G-TEAMS at Cholla High Magnet School

Chantel C. Blackburn
Department of Mathematics, University of Arizona, Tucson, Arizona

Janet Liston
Mathematics Department, Cholla High Magnet School, Tucson, Arizona

TUSD

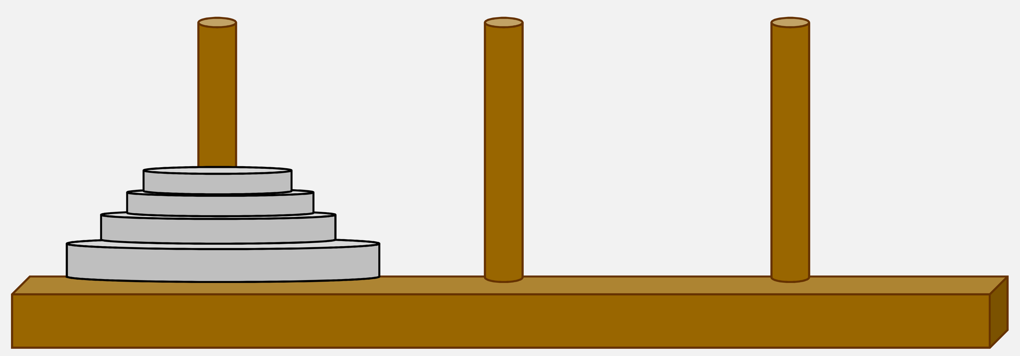
THE UNIVERSITY OF ARIZONA

G-TEAMS

INSTITUTE FOR MATHEMATICS & EDUCATION
ime.math.arizona.edu

NSF

Towers of Hanoi: Knowledge, Solutions, Connections



According to legend, a large room in a Vietnamese temple contained 3 posts and 64 golden discs. The priests were to move the discs from one post to another, never placing a larger disc on top of a smaller one. It was further said that the world would end when every disc was moved to another post.

The Question:
If the priests could move one disc per second, how long would it take for the world to end if the priests solved the 64 disc puzzle in the fewest number of possible moves?

IB Studies Standard Level: Noticing Patterns

n	m
1	1
2	3
3	7
4	15
5	31

$2 = 2^1$
$4 = 2^2$
$8 = 2^3$
$16 = 2^4$

Adding a disc increases the minimum number of moves, m, by a power of 2

Minimum number of moves for n discs is 1 less than the nth power of 2

$$m_n = 2^n - 1$$

IB Standard Level: Geometric Series

n	m
1	$1 = 2^0$
2	$3 = 2^0 + 2^1$
3	$7 = 2^0 + 2^1 + 2^2$
4	$15 = 2^0 + 2^1 + 2^2 + 2^3$
5	$31 = 2^0 + 2^1 + 2^2 + 2^3 + 2^4$

The minimum number of moves, m, for n discs is the sum of the first $n - 1$ terms of a geometric sequence with common ratio 2

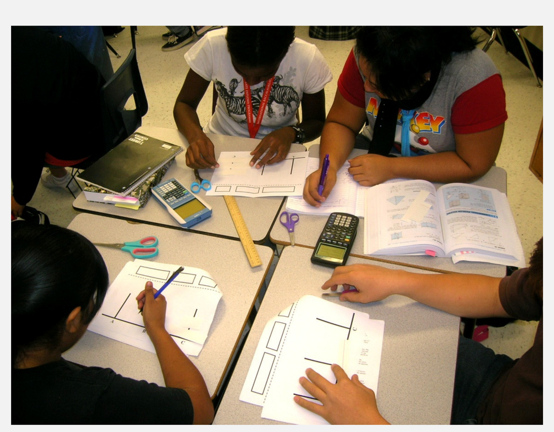
$$m_n = 2^0 + 2^1 + \cdots + 2^{n-1}$$
$$= \frac{1-2^n}{1-2} = 2^n - 1$$

IB Standard Level students can find a solution making connections to knowledge IB Studies Standard Level students do not have, namely, geometric series.

- Finding the sum of the first n terms of a geometric series shows why a sum of powers of 2 is a power of 2 minus 1.
- Rewriting the minimum number of moves for a tower n discs tall as a sum of powers of 2 provides more insight into the meaning of the solution.

