + cx^2 + dx + e.$$

The Quintic

But does there exist a formula for the quintic equation,

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f?$$

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A frenchman named Éveriste Galois answered this question in 1832.

Évariste Galois (October 25, 1811 - May 31, 1832)



- Born in the French Empire to educated and intelligent parents
- Was a radical republican during the reign of Louis the XVIII, and was jailed for his revolutionary activities
- Made major contributions to mathematics, although most of his work was not published during his lifetime
- Died in a duel in 1832 at the age of 20

Education

- Galois was educated by his mother until he was 12
- Then entered a preparatory school, the College de Louis-de-Grand
- He performed well during his first two years, but got bored and had to repeat a year because his rhetoric work was not up to standard
- During this time he first became serious about math and read Legendre's *Éléments de Géométrie* like a novel
- By age 15 he was reading research papers in mathematics and neglecting his other studies

Education (Continued)

- He took the entrance exam for the École Polytechnique, the most prestigious school for mathematics in France at the time, but failed the examination
- His father committed suicide in 1829 after a bitter political dispute with the village priest
- A few days later Galois once again took the entrance exam for the École Polytechnique, and once again failed mostly due to his poor explanation of his ideas
- He entered the École Normal instead in 1830
- He was kicked out in 1831 for his political activism

Final Years

- Galois continued his study of and scholarship in mathematics after leaving the École Normal.
- Joined the National Guard, which was disbanded shortly thereafter out of fear the guard might destabilize the government
- Galois was imprisoned for several months due to his political activism. He used this time to work on his mathematical research.
- He was killed in a duel with a friend, probably over a girl, on May 30, 1832
- The night before he wrote many letters to friends. In one of these he outlined his mathematical research and attached some manuscripts on which he had been working.

Galois' Work

Galois' work in mathematics was deep and rich and forms the basis for Galois Theory, an active area of mathematical research today. Amongst other things, this theory can be used to understand the roots of polynomial equations.

None of Galois' work was published during his lifetime. Although he gained some recognition amongst mathematicians of his time for his promise, his explanation of his ideas was poor and his papers thus unpublishable. After his death his friends and other mathematicians went through his papers and popularized his ideas.

The Quintic

Does there exist a formula for the quintic equation,

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f?$$

The Quintic

Galois answered "No". He showed there is no explicit formula for the quintic. There are no general formulas for polynomials of degree 5 or higher.

Referneces

- http://en.wikipedia.org/wiki/Evariste_Galois
- <http://www.gap-system.org/~history/Biographies/Galois.html>
- Stewart, Ian. *Galois Theory*. Third Edition. Chapman & Hall/CRC, 2004.