# Examples of Structure in the *Common Core State Standards*' Standards for Mathematical Content

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# Frontispiece: Content Domains in K–8

| | | |
|---|---|---|
| Counting and Cardinality (CC) | K | • Know number names and the count sequence<br>• Count to tell the number of objects<br>• Compare numbers |
| Operations and Algebraic Thinking (OA) | K–5 | • Concrete uses and meanings of the basic operations (word problems)<br>• Mathematical meaning and formal properties of the basic operations<br>• Prepare for later work with expressions and equations in middle school |
| Number and Operations in Base Ten (NBT) | K–5 | • Place value understanding<br>• Develop base-ten algorithms using place value and properties of operations<br>• Computation competencies (fluency, estimation) |
| Number and Operations—Fractions (NF) | 3–5 | • Enlarge concept of number beyond whole numbers, to include fractions<br>• Use understanding of the four operations to extend arithmetic to fractions<br>• Solve word problems related to the equation $ax = b$ ($a$ and $b$ fractions) |
| The Number System (NS) | 6–8 | • Build concepts of positive and negative numbers<br>• Work with the rational numbers as a *system* governed by properties of operations<br>• Begin work with irrational numbers |
| Expressions and Equations (EE) | 6–8 | • Treat expressions as objects to reason about (not as instructions to compute an answer)<br>• Transform expressions using properties of operations<br>• Solve linear equations<br>• Use variables and equations as techniques to solve word problems |
| Ratios and Proportional Relationships (RP) | 6–7 | • Extend work on multiplication and division; consolidate multiplicative reasoning<br>• Lay groundwork for linear functions in Grade 8 by studying quantities that vary together<br>• Solve a wide variety of problems with ratios, rates, percents |
| Functions (F) | 8 | • Extend and formalize understanding of quantitative relationships from Grades 3-7<br>• Lay groundwork for more extensive work with functions in High School |
| Measurement and Data (MD) | K–5 | • Emphasize the common nature of all measurement as iterating by a unit<br>• Build understanding of linear spacing of numbers and support learning of the number line<br>• Develop geometric measures<br>• Work with data to prepare for Statistics and Probability in middle school |
| Geometry (G) | K–8 | • Ascend through progressively higher levels of logical reasoning about shapes<br>• Reason spatially with shapes, leading to logical reasoning about transformations<br>• Connect geometry to number, operations, and measurement via notion of partitioning |
| Statistics and Probability (SP) | 6–8 | • Introduce concepts of central tendency, variability, and distribution<br>• Connect randomness with statistical inference<br>• Lay foundations for High School Statistics and Probability |

## I Examples of Large-Scale Structure

1 Major flows leading to Algebra

2 Major ties between OA, NBT, and NF in Grades K–6

3 Lasting achievements in K–8

4 Units as a pervasive theme in measurement, base ten, and fractions

5 Parallel structures

6 The functional thinking stream

## II Examples of Localized Structure

1 When many standards depend on one

2 When one standard depends on many

3 When many standards depend on one, upon which many depend

4 When a standard depends on many standards from previous grades or is required for many standards in later grades

5 New twists on an old standard

## III A Graph of the Content Standards

## Introduction

The delivery system for mathematics education can sometimes seem to be executing a universal master loop:

```
For i = 1 to number_of_standards;
   Do standard i;
Next i
```

The master loop processes each standard like a separate command, printing textbooks, assembling tests, and outputting the standard of the day. Nothing about the previous standard is cached in memory; every new standard wipes the register clean upon arrival.

This is obviously an exaggeration. And there is nothing wrong with focusing on a standard. Indeed, the *Common Core State Standards for Mathematics* were designed to survive being fed into the master loop, with many individual content standards that are carefully crafted and worth individual attention. Yet the Standards as a whole are also a vision for a coherent mathematical education, and as such the document is more than the sum of its parts.

As long as the entire system factors by individual standard, the coherence of the Standards as a whole will be hard to leverage for achievement gains. And yet, coherence seems like a difficult lever altogether for raising mathematics achievement, because the connections in the Standards that aim to promote coherence are not always easy to see.

Nor are those connections easy to describe. I have tried to do so here using pictures, narrative, metaphors, and quotations from the Standards themselves. I do not cover everything. Rather, by taking up a few examples, I hope to suggest a few new ways of talking vividly about the content of school mathematics and how its pieces fit together.

This document is meant to illustrate some of the coherent structures in the CCSS Standards for Mathematical Content. Among its intended audiences are:

- Curriculum designers who want to create materials with the focus and coherence implied by the standards;

- Assessment designers who want to create diagnostic, formative, or summative assessments that align with the priorities and conceptual structure of the standards;
- Professional developers who want to help teachers see connections within grades and across grades in the standards;
- Innovators in the private sector who want to create next-generation tools and platforms;
- Experts who will revise the standards in years to come, who will need a detailed understanding of the structure of the standards in order to improve upon them;
- Learning scientists and others in basic R&D in the private sector and academia.

*Final note.* In what follows, I do not take a systematic approach to describing the big ideas in the Standards and how they build across grades. While I do touch on some of those issues, the main resource for that kind of explanation is the *Progressions* documents being produced through the Institute for Mathematics and Education at the University of Arizona.[1] One could view the present document as an effort to render some of the information in the *Progressions* diagrammatically, and to give somewhat greater emphasis to cross-domain connections. But in general, I shall try here to avoid redundancy with the *Progressions*, complementing them instead. In particular, this means that what I shall have to say about any given topic is not always the first or most obvious thing one would say about it.

[1] http://math.arizona.edu/~ime/progressions. This project is being chaired by William McCallum, and co-chaired by Phil Daro and myself.

