

Problem Statement: Suppose a sheep is tied to the side of a silo with a rope of length k meters. The silo has a radius of 3 meters. What is the total amount of grazing area accessible by the sheep?

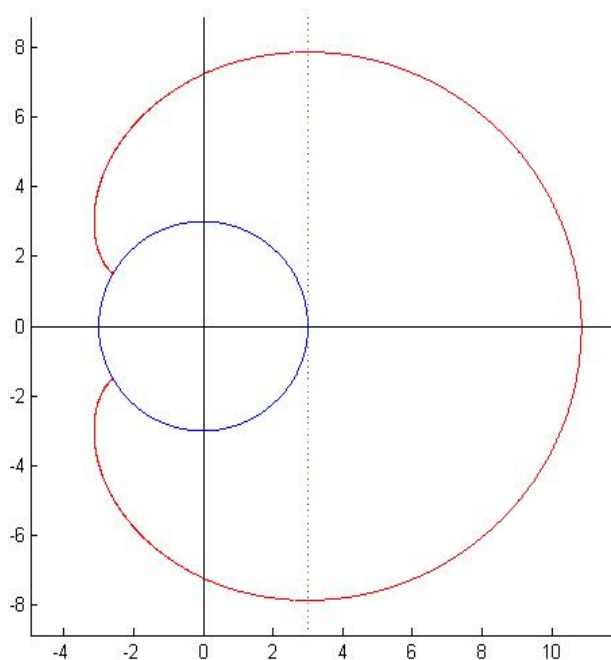


Figure 1: Here is a sample of the grazing region available to the sheep. In this case, $k = \frac{5\pi}{2}$. The blue curve represents the silo. The red curve represents the path the sheep would follow if it walked with the rope taut. The total grazing area is then the region inside the red curve but outside the blue curve.

To the right of the dashed line the red curve traces out a semicircle of radius k . To the left of the dashed line the red curve can be expressed parametrically as

$$\begin{aligned} x(t) &= 3 \cos(t) - (k - 3t) \sin(t) \\ y(t) &= 3 \sin(t) + (k - 3t) \cos(t) \end{aligned}$$

Solution using parametric equations: I have left the solutions as definite integrals. If you want explicit solutions as a function of rope length k let me know. Try to recreate the parametric equations and integrals. Let me know of any questions you have. There may be cleaner ways to express this answer (or

at least parameterize the curve) in polar coordinates but I wasn't sure if you had seen that before so here is the answer using Cartesian coordinates.

Case 1: $0 \leq k \leq \frac{3\pi}{2}$

In this case the rope will only wrap around at most a quarter of the silo. The curve traced out by the parametric equation passes the vertical line test so the total area is the area under the red curve minus the area inside the blue circle.

$$\text{Area} = \frac{\pi}{2}k^2 + 2 \left[\underbrace{\int_0^{k/3} y(t)x'(t)dt}_{\text{area under parametric curve}} - \underbrace{\int_{\cos(k/3)}^3 \sqrt{9-x^2}dx}_{\text{area under circle}} \right]$$

Case 2: $\frac{3\pi}{2} < k \leq 2\pi$

In this case the red curve fails the vertical line test. To find the area we calculate the area under the red curve, then subtrace the area under the green portion of the parametric curve and the area under the circle.

$$\text{Area} = \frac{\pi}{2}k^2 + 2 \left[\int_0^{\pi/2} y(t)x'(t)dt - \int_{\pi/2}^{k/3} y(t)x'(t)dt - \int_{\cos(k/3)}^3 \sqrt{9-x^2}dx \right]$$

Case 3: $k > 3\pi$

This case is slightly more complex as there are overlapping regions between the case where the sheep walks clockwise around the silo or counter clockwise. One would need to find the value of the parameter t where $y(t) = 0$ with $t \in (\pi/2, k/3)$. However, the solutions would look like the following.

$$\begin{aligned} \text{Area} &= \frac{\pi}{2}k^2 - 9\pi + 2 \int_0^{t^*} y(t)x'(t)dt \\ 3 \tan(t^*) + (k - 3t^*) &= 0, \quad t^* \in (\pi/2, k/3) \end{aligned}$$