

What Do Mathematicians Do?

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As a break from the tradition of this newsletter, this article is meant to provoke discussion. Most research mathematicians are quite passionate about their subject. Yet they are aware that their enthusiasm is not shared (to put it mildly) by the public at large, and even, in many cases, by research scientists. Is this just a case of "love is blind", or is it possible that mathematicians are aware of something about mathematics that outsiders are not? I'd like to investigate this matter in this article. I hope that in doing so I will stimulate others, both mathematicians and non-mathematicians, to think about these questions, and maybe even contribute their thoughts to later issues of the newsletter.

I think it's very important to start by asking the right question. Typically, academic disciplines are defined by their subject matter. So, to ask what a geologist does is more or less the same thing as to ask what a geologist studies. Thus, for the Oxford English Dictionary, it is "the science which has for its object the investigation of the earth's crust, of the strata which enter into its composition, with their mutual relations, and of the successive changes to which their present condition and positions are due". Similarly, for the OED, biochemistry is "the science dealing with the substances present in living organisms and with their relation to each other and to the life of the organism". Moving away from science, we have the OED's definition of architecture, "the art or science of building or constructing edifices of any kind for human use"; economics, the study of "the development and regulation of the material resources of a community or nation"; and linguistics, "the study of languages". These examples were chosen at random. In every case I expect that the reader's definition would be very similar to the one given by the dictionary.

Well, if such definitions are so easy for other disciplines, why not for mathematics? Like most people, the OED assumes that mathematics too can be defined by its subject matter and tries "the abstract science which investigates deductively the conclusions implicit in the elementary conceptions of spatial and numerical relations, and which includes as its main divisions geometry, arithmetic, and algebra; and, in a wider sense, so as to include those branches of physical or other research which consist in the application of this abstract science to concrete data". A good effort, but one gets the strong impression that whoever wrote it was struggling! The "spatial and numerical relations" obviously cover geometry and arithmetic, but then algebra had to be added because it wasn't dealt with. However, that's nowhere near good enough. Important "divisions" like analysis, probability, set theory and operational research are completely ignored by this definition, so clearly it's very inadequate. Should we compensate by listing the titles of, say, all mathematics modules taught at NUS, in the hope that we'll cover the subject that way? That attempt is doomed too, because a glance at the list soon reveals courses on topics like filter banks, chaos and fractals, cryptography, game theory, etc., that weren't there ten or twenty years ago. If the subject is to be defined by a list constructed at a certain time, then after that time, no newcomers can ever join. However, it's clear that mathematics is continuing to grow, its tentacles finding their way into areas of investigation previously thought beyond its reach.

The first attempt I heard to define mathematics by what it studies was by T G Room (a geometer commemorated by Room squares in combinatorics). He reckoned that mathematics is the study of relationships between concepts. Although this is helpful to the non-mathematician, it is clearly inaccurate. There are many concepts, like punishment and retribution, love and fidelity, whose relationships have failed to attract mathematical interest. In order to nail down those concepts that might yield mathematical investigation, the topologist D H Gottlieb claimed that mathematics is the study of well-defined things. This notion has some appeal to mathematicians, for whom the expression "well-defined" is part of the lingo, and for whom it excludes the above philosophical concepts. Yet I fear that an attempt to explain it to a non-mathematician would result in a "well-defined thing" as being "one that is amenable to mathematical inquiry". In other words, mathematicians study what mathematicians study.

Mathematicians tend to be pretty stubborn (we like to say that we persevere), but there comes a stage when one has unsuccessfully battled against a tough question for so long that one realises that the difficulty was simply that it was

the wrong question in the first place. I believe that's what's happened here.

We shouldn't ask **what** a mathematician studies, we should ask **how**.

Put it in another way; instead of the question, "What do mathematicians study?" we should ask, "What do mathematicians do?" Interestingly, when one examines the OED's attempt at a definition, one sees that, in contrast to the definitions of the other disciplines, there's a how answer only partly suppressed, "Investigates deductively the conclusions implicit in the elementary conceptions ...". Since mathematicians get their fingers into pies that often have names very different from geometry, arithmetic and algebra, it's more fruitful to clarify the process of doing mathematics. Then, when a new topic is proposed for the next revision of the mathematics curriculum, one has some hope of answering the question, "But is it mathematics?" My guess is that the correct answer would be, "Yes, when you look at it the right way." No geology lecture would be about zebras, or blood vessels, or language grammars, or DNA. By definition, it would fail to be a geology lecture. However, I've known mathematics lectures about these four topics.

Specifically, the lecture about **zebras** was interested in how they and other quadrupeds move. Different speeds of walking or running result in different sequences of hooves hitting the ground. Which sequences can occur, and what is the relation between the sequence and the speed?

Blood vessels can be studied for the way in which cells move along them; this is the dynamics of fluid motion where the walls are not rigid. And what governs the shape of the vessel itself? Can one predict when the forces will be so great as to lead to rupture?

Are there common rules of manipulation of words and phrases that apply across different **languages**? What does the similarity of such rules suggest about the cultural or genetic links between the speakers of such languages?

It's recently been discovered that in the process of replication, the enormously long **DNA** molecules get tied into knots, which partly dissolve and recombine as different knots. By inspecting the knots that appear, one can attempt deductions about the biochemical process that is leading from one knot to the next.