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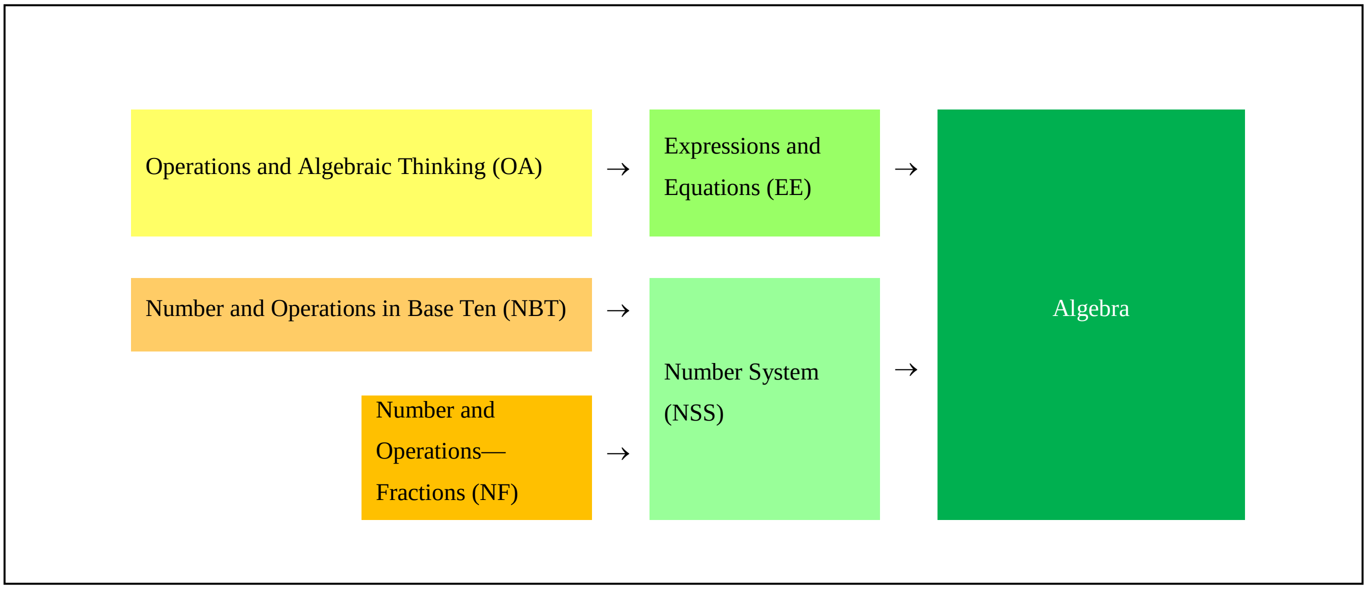
I.1. Major flows leading to Algebra




K 1 2 3 4 5 6 7 8 High School

Operations and Algebraic Thinking leads to explicit work with Expressions and Equations beginning in Grade 6. Grade 6 is also when Base Ten and Fractions are merging to become The Number System. By the end of middle school, Expressions and Equations and The Number System have merged to become high school Algebra.

This picture is a good start. But the harder one thinks, the less simple things become. For example, middle school work in rational numbers relies heavily on the properties of operations. This would be represented by a “crossover arrow” diagonally downward from OA to NS. Likewise, to the extent that the variables which students study in middle school are “general numbers” (an old-fashioned term for them), there might also be a vertical arrow from NS upward to EE. We begin to see that there are important ties between domains.

## I.2. Major ties between OA, NBT, and NF in Grades K–6

The Standards' various content domains serve to focus attention on important themes that span several grades. Each domain has its preoccupations. But just as the energy of a molecule is in its bonds, there is sometimes as much going on in the connections between domains as there is within the domains themselves. To put it another way, if we think of the domains as being like rails, then we also have to be able to see the ties that bind the rails together. This is certainly the case for OA, NBT, and NF.

In this section, we describe ties that bind these three domains together. First, however, given that OA, NBT, and NF have traditionally been considered as a single strand (Number and Operations), it might be worthwhile to begin with a few words about how these domains *differ*. Why are there three different domain headings in the first place?

We begin with OA.

In OA, addition is an operation of putting together or adding to. Subtraction is an operation of taking apart or taking away. These ideas are introduced and developed in Grades K–2.

Multiplication and division then appear in Grade 3, as operations of replication and partitioning, in the sense that $m \times n$ is the number of things in $m$ groups of $n$ things each, while $p \div b$ is the number of things in each share when $p$ things are divided into $b$ equal shares—or the number of shares when $p$ things are divided into equal shares of $b$ things each.

Operations having these meanings prove to be useful in solving word problems. Conversely, the solving of word problems helps to develop the meanings of the operations. The important point for our purposes is that operations model the same quantitative relationships regardless of whether the numbers involved are whole numbers, fractions, decimals, or any combination of these. (Or even a variable standing for any of these.) (Or even a complicated expression standing for any of these.) For example, if a bicyclist travels at speed $v_1$ for time $T$, and then travels at speed $v_2$ for another time $T$, then the total distance traveled is $v_1T + v_2T$. The joining of lengths that is represented here by the + sign is an example of middle grades algebra (e.g., 6.EE.6 or 7.EE.4) yet is already present conceptually in Grade 2. (2.MD.5; 2.MD.6) We don't need to know whether the variables stand for whole numbers or fractions in order to reason this way.

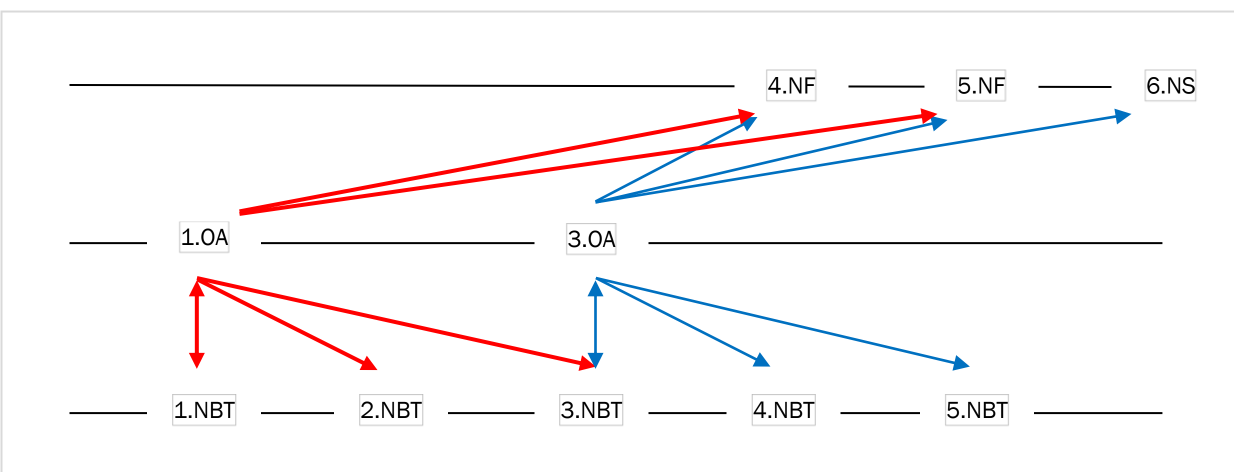
The mathematical properties of operations also hold regardless of the form of the numbers involved. For example, I might use the distributive property in everyday life to compute figures such as 7 × 22 mentally. But I can also use the distributive property algebraically, for example to express the total distance traveled by the bicyclist as $(v_1 + v_2)T$. This has the interesting interpretation that the bicyclist would have arrived at her destination in half the time had she traveled at speed $v_1 + v_2$. (7.EE.1; 7.EE.2; A.SSE.3)

Both aspects of operations—the quantitative relationships they model and their mathematical properties—are introduced in early grades. Yet both aspects remain important beyond the early grades into middle school and high school algebra.

