

A Lesson on Limits

Qiyam Tung

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1 Goals

- Learn about models.
- Understand limits
- Practice independent thinking.

2 Pupils

Pupils constrict (become small) when there is a lot of light and dilate (become big) when there isn't. However, the amount it increases by its diameter is not constant relative to the amount of light there is.



Figure 1: An eye constricts or dilates depending on whether the sphincter pupillae muscles or the dilator pupillae contract.

2.1 Graphs

- Describe as precisely as possible when a pupil dilates and constricts. In other words, How does the size change with the intensity of the light.
 - Your graph should indicate that as it gets really bright, your pupil can only constrict a little. Likewise for darkness and dilation.
- Create a graph (with intensity as the x -axis and pupil diameter as the y -axis). You do not need to add units, but you should instead give a basic idea of how the size of the pupil changes as light increases.

Now the question is, how far can you enlarge or constrict your pupils? The answer to that, which is a number, is called the limit and is the essence of calculus.

3 Limits

In the course, you will be studying differentiation and integration, two techniques that require the use of the concept of limits.

Here is a little history. Who is usually credited with the invention of calculus? Newton and Leibniz. As you may know, Newton is responsible for developing early physics and is often seen as one of the most influential of not most influential scientist to have ever lived. Newton invented calculus because he need the mathematics necessary to describe the change in motion of objects.

In order to calculate this change, one would need to look at the limits of a function. Newton's method was not very rigorous and used something called infinitesimal values. My understanding is that is something of a hand wavy justification of getting derivatives. What we will study is the concept of limits, which did not come until much later after Newton had died.

3.1 Key Concepts of a Limit

- Limit is something that describes the behavior of a function near some point, but not at that actual point.
- For a small change in your independent variable, you get a small change in your dependent variable.

We start by analyzing a few functions. When we ask questions like

Does the limit exist at the $x = 0$ for the function $\frac{1}{x}$ we can also ask the question by writing it as

$$\lim_{x \rightarrow 0} \frac{1}{x} \tag{1}$$

3.2 Problems

See other worksheet.

4 Definition of a Limit

Without a proper definition, you will not be able to *prove* to me that the limit for any function exists. The reason for that is simply because you only have a vague notion of what a limit is.

Why is it important to have a precise definition? Consider the following troll comic.

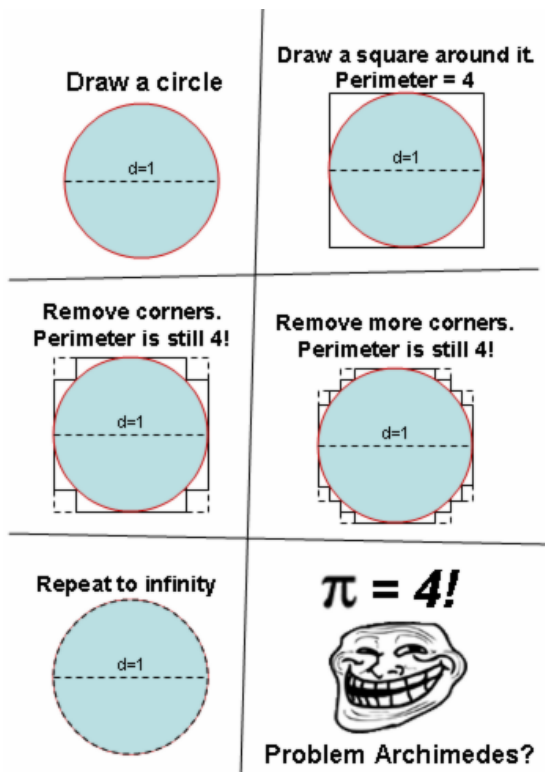


Figure 2: Can you disprove it? It takes a surprising amount of insight to refute it, but it can be done. To do it, you have to understand how to take the limit of functions and sequences.

A more typical example, however, would be to ask you to **prove** something simple like

$$\lim_{x \rightarrow 2} x - 2 = 0 \quad (2)$$

To **prove** something, you cannot make approximations or heuristic arguments like “we tried $x = 2.1, 2.05, 2.025$, and $x = 0.5, 0.75$, and it seems to get closer to 0”. You have to show logically that we are *always* getting *closer* to 0 as we approach 2.

4.1 Define a Limit

To be able to do prove that functions have limits, give a precise definition of what a limit is in your own words.

- Get into groups of 2.
- Individually, come up with a definition of a limit.
- Test the definition with your partner. Your partner should try to find an examples that show that your definition doesn’t cover the intuitive understanding of a limit.

Here are some ideas on how to potentially “break” a definition. Let’s assign L to be the limit when $f(x)$ gets arbitrarily close to a . That is, $\lim_{x \rightarrow a} f(x) = L$

- Will the definition be satisfied if the limit exists at some point other than a ?
- Does the definition imply that as x gets further away from a , $f(x)$ is getting further away from the limit? This should not be the case.