

Smooth Manifolds and Minkowski Spacetime

Jordan Schettler

Department of Mathematics
University of Arizona

3/2/11

Outline

- 1 Topology and Curves
- 2 Surfaces
- 3 Spaces and the Schwarzschild Metric

Topology and Curves

Definition

to•pol•o•gy |tə'päləjē|

noun

the study of geometric properties and spatial relations
unaffected by the continuous change of shape or size of figures.

Definition

to•pol•o•gy |tə'päləjē|

noun

the study of geometric properties and spatial relations
unaffected by the continuous change of shape or size of figures.

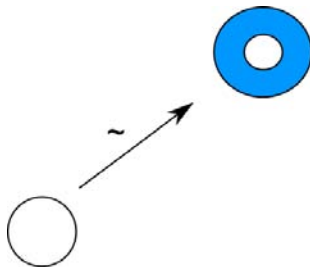
Remark

Actually, **topology** is the study of topological spaces (X, T) where X is a set and $T \supseteq \{\emptyset, X\}$ is a collection of subsets of X which is closed under arbitrary unions and finite intersections.

Types of Topological Equivalence

- Homotopy equivalent

circle $\sim$ annulus



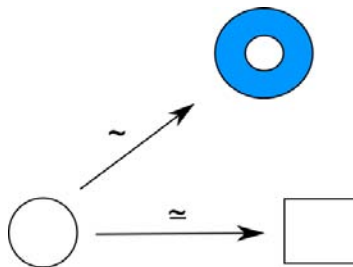
Types of Topological Equivalence

- Homotopy equivalent

circle $\sim$ annulus

- Homeomorphic

circle $\cong$ square



Types of Topological Equivalence

- Homotopy equivalent

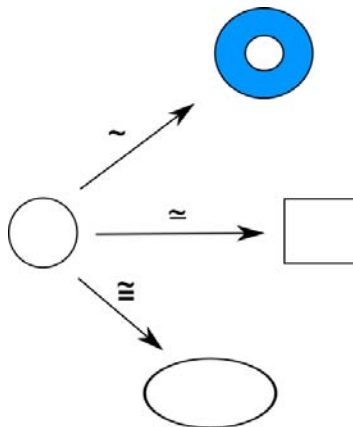
circle $\sim$ annulus

- Homeomorphic

circle $\cong$ square

- Diffeomorphic

circle $\cong$ ellipse



Here we'll be interested in “manifolds” that are

Here we'll be interested in “manifolds” that are

- connected (all one piece)

Here we'll be interested in “manifolds” that are

- connected (all one piece)
- locally Euclidean (if we zoom in, things look flat)

Here we'll be interested in “manifolds” that are

- connected (all one piece)
- locally Euclidean (if we zoom in, things look flat)
- smooth (no sharp corners, so we can do calculus)

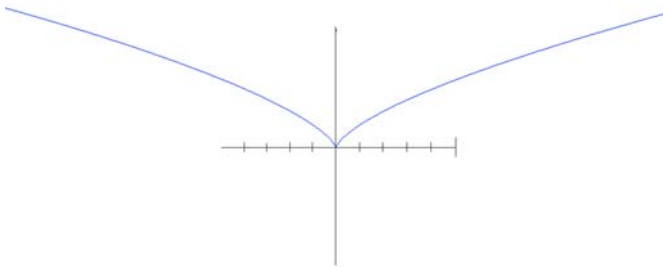


Figure: $y = x^{2/3}$ has a “cusp” at $(0,0)$

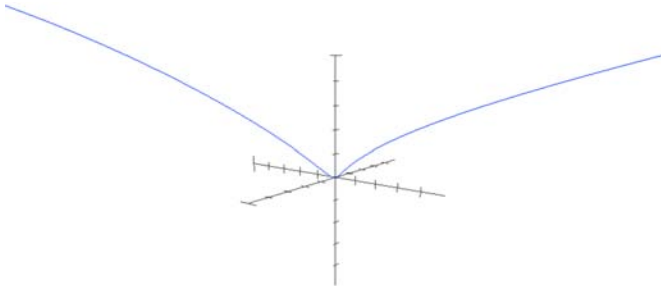
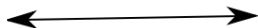
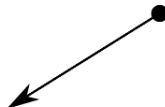