The longevity of OA's ideas makes OA importantly different from NBT.[2] While fluency in multi-digit computation continues to be an important skill throughout one's life, the trajectories of learning that are required to express a sum, difference, product, or quotient of multi-digit numbers as another multi-digit number come to an end by Grade 6—and really by Grade 5, as Grade 6 introduces no new ideas but only brings students to fluency who were not fluent already. From middle school onward, place value techniques then recede as algebraic thinking about operations comes to the fore. In brief: you don't carry the 1 when you add *x* to *y*. One reason to have a separate domain for OA is to help ensure that the right algebraic foundations are being laid during students' study of arithmetic with whole numbers, decimals and fractions.

The NF domain is arguably more directly related to algebra than NBT is. The quotient of *x* and *y* cannot be found by long division—but this quotient can be expressed as a fraction, $x/y$.

So much for the distinctions. Some of the connections between OA, NBT, and NF are summarized in the diagram below. (This diagram only refers to whole numbers in NBT.)




[2] While it is true that polynomial arithmetic is mathematically analogous to base-ten arithmetic, this point is somewhat academic for our purposes. There are several years of algebra to do between the end of the NBT progression and the advent of polynomial arithmetic—most of it at the core of college and career readiness.

Thick red arrows represent addition and subtraction, while thin blue arrows represent multiplication and division. Arrows in the bottom half of the diagram represent the following relationships between OA and NBT.

- The 1.OA standards introduce the properties of addition and the mathematical relationship between addition and subtraction. These ideas combine with place value in 1.NBT as algorithms begin to be developed for adding and subtracting multi-digit numbers.
- No new properties of addition and subtraction are introduced in the Grade 2 standards as multi-digit addition and subtraction algorithms continue to develop in 2.NBT.
- Multi-digit addition and subtraction algorithms continue to develop in 3.NBT.
- In the 3.OA standards, multiplication and division are introduced, including the properties of multiplication and the relationship between multiplication and division. These ideas combine with place value in 3.NBT to begin the development of algorithms for multiplication of multi-digit numbers.
- Multi-digit algorithms for multiplication and division continue to develop in 4.NBT and 5.NBT.

Arrows in the top half of the diagram represent the following relationships between OA and NF.

- The notions of addition as joining or adding to, and subtraction as separating or taking away, are leveraged to extend addition and subtraction to fractions. In Grade 4, this is limited to like fractions, to give students time to build strength with fraction equivalence; in Grade 5, students use equivalent fractions as a strategy to add and subtract unlike fractions.
- The meanings and properties of multiplication are leveraged to extend multiplication to fractions in Grade 4 and Grade 5.
- The relationship between multiplication and division is leveraged to extend division to fractions in Grade 6.

(These connections are to be described at greater length in the *Progressions* documents referred to in the introduction.)

Earlier in this section, I spoke as if the meanings of operations remain static from kindergarten to college. But the meanings of operations evolve and generalize as students' notion of quantity itself evolves and generalizes from the discrete to the continuous. The most important example of this is the way the meaning of multiplication evolves from Grade 3 to Grade 5. The properties of operations play an important part in this development, as does the area model of multiplication that is first introduced in Grade 3.

- The multiplication concept begins in Grade 3 with the entirely discrete notion of "equal groups" (3.OA.1)
- By Grade 4, students can also interpret a multiplication equation as a statement of comparison involving the notion "times as much." (4.OA.1) For example, the equation $3 = 4 \times \frac{3}{4}$ might describe how 3 cups of flour are 4 times as much as ¾ cup of flour. (4.NF.4, 4.MD.2)
- By Grade 5, when students multiply fractions in general, products can be larger or smaller than either factor, and multiplication by a number becomes thought of as an operation that "stretches or shrinks" by a scale factor. This view of multiplication as scaling is the appropriate notion for reasoning multiplicatively with continuous quantities.
- Note that by Grade 5, the main action in multiplicative reasoning has jumped the rails from OA to NF. The next step is reasoning with unit rates—thus, the main action next shifts to RP in Grades 6 and 7, with just one important coda located in 6.NS, division of a fraction by a fraction.

*The supporting role of measurement of continuous quantities*. Work involving measurement of continuous quantities (such as mass, weight, time, and capacity) is obviously relevant to science instruction, but it also plays a conceptual role in the mathematical progressions of multiplication, division, and fractions. Notice that many example problems in the NF standards refer to such quantities:

> *if each person at a party will eat 3/8 of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed?* (whole number × fraction)
>
> 4.NF.4.c

> *how much chocolate will each person get if 3 people share 1/2 lb of chocolate equally?* (whole number ÷ unit fraction)
>
> 5.NF.7.c

> *how many 1/3-cup servings are in 2 cups of raisins?* (unit fraction ÷ whole number)
>
> 5.NF.7.c

> *if 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get?* (fraction as a quotient of whole numbers)
>
> 5.NF.3

## I.3. Lasting achievements in K–8

Most of the K–8 content standards trace explicit steps A $\rightarrow$ B $\rightarrow$ C in a progression. This can sometimes make it seem as if any given standard only exists for the sake of the next one in the progression. There are, however, culminating or capstone standards (I sometimes call them "pinnacles"), most of them in the middle grades, that remain important far beyond the particular grade level in which they appear. This is signaled in the Standards themselves (p. 84):

> The evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8. This body of material includes powerfully useful proficiencies such as applying ratio reasoning in real-world and mathematical problems, computing fluently with positive and negative fractions and decimals, and solving real-world and mathematical problems involving angle measure, area, surface area, and volume. Because important standards for college and career readiness are distributed across grades and courses, systems for evaluating college and career readiness should reach as far back in the standards as Grades 6–8. It is important to note as well that cut scores or other information generated by assessment systems for college and career readiness should be developed in collaboration with representatives from higher education and workforce development programs, and should be validated by subsequent performance of students in college and the workforce.