  Each term in the sum represents the number of times an individual disc is moved:

  The largest disc is moved $2^0 = 1$ time
  The next largest disc is moved $2^1 = 2$ times
  …
  The smallest disc is moved 2^{n-1} times



Answer: The world would end in approximately

585 billion years

Where do Models Come From? A Search for Understanding

Formula (from assumptions) → Data → Regression

Modeling Flour Beetle Populations

Number of Insects = **Number of New Insects** (New Recruits) + **Number of Surviving Insects** (Survivors)

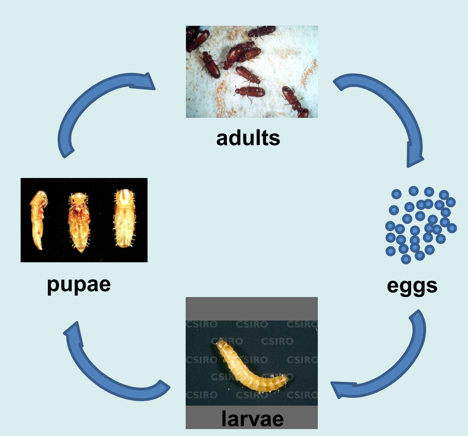
$$\underbrace{x_{t-1}\, b}_{\text{birthrate} \times \text{population size last census}} \times \underbrace{e^{-c\, x_{t-1}}}_{\text{reduced (negative exponent) by cannibalism}} = \textit{Number of new insects}$$

$$\underbrace{\mu}_{\text{probability of death}} \rightarrow \underbrace{1-\mu}_{\text{probability of survival}} \rightarrow x_{t-1}(1-\mu) = \textit{Number of surviving insects}$$

$$x_t = x_{t-1}\, b e^{-c\, x_{t-1}} + x_{t-1}(1-\mu)$$

General Ricker Population Model

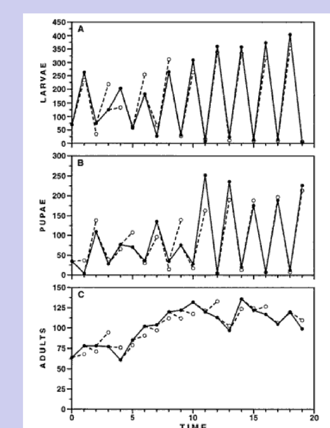
Flour Beetle Life Cycle



$$L_t = b\, A_{t-1}\, e^{-c_{el} L_{t-1} - c_{ea} A_{t-1}}$$
$$P_t = L_{t-1}(1-\mu_l)$$
$$A_t = P_{t-1}\, e^{-c_{pa} A_{t-1}} + A_{t-1}(1-\mu_a)$$

Modeling flour beetle populations 1) given assumptions about life cycles and cannibalism (of eggs and pupae by adults and larvae) and 2) pieces of a general model.

L = Larvae, P = Pupae, A = Adults
b = birth rate, c = cannibalism, μ = death rate
t = time (two weeks)



Test model against data. Do our assumptions explain the major influences on the population? If not, revise assumptions and try again.

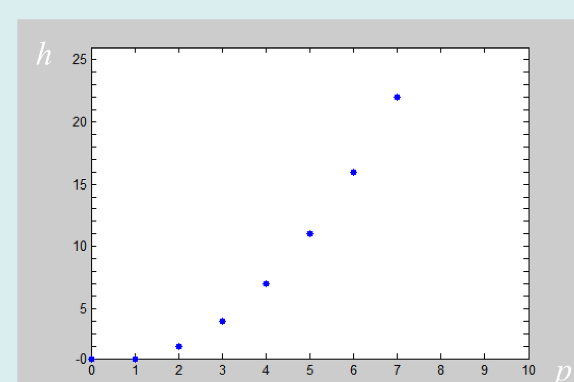
Data → Formula (from "shape") → Regression

The Handshake Problem

How many handshakes, h, take place if every person in a group of size p shake hands with every person exactly once?