Figure: Maybe we're looking at a shadow of the twisted cubic

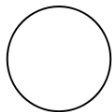
1-Dimensional Manifolds



line



ray



circle



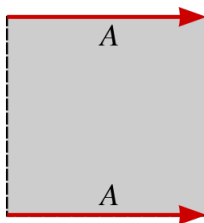
segment

Surfaces

Polygon

Picture

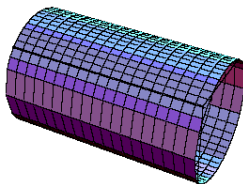
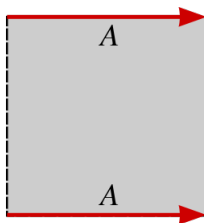
Boundary



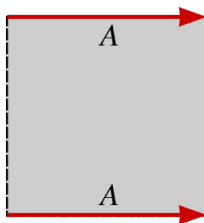
Polygon

Picture

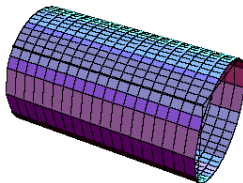
Boundary

Table: Finite Cylinder C

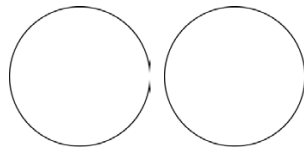
Polygon



Picture



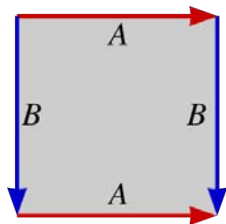
Boundary

Table: Finite Cylinder C

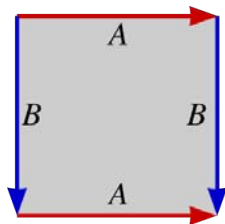
Polygon

Picture

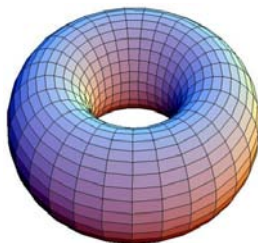
Boundary



Polygon



Picture



Boundary

Table: Torus $\mathbb{T}^2$

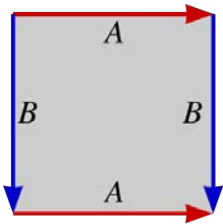
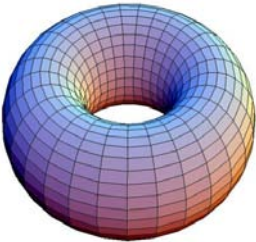
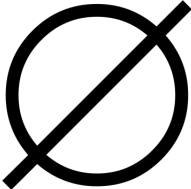
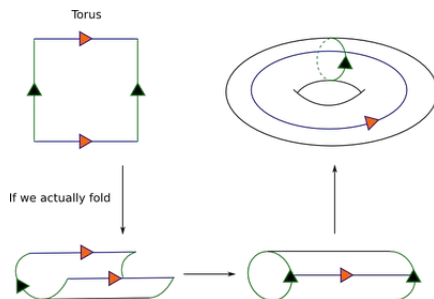
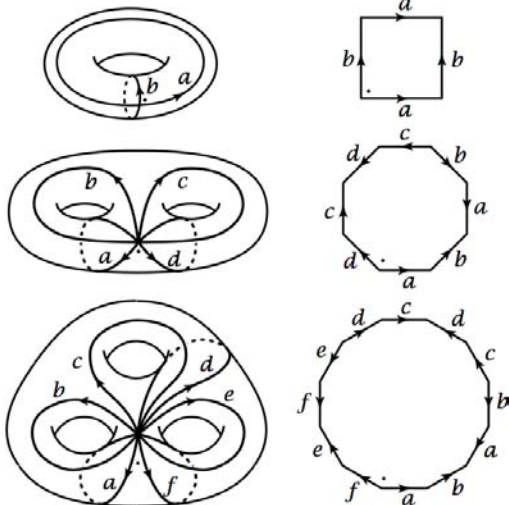
Polygon	Picture	Boundary
		