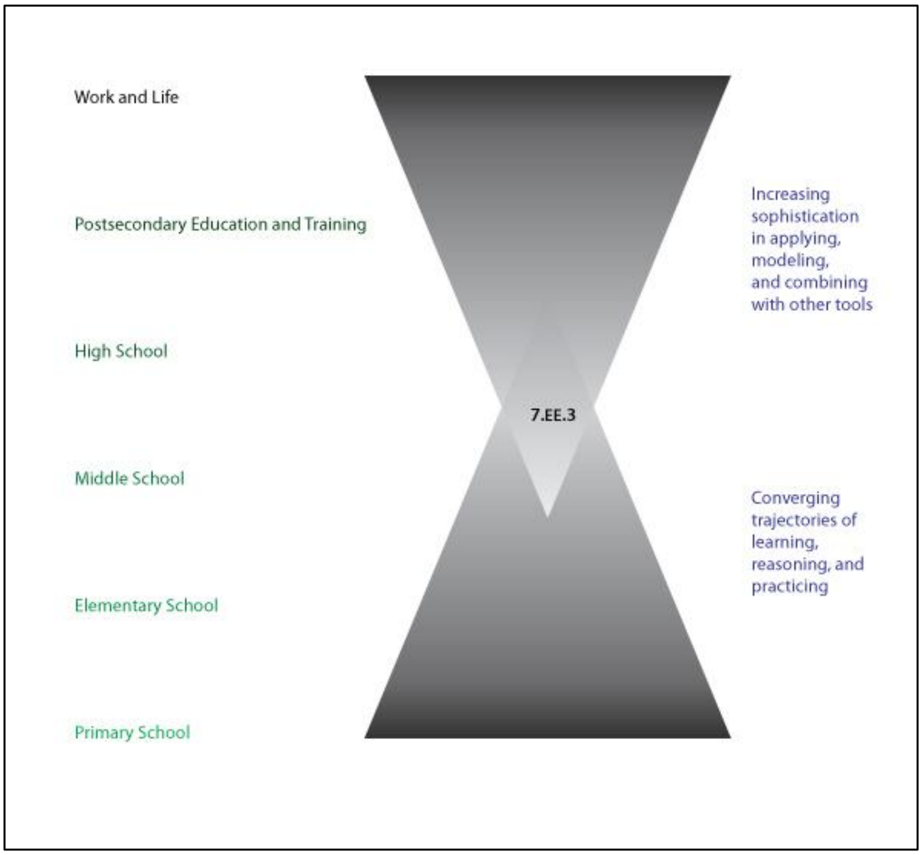
One example of a standard that refers to skills that remain important well beyond middle school is 7.EE.3:

> Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. *For example: If a woman making $25 an hour gets a 10% raise, she will make an additional 1/10 of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar 9 3/4 inches long in the center of a door that is 27*

*1/2 inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.*

Other lasting achievements from K–8 would include working with proportional relationships and unit rates (6.RP.3; 7.RP.1,2); working with percentages (6.RP.3e; 7.RP.3); and working with area, surface area, and volume (7.G.4,6).

As indicated in the quotation from the Standards, skills like these are crucial tools for college, work and life. They are not meant to gather dust during high school, but are meant to be applied in increasingly flexible ways, for example to meet the high school standards for Modeling. The illustration below shows how these skills fit in with both the learning progressions in the K–8 standards as well as the demands of the high school standards and readiness for careers and a wide range of college majors.




As shown in the figure, standards like 7.EE.3 are best thought of as descriptions of component skills that will be applied flexibly during high school in tandem with others in the course of modeling tasks and other substantial applications. This aligns with the demands of postsecondary education for careers and for a wide range of college majors. Thus, when high school students work with these skills in high school, they are not working below grade level; nor are they reviewing. Applying securely held mathematics to open-ended problems and applications is a *higher*-order skill valued by colleges and employers alike.

One reason middle school is a complicated phase in the progression of learning is that the pinnacles are piling up even as the progressions A → B → C continue onward to the college/career readiness line. One reason we draw attention to lasting achievements here is that their importance for college and career readiness might easily be missed in this overall flow.

## I.4. Units as a pervasive theme in measurement, base ten, and fractions

Think about a 4-by-5 rectangle. The rectangle contains infinitely many points—you could never count them. But once you decide that a 1-by-1 square is going to be "one unit of area," you are able to say that a 4-by-5 rectangle amounts to twenty of these units. A choice of unit makes the uncountable countable.

A more ancient example: think about two intervals of time. One is the period of the earth's rotation about its axis; the other, the period of the earth's revolution around the sun. Both intervals are infinitely divisible—a continuum of moments. But once you decide that the first period is going to be a "unit of time," you are able to say that the second period of time amounts to 365 of these units. A choice of unit makes the uncountable countable.

More abstractly, think of a number line. Zero marks its center. The line is an infinitely divisible continuum of points. Now make a mark for the number 1. The interval from 0 to 1 is the unit that makes the uncountable countable. We mark off these units along the line as 2, 3, 4, 5, …. (Later, we go the other way from zero marking off units: −1, −2, −3, −4, −5, ….) It is not for nothing that mathematicians call 1 "the unit."

To a physicist, measurement is an active idea about using one empirical quantity (such as the earth's rotation period) to "measure" (divide into) another empirical quantity of the same general kind (such as the earth's orbital period). Mathematically, one can see that measurement is linked to division: how many units "go into" the quantity of interest.

Another way to say it is that measurement is linked to multiplication, in particular to a mature picture of multiplication called "scaling," (5.OA) in which we reason that one quantity is so many times as much as another quantity. The concept of "times as much" enters the Standards in Grade 4 (4.OA:1,2; 4.NF.4), but only with whole-number scale factors. By Grade 5, scale factors and the quantities they scale may both be fractional; the flow of ideas extends into Grade 6, when students finally divide fractions in general. At that point, we may consider a deep measurement problem such as, "2/3 of a cup of flour is how many quarter-cups of flour?"