p	h
1	0
2	1
3	3
4	6
5	10
6	15
7	21

Generate Data



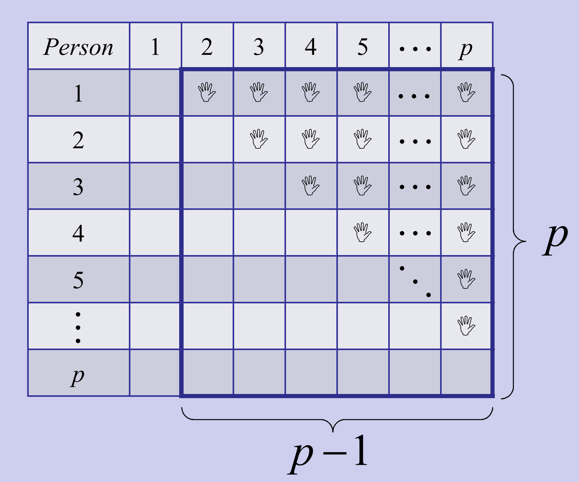
Plotted Data
Looks Quadratic

3) Quadratic Regression $h(p) = 0.5p^2 - 0.5p$

"Answers" Are Not Enough, We Want Understanding

Now we have a formula.
What does it mean?

$$h(p) = 0.5p^2 - 0.5p$$
$$= \frac{1}{2}p(p-1)$$



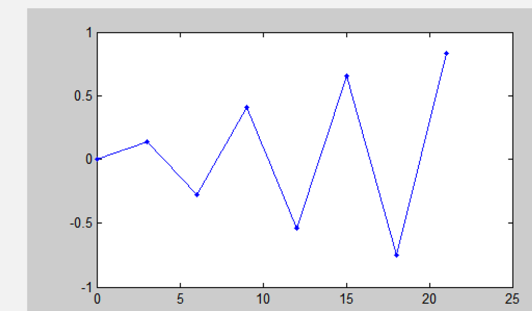
Changing the form of the equation we can see that $p(p-1)$ is the number of handshakes if each person in a group of p people shook hands twice!

Cholla High Magnet School

Honors Intermediate Algebra
International Baccalaureate Studies Standard Level (Juniors/Seniors)
International Baccalaureate Standard Level (Juniors)

What is the Graph of a Function?

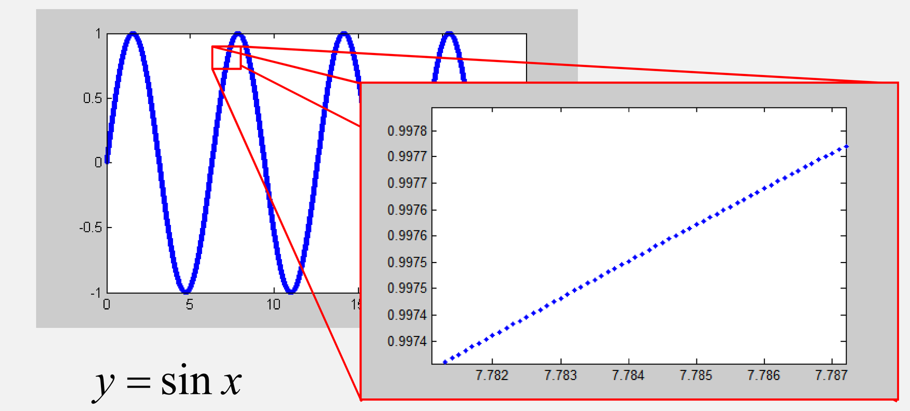
Not a line connecting points!



$y = \sin x$

It is dangerous to treat a graph as points connected by a line. If we don't plot enough points, we don't get a true graphical representation of the function.

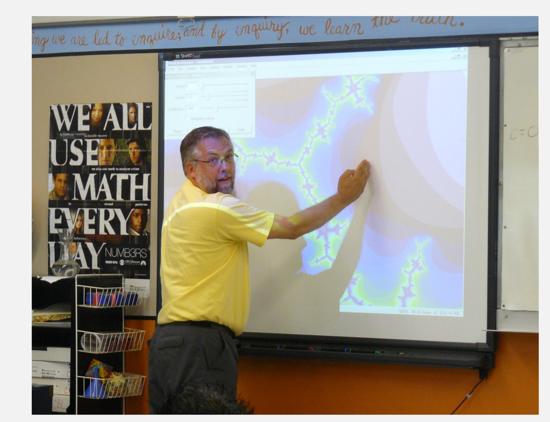
Graphs (that are curves) are collections of points!