Table: Torus $\mathbb{T}^2$

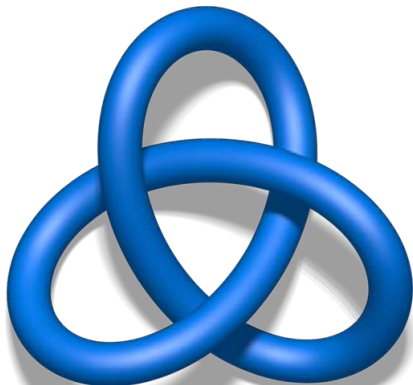
$$C \rightsquigarrow \mathbb{T}^2$$



Other Polygons Produce Multi-Holed Tori



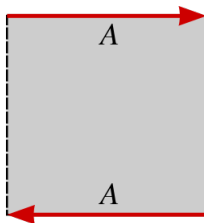
A Knotted Torus is Still a Torus



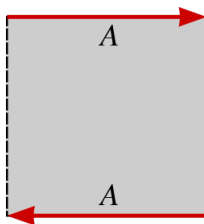
Polygon

Picture

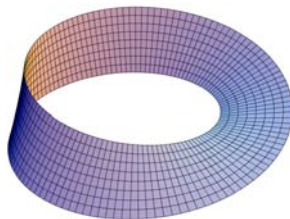
Boundary



Polygon



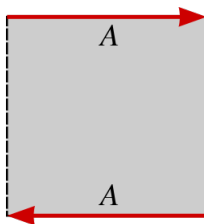
Picture



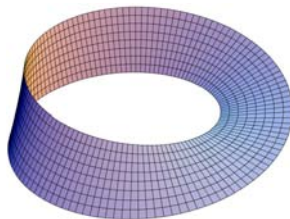
Boundary

Table: Möbius Strip M

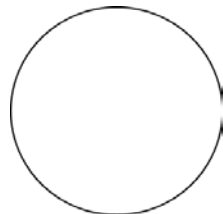
Polygon



Picture



Boundary

Table: Möbius Strip M

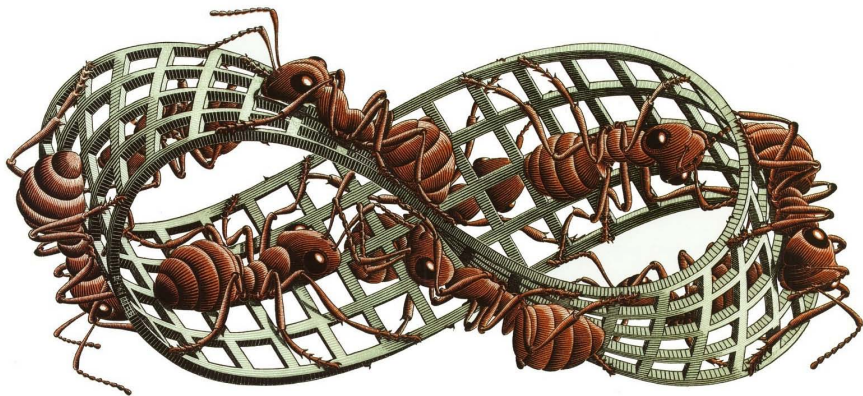
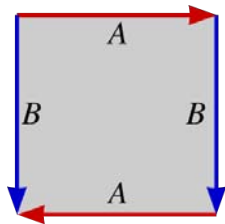


Figure: M. C. Escher, "Möbius Strip II"

Polygon

Picture

Boundary



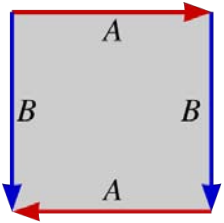
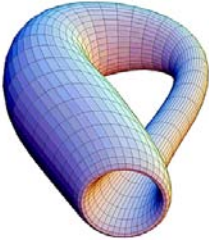
Polygon	Picture	Boundary
		