There's a pattern to what is happening in each of the above examples. The mathematician immediately ignores many specific features of the object in question. He or she is unlikely to care about whose body the blood vessel inhabits, or the age of the zebra. But pretty soon he or she may even forget

that it's a blood vessel or zebra that's being studied, and may talk to a colleague about fluid in a tube or configurations of moving rods. The process of abstraction (OED: "of considering ... an attribute or quality independently of the substance to which it belongs") takes on a life of its own, so that before long two mathematicians may be discussing the problem in such a way that a third mathematician listening in would find it difficult to guess its physical origins. (The degree of difficulty is probably the distinction between pure and applied mathematics. Put like this, it's apparent that the distinction is more arbitrary and less clear-cut than generally recognised.)

After reading the above, the Japan-based mathematician A Kozlowski observed: "I think it a very important point that mathematics is probably the only subject whose content could change entirely and yet we would still recognize it as mathematics. We would probably recognize mathematics of beings from another universe, though we may have problems in distinguishing their physics from their philosophy, their history from their mythology etc."

I believe that the process of abstraction is a vital characteristic of mathematical thought, probably more distinctive than the method of deduction that the OED emphasises. Most scientists practise deduction, although not necessarily to the extent of mathematicians. However, other disciplines are comparatively restricted in the amount of abstraction that they allow themselves. While physiologists might be comfortable with the general properties of all blood vessels belonging to humans, or of all those with a certain condition, would they still be at ease with a level of abstraction that considered equally other fluids moving in inorganic tubes? Or would they feel that such generalizations were no longer in the realm of physiology?

In fact, the desire for abstraction seems to be an essential part of a mathematician's psyche. It's not just a matter of abstracting from the physical world to the mathematical; many mathematicians commence work only long after that process has been completed. Within mathematics, researchers are all the time striving to find just the right level of abstraction for a given setting, seeking the perfect balance between the twin goals of utility and generality.

Another feature of scientific method is of course induction, the attempt to generalize conclusions from a number of particular instances. Mathematicians practise this more often than is usually realised, however, in a special way. For a scientist such a conclusion has the status of a probationary law. If it stands the test of time, that is, accords with, and even predicts, subsequent observations, then it becomes more widely accepted. This tends to be a gradual process that can be partly or wholly reversed (medical science

provides many examples of reversibility and controversy). For a mathematician, the result of induction is just a hunch. A strongly held hunch is honoured with the title of conjecture. For example, there was the Fermat Conjecture:

If x, y, z, n are whole numbers greater than 1 and $x^n + y^n = z^n$, then $n = 2$.

Evidently, Fermat produced this statement by induction after examining many specific cases (with no help from an electronic computer). (In the event that $n = 2$, then for any whole number k we can always take $x = 2k + 1$, $y = 2k^2 + 2k$ and $z = y + 1$.) After succeeding generations of mathematicians had played with it, the formulation received the status of a conjecture. For further generations it was widely believed to be a correct assertion, yet no mathematician would admit to the list of proved statements anything that was logically dependent on it. That all changed in this decade, when Andrew Wiles' famous proof (a chain of deductions occupying hundreds of printed pages) survived the rigorous checks of his peers. The statement is now called a theorem, and will always remain so. "Elevation to the theoreage" is an irreversible process. The progression from hunch to conjecture is an example of scientific induction, but the final, irreversible graduation from conjecture to theorem has no parallel outside mathematics.

We have now distinguished three modes of thinking that highlight the difference between mathematics and other disciplines:

Abstraction, Deduction, Induction.

They are listed above in decreasing order of importance to mathematical research, but I would guess in increasing order of importance for scientists generally. One might object that if deduction and induction are seen as opposites, then why doesn't an opposite of abstraction appear in the list? Well, since abstraction consists in seeing common properties and patterns among different situations, then once one has obtained conclusions about the general, abstract setup, the process of deduction is enough to get one back to conclusions about the original, more concrete setting. So our list of three seems to be enough.

Just what the process of abstraction involves is a big topic, and not central to this article. At its heart seems to be one of the greatest joys of mathematical research - pattern recognition. The patterns are not usually the visual ones of everyday experience. Recall that humans also get excited by more subtle threads of similarity, for instance in hearing in a Wagnerian opera a musical

motif that occurs elsewhere in the Ring Cycle. Mathematicians have available for painting their patterns the whole canvas of human experience. For example, the kinship ties of the Warlpiri people of the Australian outback exhibit the same pattern as the symmetries of a square (known algebraically as the dihedral group of order 8).

The drive to find common themes from disparate areas seems to be part of the mathematician's temperament. At its most banal, it's a source of painful puns (like the one I had to resist earlier, after including the words "pie" and "fruitful" in the same sentence). Used more creatively, it helps to explain what has been called the "unreasonable effectiveness" of mathematics. Consider the following title of a research paper, by R Ghrist, that reached me today.

Configuration spaces ... it begins. I think: Yes, it's about topology.

... and **braid groups** ... Okay, about algebra too.

on **graphs** ... Combinatorics as well. This is getting pretty interesting. But now for the knockout blow ...

in **robotics**.

The discovery of a unifying pattern can be like lightning flashing from one discipline to another. The difference is that it can illuminate both subjects forever. So there is a simple message for the nonmathematical researcher reading this article. When all seems cloudy, contact a mathematician (as broadminded as possible). Then stand by for flashes of lightning!

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Return to List of Articles