The roots of all this in K–2 are 1.MD.2 and 2.MD:1–7, and 2.G:2,3.

When we reflect back on the geometry in K–2 from this perspective, we see that some of what is going on is learning to "structure space" by, for example, seeing a rectangle as decomposable into squares and composable from squares. Researchers show interesting pictures of the warped grids that students make until they get sufficient practice.

By Grade 3 we are dissatisfied with measurement. We want to know what happens when the unit "doesn't go evenly into" the quantity of interest. So we create finer units called thirds, fourths, fifths, and so on. This is the intuitive concept of a unit fraction, $^1/_b$—a quantity whose magnitude is equal to one part of a partition of a unit quantity into *b* equal parts. (3.NF) We reason in applications by thinking of the unit quantity as a bucket of paint, or an hour of time. We reason about fractions as numbers by thinking of the unit quantity as that portion of the number line lying between 0 and 1. Then $^1/_b$ is the number located at the end of the rightmost point of the first partition.

Because you can count with unit fractions, you can also do arithmetic with them (4.NF:3,4). You can reason naturally that if Alice has $^2/_3$ cup of flour (two "thirds") and Bob has $^5/_3$ cup of flour (five "thirds"), then together they have $^7/_3$ cup (seven "thirds," because two things plus five more of those things is seven of those things). The meanings you have built up about addition and subtraction in K–2 morph easily to give you the "algorithm" for adding fractions with like denominators: just add the numerators.

Likewise, multiplying a unit fraction by a whole number is a baby step from Grade 3 multiplication concepts. If there are seven Alices who each have $^2/_3$ cup of flour, it is a bit like when we reasoned out the product $7 \times 20$ in third grade: seven times two tens is fourteen tens; likewise seven times two thirds is fourteen thirds. Again the meanings you have built up about multiplication in Grade 3 morph easily to give you the "algorithm" for multiplying a fraction by a whole number: $n \times {}^{a}/_{b} = {}^{(n \times a)}/_{b}$.

The associative property of multiplication $x \times (y \times z) = (x \times y) \times z$ is implicit in the reasoning for both $7 \times 20$ and $7 \times {}^{2}/_{3}$. So is unit thinking. In $7 \times {}^{2}/_{3}$, the unit of thought is the unit fraction ${}^{1}/_{3}$. In $7 \times 20$ and other problems in NBT, the units of thought are the growing sequence of tens, hundreds, thousands and ever larger units, as well as the shrinking sequence of tenths, hundredths, thousandths, and ever smaller *unit fractions*.

The conceptual shift involved in progressing from multiplying with whole numbers in Grade 3 to multiplying a fraction by a whole number in Grade 4 might be aided by the multiplication work in Grade 4 that extends the whole number multiplication concept a nudge beyond "equal groups" to a notion of "times as many" or "times as much" (4.OA:1,2). The reason this meshes with the problem of the seven Alices is that those seven Alices don't exactly have among them seven "groups of things," yet they do among them have seven "times as much" as one Alice.

The step in Grade 4 from "equal groups" to "times as much," along with the coordinated step of multiplying a fraction by a whole number, represents the first major step toward viewing multiplication as a scaling operation that magnifies or shrinks. Multiplying by 7 has the effect of "magnifying" the amount of flour that a single Alice has. In Grade 5, we will "magnify" by non-whole numbers, for example by asking how many tons 4 ½ pallets weigh, if one pallet weighs ${}^{3}/_{4}$ ton. We will find, during the course of that study, that a product can sometimes be smaller than either factor.

This kind of thinking about scaling is connected to proportional reasoning, as when we use a "scaling factor" to get answers to compound multiplicative problems quickly. For example, 1500 screws in 6 identical boxes…how many in 2 of the boxes? What would be a multi-step multiplication and division problem to a fifth grader becomes, for a more mature student, a proportional reasoning problem: a third as many boxes, so scale the number screws of by a third. We quickly have the answer 500.

A year isn't exactly 365 days—nor is it exactly 365.25 days. Could any rational number express the number of days in a year? Students of mathematics run up against a similar problem when they ask how many times the side of a square "goes into" its diagonal. By Grade 8, we learn without proof that the diagonal cannot be written as any rational multiple of the side. In this way, irrational numbers such as √2 enter the discussion, and likewise π for the quotient of the circumference of a circle by its diameter. The Greeks called the diagonal and the side, or the circumference and the diameter, *incommensurable quantities*. This ancient idiom, meaning *not measurable by a common unit*, underscores the importance of measurement thinking to arithmetic.

## I.5. Parallel structures

Parallel structures in the Standards emphasize family resemblances and recurring themes. Curricula can take advantage of these parallels and recurrences to make the subject as a whole more coherent. For example, as middle grades students enter into algebra, they should be able to appreciate that addition and multiplication have parallel mathematical properties, and that subtraction and division are mathematically derivative of them.

Thus, one important parallel structure occurs in Grades 1 and 3, where fundamentals of addition and subtraction and multiplication and division are respectively introduced:

| Addition and Subtraction, Grade 1 | Multiplication and Division, Grade 3 |
| --- | --- |
| Represent and solve problems involving addition and subtraction. | Represent and solve problems involving multiplication and division. |
| Apply properties of operations as strategies to add and subtract. | Apply properties of operations as strategies to multiply and divide. |
| Understand subtraction as an unknown-addend problem. | Understand division as an unknown-factor problem. |
| Add and subtract within 20. | Multiply and divide within 100. |
| Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. | Determine the unknown whole number in a multiplication or division problem relating three whole numbers. |
| Use place value understanding and properties of operations to add and subtract. | Use place value understanding and properties of operations to perform multi-digit arithmetic. |

An echo of this is in Grade 7, where the fundamentals of addition and subtraction and multiplication and division with rational numbers are introduced:

| Addition and Subtraction, Grade 7 | Multiplication and Division, Grade 7 |
|---|---|
| Apply properties of operations as strategies to add and subtract rational numbers. | Apply properties of operations as strategies to multiply and divide rational numbers. |

Beneath these parallels in standards language are of course deeper parallels that exist at the heart of arithmetic itself—as can be seen from a side-by-side comparison of the properties of operations:

| Properties of Addition | Properties of Multiplication |
|---|---|
| $(a + b) + c = a + (b + c)$ | $(a \times b) \times c = a \times (b \times c)$ |
| $a + b = b + a$ | $a \times b = b \times a$ |
| $a + 0 = 0 + a = a$ | $a \times 1 = 1 \times a = a$ |
| For every $a$ there exists $-a$ so that $a + (-a) = (-a) + a = 0$. | For every $a \neq 0$ there exists $^1/a$ so that $a \times {}^1/a = {}^1/a \times a = 1$. |
| Distributive Property | |
| $a \times (b + c) = a \times b + a \times c$ | |

The properties of operations are a reliable anchor for reasoning; they hold for whole numbers, rational numbers, real numbers, complex numbers, variables standing for any of these things, or complicated expressions standing for any of these things. The distributive property, the only property that refers to both addition and multiplication, connects these two operations and expresses how they relate mathematically.