Table: Klein Bottle K

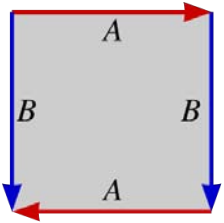
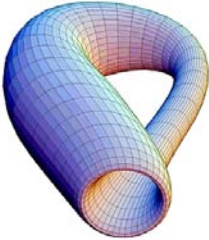
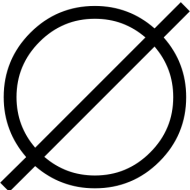
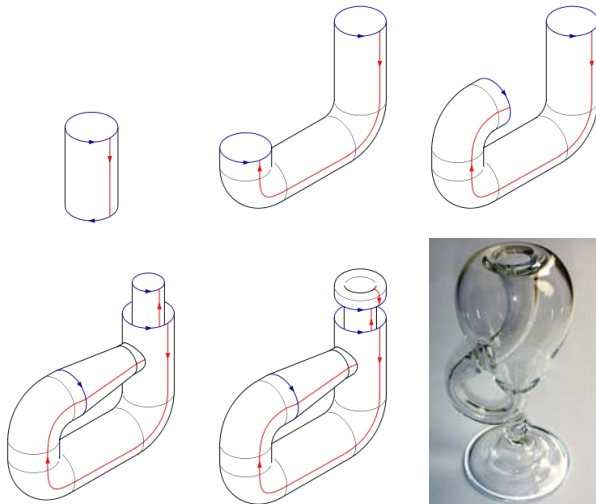
Polygon	Picture	Boundary
		

Table: Klein Bottle K

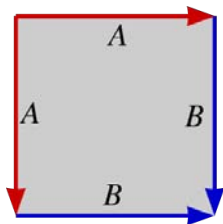
$$C \rightsquigarrow K$$



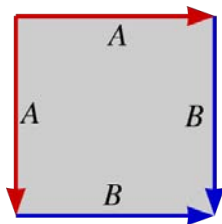
Polygon

Picture

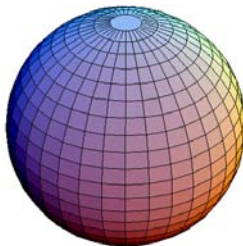
Boundary



Polygon



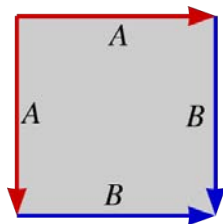
Picture



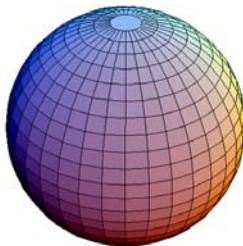
Boundary

Table: Sphere $\mathbb{S}^2$

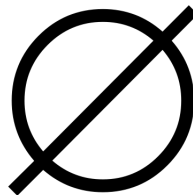
Polygon



Picture



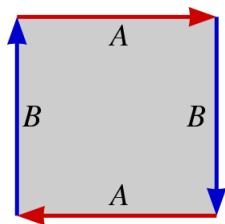
Boundary

Table: Sphere $\mathbb{S}^2$

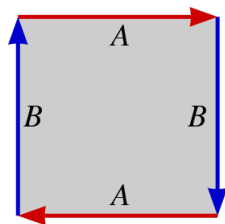
Polygon

Picture

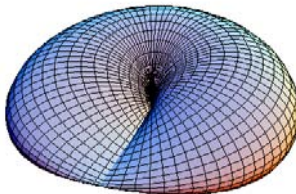
Boundary



Polygon



Picture



Boundary

Table: Real Projective Plane $\mathbb{RP}^2$

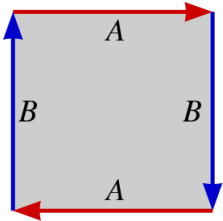
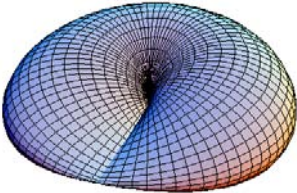
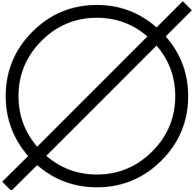
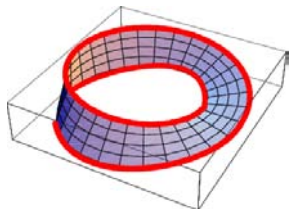
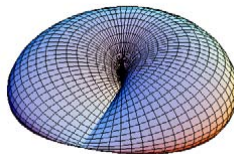
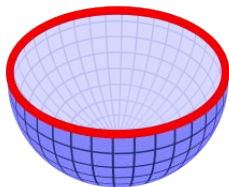
Polygon	Picture	Boundary
		