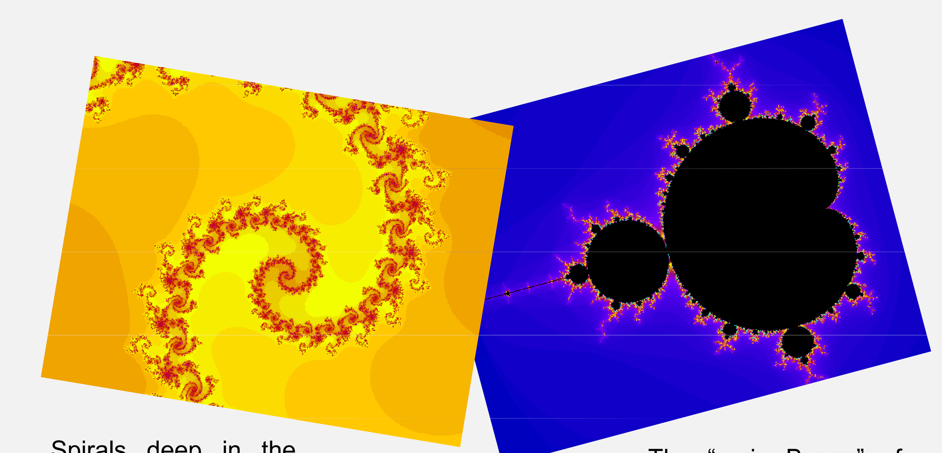
$y = \sin x$

When we plot enough points, a graph will start to look like a curve. Once we have a good sense of the shape of the graph, we can be confident our curve will pass through all the points on the graph.

Fractals: Mandelbrot, Sierpinski Triangle

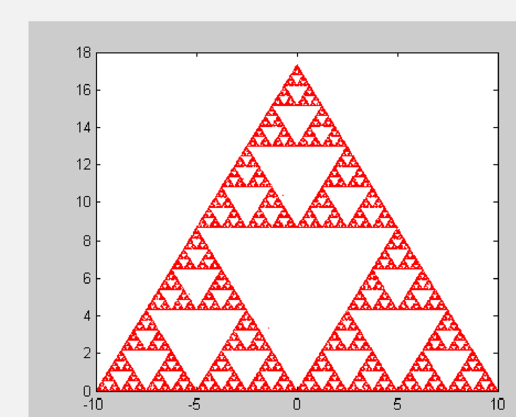


Guest speaker Terry Blackburn explores the Mandelbrot fractal with IB Standard Level students. He explains that at this depth the main "Benny" is light years across.



Spirals deep in the Mandelbrot "meet" at infinity

The "main Benny" of the Mandelbrot set



The Sierpinski Triangle beautifully illustrates the fractal property of self similarity.

Students learn about computer programming to help write the code that generates the triangle.

```
A = 0;

% define vertices of triangle

Px = [];
Py = [];

Px(1) = 10;
Py(1) = 0;

Px(2) = 0;
Py(2) = 10*sqrt(3);

Px(3) = -10;
Py(3) = 0;

% plot triangle

plot(Px,Py);

% define (x,y) coordinates to plot

X = [];
Y = [];

% get user input for initial point

[X(1), Y(1)] = ginput(1);

% start loop

for i=1:50000

    A = randi([1,3]);

    if A == 1

        X(i+1) = (X(i) + Px(1))/2;
        Y(i+1) = (Y(i) + Py(1))/2;

    elseif A == 2

        X(i+1) = (X(i) + Px(2))/2;
        Y(i+1) = (Y(i) + Py(2))/2;

    elseif A == 3

        X(i+1) = (X(i) + Px(3))/2;
        Y(i+1) = (Y(i) + Py(3))/2;

    end

end

plot(X,Y,'.r','MarkerSize',.5);
```