Another recurring structure can be seen in the parallel language used in Grades 3–5 for area, volume, and angle:

| Area | Volume | Angle |
|---|---|---|
| Recognize area as an attribute of plane figures | Recognize volume as an attribute of solid figures | Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, |
| and understand concepts of area measurement. | and understand concepts of volume measurement. | and understand concepts of angle measurement: |
| a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area. | a. A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume. | a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles. |
| b. A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units. | b. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units. | b. An angle that turns through n one-degree angles is said to have an angle measure of n degrees. |
| Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units). | Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units. | Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure. |
| Relate area to the operations of multiplication and addition. | Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume. | |
| a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths. | a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication. | |
| b. Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning. | b. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems. | |

| Area | Volume | Angle |
| --- | --- | --- |
| c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths *a* and *b* + *c* is the sum of *a* × *b* and *a* × *c*. Use area models to represent the distributive property in mathematical reasoning. | | |
| d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems. | c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems. | Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure. |

## I.6. The functional thinking stream

Many topics in the standards develop over several grade levels, which means that they are not particularly well thought of as topics at all. One might rather call them streams. Streams are not like buckets, or like drawers in a file cabinet. For example, there is no sense arguing about whether a given Standard “fits better in” Stream X or Stream Y. Rather, streams intersect, like rivulets coursing through the Standards.

Streams can be hard to see, because they don’t have headers of their own, and because they don’t always run along the channels defined by domains. For example, here is a stream for functional thinking in the Standards:

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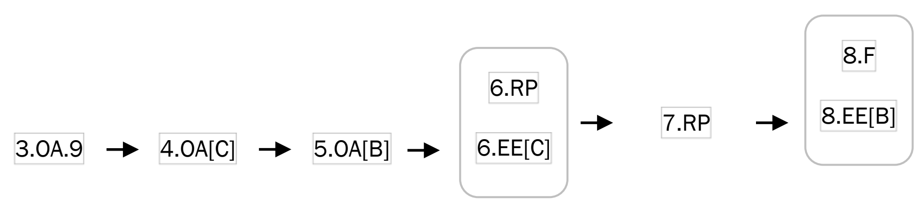



Functional Thinking. Capital letters in square brackets refer to clusters. For example, 6.EE[C] refers to the third cluster in 6.EE.

The following tables show selected standards arranged into a stream for Functional Thinking.

| Emphasis | 3 | 4 | 5 | 6 |
|---|---|---|---|---|
| Patterns | Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.* | **Generate and analyze patterns.**<br>Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.* | | |
| Relationships | | | **Analyze patterns and relationships.**<br>Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. *For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.* | |
| Modeling relationships with variables and equations | | | | **Represent and analyze quantitative relationships between dependent and independent variables.**<br>Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. *For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation $d = 65t$ to represent the relationship between distance and time.*<br>**Understand ratio concepts and use ratio reasoning to solve problems.**<br>Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."*<br>Understand the concept of a unit rate $^{a}/_{b}$ associated with a ratio $a:b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is ¾ cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."* [3]<br>Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations. (a) Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios. (b) Solve unit rate problems including those involving unit pricing and constant speed. *For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed? (c)* Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means $^{30}/100$ times the quantity); solve problems involving finding the whole, given a part and the percent. (d) Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities. |

[3] Expectations for unit rates in this grade are limited to non-complex fractions.

| Emphasis | 6 | 7 | 8 |
|---|---|---|---|
| Modeling relationships with variables and equations | **Represent and analyze quantitative relationships between dependent and independent variables.** Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. *For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation d = 65t to represent the relationship between distance and time.*<br><br>**Understand ratio concepts and use ratio reasoning to solve problems.**<br><br>Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."*<br><br>Understand the concept of a unit rate $^a/_b$ associated with a ratio *a:b* with $b \neq 0$, and use rate language in the context of a ratio relationship. *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is ¾ cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."* [4]<br><br>Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations. (a) Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios. (b) Solve unit rate problems including those involving unit pricing and constant speed. *For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed? (c)* Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means $^{30}/100$ times the quantity); solve problems involving finding the whole, given a part and the percent. (d) Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities. | Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, a + 0.05a = 1.05a means that "increase by 5%" is the same as "multiply by 1.05."*<br><br>Recognize and represent proportional relationships between quantities.<br><br>a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.<br><br>b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.<br><br>c. Represent proportional relationships by equations. *For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as t = pn.*<br><br>d. Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, r) where r is the unit rate. | **Understand the connections between proportional relationships, lines, and linear equations.**<br><br>Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. *For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.*<br><br>Use similar triangles to explain why the slope *m* is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at *b*. |
| Functions | | | **Define, evaluate, and compare functions.**<br><br>Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.[5]<br><br>Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*<br><br>Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. *For example, the function $A = s^2$ giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4) and (3,9), which are not on a straight line.*<br><br>**Use functions to model relationships between quantities.**<br><br>Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (*x*, *y*) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.<br><br>Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally. |

[4] Expectations for unit rates in this grade are limited to non-complex fractions.
[5] Function notation is not required in Grade 8.