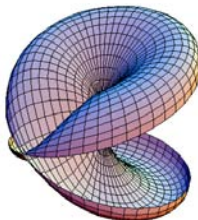
Table: Real Projective Plane $\mathbb{RP}^2$



↓ paste

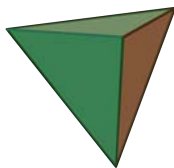


↓ slice

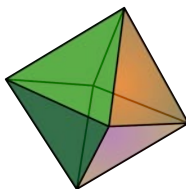


Spaces and the Schwarzschild Metric

We can also take polyhedra (instead of polygons) and attach faces (instead of sides) to get 3-dimensional manifolds



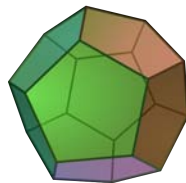
tetrahedron



octahedron



truncated oct.



dodecahedron

Table: Some Polyhedra

Minkowski Spacetime

We can indicate geometries with “line elements”

ds = an infinitesimal element of length

This allows us to do calculus: tangents, areas

Flat Spaces

Minkowski Spacetime

We can indicate geometries with “line elements”

ds = an infinitesimal element of length

This allows us to do calculus: tangents, areas

Flat Spaces

$$ds^2 = dx^2$$

linear

Minkowski Spacetime

We can indicate geometries with “line elements”

ds = an infinitesimal element of length

This allows us to do calculus: tangents, areas

Flat Spaces

$$ds^2 = dx^2$$

linear

$$ds^2 = dx^2 + dy^2$$

planar

Minkowski Spacetime

We can indicate geometries with “line elements”

ds = an infinitesimal element of length

This allows us to do calculus: tangents, areas

Flat Spaces

$$ds^2 = dx^2$$

linear

$$ds^2 = dx^2 + dy^2$$

planar

$$ds^2 = dx^2 + dy^2 + dz^2$$

spatial

Minkowski Spacetime

We can indicate geometries with “line elements”

ds = an infinitesimal element of length

This allows us to do calculus: tangents, areas

Flat Spaces

$$ds^2 = dx^2$$

linear

$$ds^2 = dx^2 + dy^2$$

planar

$$ds^2 = dx^2 + dy^2 + dz^2$$

spatial

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2$$

spatial-temporal

$$= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)$$

General relativity describes gravity as the warping of this spacetime by matter and energy.

General relativity describes gravity as the warping of this spacetime by matter and energy.

Einstein did this in 1915 via 10 “field equations.”

General relativity describes gravity as the warping of this spacetime by matter and energy.

Einstein did this in 1915 via 10 “field equations.”

In the case of a non-charged, non-rotating, uniformly dense, spherical body of mass M we get...

General relativity describes gravity as the warping of this spacetime by matter and energy.

Einstein did this in 1915 via 10 “field equations.”

In the case of a non-charged, non-rotating, uniformly dense, spherical body of mass M we get...

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)$$

Embedding diagrams help us visualize this geometry

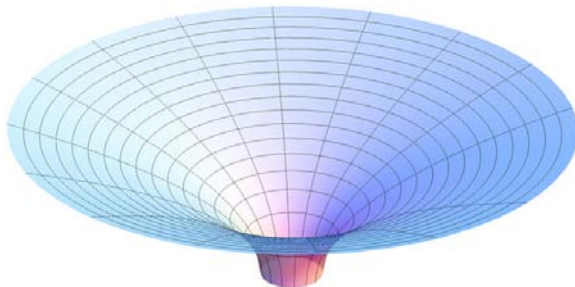


Figure: Flamm's Paraboloid

At a point in the life of a star, it runs out of thermonuclear fuel.

At a point in the life of a star, it runs out of thermonuclear fuel.

It could reach an equilibrium supported by internal forces, or it may be unsupported against further gravitational collapse...

At a point in the life of a star, it runs out of thermonuclear fuel.

It could reach an equilibrium supported by internal forces, or it may be unsupported against further gravitational collapse...

If the radius shrinks below $2M$, a black hole is formed.

At a point in the life of a star, it runs out of thermonuclear fuel.

It could reach an equilibrium supported by internal forces, or it may be unsupported against further gravitational collapse...

If the radius shrinks below $2M$, a black hole is formed.



Figure: An artist's interpretation of a black hole