What is i^i?

Ryan Smith

Department of Mathematics
University of Arizona

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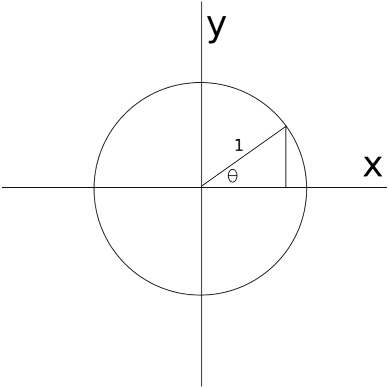
Definition

The imaginary number i is given by $i = \sqrt{-1}$. Complex numbers are numbers of the form $a + bi$ where a, b are real numbers. The set of complex numbers is written $\mathbb{C}$.

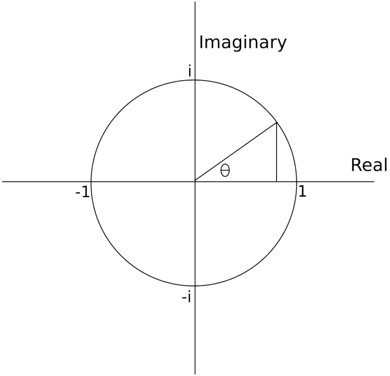
You can plot points on the complex plane using their real and imaginary parts.

Points on the Unit Circle



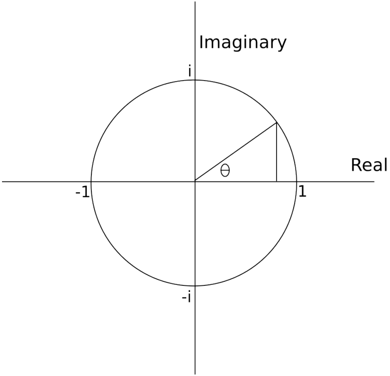
Recall from trigonometry that the x coordinate of the point is $\cos(\theta)$, and the y coordinate is $\sin(\theta)$, i.e., the point is at $(\cos(\theta), \sin(\theta))$.

Points on the Complex Unit Circle



The coordinate in the complex plane is $(\cos(\theta), \sin(\theta))$. However, the y-axis here is given by multiples of i. So the point is located at $\cos(\theta) + i\sin(\theta)$.

Points on the Complex Unit Circle



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Theorem (Euler's Formula)

Let θ be a real number. Then

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

"[Euler's formula is] one of the most remarkable, almost astounding, formulas in all of mathematics." - Richard Feynman

Example

By the laws of exponents, $(e^{iz})^2 = e^{2iz}$.

$$\begin{aligned}(e^{iz})^2 = (\cos(\theta) + i\sin(\theta))^2 &= \cos^2(\theta) + 2i\sin(\theta)\cos(\theta) + i^2\sin^2(\theta) \\ &= (\cos^2\theta - \sin^2\theta) + i(2\sin(\theta)\cos(\theta)) \\ &= \cos(2\theta) + i\sin(2\theta) \\ &= e^{2iz}\end{aligned}$$

Remark

One of the most famous formulas in all of mathematics is

$$e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1$$

Euler was so proud of this that he asked for it to be written on his grave.

How to take e to any power

Let $a + bi$ be any complex number. Then

$$\begin{aligned} e^{a+bi} &= e^a e^{bi} \\ &= e^a \left(\cos(b) + i\sin(b)\right) \\ &= e^a\cos(b) + ie^a\sin(b) \end{aligned}$$

Any complex number can be written in polar form, $z = re^{i\theta}$ for some $r \geq 0$ and $0 \leq \theta < 2\pi$.

Remark

Unlike with real numbers, e^z is not a 1-1 function! In particular,

$$
\begin{aligned}
e^{2\pi i} &= \cos(2\pi) + i\sin(2\pi) = 1 \\
e^{4\pi i} &= \cos(4\pi) + i\sin(4\pi) = 1 \\
e^{6\pi i} &= \cos(6\pi) + i\sin(6\pi) = 1 \\
e^{8\pi i} &= \cos(8\pi) + i\sin(8\pi) = 1 \\
&\dots
\end{aligned}
$$

This is a very big deal.

Since $e^{\pi i} = -1$, it seems reasonable to define

$$\ln(-1) = \pi i$$

However there is a 'minor' problem since

$$e^{\pi i} = e^{3\pi i} = e^{5\pi i} = e^{7\pi i} = \cdots = -1$$

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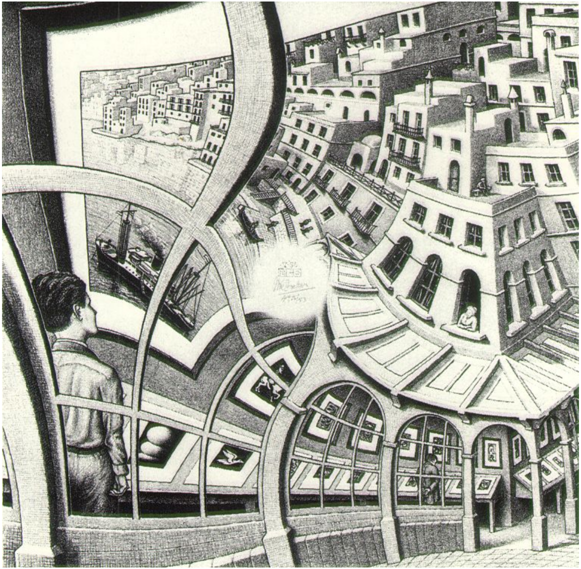

General complex exponents

FIrst we have to choose a branch of the logarithm. Then we use the fact that e^z and $\ln(z)$ are inverses! Let $z \neq 0$ and w be any complex numbers now.

$$\begin{aligned} z^w &= e^{\ln(z^w)} \\ &= e^{w\ln(z)} \end{aligned}$$

We can now compute i^i: First we have to choose a branch of the logarithm. For simplicity assume that we take the branch between 0 and 2π.

$$
\begin{aligned}
i^i &= e^{\ln(i^i)} \\
&= e^{i\ln(i)} \\
&= e^{i(\pi i/2)} \\
&= e^{-\pi/2} \\
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Complex Trig Functions

Since $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we can use

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta)$$

to solve for $\sin(\theta)$:

$$\begin{aligned} e^{i\theta} - e^{-i\theta} &= (\cos(\theta) + i\sin(\theta)) - (\cos(\theta) - i\sin(\theta)) \\ &= 2i\sin(\theta) \end{aligned}$$

Definition

The sine function is

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Using the same trick but adding $e^{i\theta}$ and $e^{-i\theta}$ we solve to get

Definition

The cosine function is

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

Definition

Recall that

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

If you take $\cos(ix)$ you get

$$\cos(ix) = \frac{e^{i^2x} + e^{-i^2x}}{2} = \frac{e^{-x} + e^x}{2} = \cosh(x)$$

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Why bother with triangles when you can use complex exponents to prove your trig identities!

$$
\begin{aligned}
2\sin(\theta)\cos(\theta) &= 2\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) \\
&= \frac{e^{i\theta+i\theta} - e^{0} + e^{0} - e^{-i\theta-i\theta}}{2i} \\
&= \frac{e^{2i\theta} - e^{-2i\theta}}{2i} \\
&= \sin(2\theta)
\end{aligned}
$$