## II Examples of Localized Structure

1. When many standards depend on one
2. When one standard depends on many
3. When many standards depend on one, upon which many depend
4. When a standard depends on many standards from previous grades or is required for many standards in later grades
5. New twists on an old standard

II.1. When many standards depend on one

When many standards depend on one, it makes sense to be doubly sure students are meeting the one. These precursors may be opportunities for extended deep treatments in class; for lots of practice; for multiple forms of assessment; and for integration with other material. An example of such a standard might be 2.NBT.1:

| A student may not be able to meet these expectations: | Without being able to meet this one: |
|---|---|
| *Understand Place Value.*<br><br>2.NBT.3. Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.<br><br>2.NBT.4. Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons.<br><br>*Use Place Value Understanding And Properties Of Operations To Add And Subtract.*<br><br>2.NBT.6. Add up to four two-digit numbers using strategies based on place value and properties of operations.<br><br>2.NBT.7. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.<br><br>2.NBT.8. Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.<br><br>*Use Place Value Understanding And Properties Of Operations To Perform Multi-Digit Arithmetic*.<br><br>3.NBT.1. Use place value understanding to round whole numbers to the nearest 10 or 100.<br><br>3.NBT.3. Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9×80, 5×60) using strategies based on place value and properties of operations.<br><br>*Generalize Place Value Understanding For Multi-Digit Whole Numbers*.<br><br>4.NBT.1. Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. For example, recognize that 700 ÷ 70 = 10 by applying concepts of place value and division. | 2.NBT.1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:<br><br>a) 100 can be thought of as a bundle of ten tens — called a “hundred.”<br><br>b) The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones). |

II.2. When one standard depends on many

When one standard depends upon many, the one can be a referendum on the many. We might expect that performance will be low on such a standard, with lots of ways for students to be missing pieces of the puzzle. These postcursors may also be resistant to specific intervention; instead, they may be important opportunities for formative assessment that casts a sufficiently wide net to consider the contributing factors. An example might be 1.OA.6:

| A student may not be able to meet this expectation: | Without being to meet these: |
|---|---|
| 1.OA.6. Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13). | K.OA.2. Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.<br><br>K.OA.3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).<br><br>K.OA.4. For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.<br><br>K.OA.5. Fluently add and subtract within 5.<br><br>1.OA.3. Apply properties of operations as strategies to add and subtract.3 Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)<br><br>1.OA.4. Understand subtraction as an unknown-addend problem. For example, subtract 10 – 8 by finding the number that makes 10 when added to 8.<br><br>1.OA.5. Relate counting to addition and subtraction (e.g., by counting on 2 to add 2). |

II.3. When many standards depend on one, upon which many depend

Some standards have important dependencies in both directions. An example might be 7.NS.3:

| Some precursors | 7.NS.3 | Some postcursors |
|---|---|---|
| 4.OA.3. Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.<br><br>6.NS.3. Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.<br><br>7.NS.1.d. Apply properties of operations as strategies to add and subtract rational numbers.<br><br>7.NS.2.c. Apply properties of operations as strategies to multiply and divide rational numbers.<br><br>7.NS.2.d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats. | Solve real-world and mathematical problems involving the four operations with rational numbers | 7.EE.3. Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. For example: If a woman making $25 an hour gets a 10% raise, she will make an additional 1/10 of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar 9 3/4 inches long in the center of a door that is 27 1/2 inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.<br><br>7.EE.4.a. Solve word problems leading to equations of the form px + q = r and p(x + q) = r, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width? |

II.4. When a standard depends on many standards from previous grades or is required for many standards in later grades

When a standard in one grade depends on many standards from previous grades, it may be especially important for teachers in the earlier grades to know where their students' work is headed, for teachers in later grade to be aware of key precursors from previous grades, and for diagnostic assessment to reach as far back as necessary. An example might be 4.NF.3:

| A student may not be able to meet this expectation: | Without being to meet these: |
|---|---|
| 4.NF.3. Understand a fraction $^{a}/_{b}$ with $a > 1$ as a sum of fractions $^{1}/_{b}$.<br><br>a. Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.<br><br>b. Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model. Examples: $^{3}/_{8} = ^{1}/_{8} + ^{1}/_{8} + ^{1}/_{8}$ ; $^{3}/_{8} = ^{1}/_{8} + ^{2}/_{8}$ ; $2\ ^{1}/_{8} = 1 + 1 + ^{1}/_{8} = ^{8}/_{8} + ^{8}/_{8} + ^{1}/_{8}$.<br><br>c. Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.<br><br>d. Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem. | 1.OA.3. Apply properties of operations as strategies to add and subtract. Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)<br><br>1.OA.4. Understand subtraction as an unknown-addend problem. For example, subtract 10 – 8 by finding the number that makes 10 when added to 8.<br><br>2.OA.1. Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.<br><br>3.NF.1. Understand a fraction $^{1}/_{b}$ as the quantity formed by 1 part when a whole is partitioned into $b$ equal parts; understand a fraction $^{a}/_{b}$ as the quantity formed by $a$ parts of size $^{1}/_{b}$.<br><br>3.NF.2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.<br><br>a. Represent a fraction $^{1}/_{b}$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into $b$ equal parts. Recognize that each part has size $^{1}/_{b}$ and that the endpoint of the part based at 0 locates the number $^{1}/_{b}$ on the number line.<br>b. Represent a fraction $^{a}/_{b}$ on a number line diagram by marking off a lengths 1/b from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.<br><br>4.NF.1. Explain why a fraction $^{a}/_{b}$ is equivalent to a fraction $^{(n \times a)}/_{(n \times b)}$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions. |

When many standards from later grades depend on a standard in an earlier grade, it may be important for teachers in the earlier grade to realize that these "key takeaways" will be important for students later on. An example might be 7.EE.1:

| A student may not be able to meet these expectations: | Without being able to meet this one: |
|---|---|
| 8.EE.7.b. Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.<br><br>A-SSE.2. Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.<br><br>A-REI.3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.<br><br>A-CED.4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. For example, rearrange Ohm's law V = IR to highlight resistance R. | 7.EE.1. Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients. |

## II.5. New twists on an old standard

Small but important phrases are sometimes tucked into individual standards statements. These may indicate shifts in the traditional approach to the topic in question. Such cases might be good starting points for professional development activities—simple ways to get into the document by looking at a single detail that quickly opens up to larger issues.

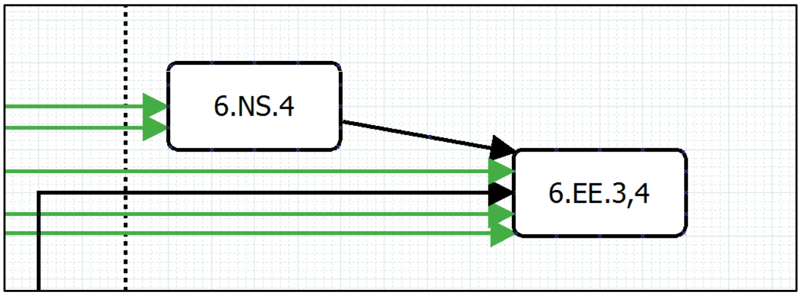
There are any number of cases in which the specific wording of individual standards suggests a shift in approach. Here are just a couple of perhaps less obvious examples:

### *6.NS.4*

This standard refers to greatest common factor—a staple of in-class worksheets. But note the second sentence and the example given:

> Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. **Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express 36 + 8 as 4 (9 + 2).**

The example given is to express 36 + 8, a sum of two numbers with a common factor, as 4(9 + 2), a multiple of a sum of two numbers with no common factor.




This example sends a signal about how "greatest common factor" can be piped into the main flow of ideas leading to algebra. For example, consider what we do when we express $ax + a$ as $a(x + 1)$. We are using the distributive property to express a sum of terms with a common factor as a multiple of a sum of terms with no common factor (6.EE.3,4). Standard 6.NS.4 is a kind of rehearsal for this, with numbers instead of variables.

*3.OA.1,2,6*

> 3.OA.1. Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as 5 × 7.

> 3.OA.2. Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.

> 3.OA.6. Understand division as an unknown-factor problem. For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8.

These three standards are notable for what they don't say. For example, nowhere in the Standards is there an explicit expectation that students will interpret products as repeated addition, jumps on a number line, skip counting, or the like. The Grade 3 standards only explicitly require students to interpret a product in terms of equal groups, and to interpret a quotient as a number of groups, a number in each group, or an unknown factor.

Of course, students come into Grade 3 knowing only addition and subtraction, so at first they will have to use addition in order to find products. But that makes repeated addition a strategy for finding a product, not what the product itself fundamentally means. (And likewise for repeated subtraction as a strategy for finding a quotient, not what a quotient fundamentally means.) The distinction between the meaning of a quotient vs. the steps one takes to compute that quotient is of course an important one. Consider that fractions and decimals have distinct algorithms for computing quotients, yet the operation and its meaning are the same in either case.

Multiplicative reasoning differs in kind from additive reasoning. This difference becomes acute by Grade 5.NF, and it connects to proportional reasoning in Grades 6 and 7. While proportional reasoning problems can sometimes be solved by skip-counting, that is certainly not the goal. Moreover, repeated addition is pretty hopeless as a way to understand a Grade 7 problem such as $-(8/3)\,x = 3/4$.

*6.EE.3*

> Apply the properties of operations to generate equivalent expressions. For example, apply the distributive property to the expression $3\,(2 + x)$ to produce the equivalent expression $6 + 3x$; apply the distributive property to the expression $24x + 18y$ to produce the equivalent expression $6\,(4x + 3y)$; apply properties of operations to $y + y + y$ to produce the equivalent expression $3y$.

Normally, the three examples given in this standard would be called, respectively, distributing, factoring, and collecting like terms. But in the standard itself, none of those terms are used. The implication is that instead of thinking of these techniques as a to-do list of disconnected items, they are all applications of a few fundamental and familiar principles. This is how the Standards aim to approach both arithmetic and algebra altogether; this is coherence in a nutshell.

We end this section on shifts by calling attention to a couple of "eloquent silences" in the Standards. Granting that eloquence is probably an overstatement, one can at least say that certain silences were meant to be audible. Two of these are:

*Fractions in lowest terms*. There is no explicit requirement in the Standards about simplifying fractions or putting fractions into lowest terms. What there is instead is an important progression of concepts and skills relating to fraction equivalence (e.g., 4.NF.4). Observe that putting a fraction into lowest terms is a special case of generating equivalent fractions.

(Of course, generating equivalent fractions can go "either way" – starting from $^4/_{12}$, we might generate an equivalent fraction $^1/_3$, or we might generate an equivalent fraction $^{40}/_{120}$.)

While the standards don't make an explicit demand that answers to fraction problems be put in lowest terms, teachers are of course free to impose that requirement if they wish. If students have a good understanding of fraction equivalence, and fluency with multiplication and division and knowledge of the times table, then putting fractions into lowest terms is presumably not too much of a problem. But in any case, $^4/_{12}$ and $^1/_3$ are equally correct ways to express $4 \times {}^1/_{12}$ as a fraction.

*"Simplifying."* Putting fractions into lowest terms is related to the issue of simplifying expressions in general. Apart from a single instance in the Mathematical Practices, the word "simplify" does not appear in the Standards. One reason for this is that "simple" is sometimes in the eye of the beholder. Is $x^2 + x$ simpler than $x(x + 1)$? (Is adding to a product simpler than multiplying by a sum? Perhaps the edge goes to the latter expression, because one of its atomic expressions is a number and not a variable? Perhaps the edge goes to the former expression, because all of its atomic expressions are the same? Or cosmetically because it has no parentheses?)

Sometimes it can also be important to "complexify" an expression. Depending on the context, writing $1.05P$ in the less(?) simple form $P + 0.05P$ could add insight.

Even in the case of numerical fractions, putting into lowest terms might not always be best—and might not always be particularly simple either. I think for example about my father, who was a screw machine operator at the end of his career. In the shirt pocket of his work uniform, he always carried a metal machinist's scale marked in hundredths of an inch. For him, eight hundredths was a perfectly useful measure. In lowest terms, this is two twenty-fifths—an observation which is pretty academic given that no such number was etched onto his scale. I'm not even sure if he could locate $2/25$ on a hundredths scale; but if he could, it would be because of a good understanding of fraction equivalence.

Of course, none of this is meant to suggest that simplification is never straightforward, or never useful. And mathematics itself is a subject with a very strong aesthetic favoring elegance and simplicity. $(xy^2 + y^3)/y^5$ is an ugly way to express $x/y^3 + 1/y^2$ as a fraction. But a student well-trained in the habit of seeing structure in expressions (A-SSE.2) would quickly see an opportunity to rewrite the numerator of the former expression as $y^2(x + y)$ en route to rewriting finally as $(x + y)/y^3$. Anybody would agree that this is a simplification. The point is that simplification, least common denominators, and putting into lowest terms shouldn't become little tin gods that mock the fundamental mathematical ideas involved in generating equivalent fractions and generating equivalent expressions.