Rates of Change
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1 Introduction

Consider the following problem involving rates of change:

Problem: *A mine tunnel goes into the side of a hill. Suppose $G(s)$ is the total amount of gold (in ounces) extracted from the tunnel when the tunnel is s feet long.*

(a) *What is the practical meaning of the quantity* $\frac{G(200)-G(150)}{50}$*?*

(b) *What is the practical meaning of the quantity* $G'(100)$*?*

Be sure to use correct units in your answer.

It can be expected that a significant number of calculus students would find this problem challenging. Many students have difficulty relating the symbolic and graphical representations of rate of change to a description in words in a real-life situation (especially using the words "average" and "instantaneous"), and tend not to use units to guide them. Some students also have a misconception that because the average rate of change is over an interval, the instantaneous rate of change is also over an interval.

2 Key issues and common misconceptions

2.1 Centrality

The concept of rate of change is perhaps the central concept in differential calculus. It is key to understanding approximation and maximization methods, and the fundamental theorem of calculus, among other things. Yet students consistently have misconceptions about rates of change.

Common misconceptions:

- Many students have a poor grasp of the concept of rate of change, and end up identifying a rate of change with a slope of a tangent line or a chord without being able to understand it in a real-world or algebraic context.
- Students tend to compartmentalize their knowledge, and may not realize that topics such as the chain rule, related rates, implicit differentiation, *etc.*, are all about rates of change. They have a difficult time recalling the concept of rate of change in the context of the Fundamental Theorem of Calculus.

2.2 Different kinds of rate of change

There are different notions that involve rates of change of a function f:

1. Average rate of change of f between two points.
2. Instantaneous rate of change of f at a point.
3. Average rate of change of f between a and x, as a function of x.
4. Instantaneous rate of change of f at x, as a function of x.

There are significant linguistic and notational issues associated with these distinctions.

Common misconceptions:

- Students tend not to distinguish among the different kinds of rates of change, especially between average and instantaneous. For example, some students may think it is possible to compute an instantaneous rate exactly from a table of values of a function, or that it is possible to compute the change of a function over an interval using only the instantaneous rate of change at a point.
- "Average rate of change" is not an average in the usual (arithmetic) sense, which can lead to incorrect reasoning.

2.3 Rates as quantities

Calculus not only requires students to find rates of change, but also to reason with rates as they do with other quantities (functions).

Common misconceptions:

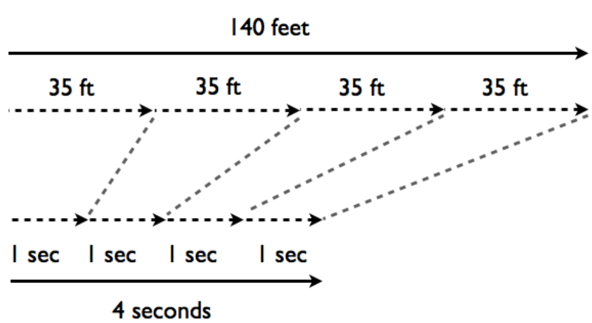
- Students have a difficult time transitioning from a rate being the result of a computation (an "answer") to a rate being a variable they can manipulate and solve for. As an example, students may not be able to understand the statement "the rate of growth of a population is proportional to its size."

3 Teaching suggestions

3.1 Teaching different kinds of rates

3.1.1 Average rate of change

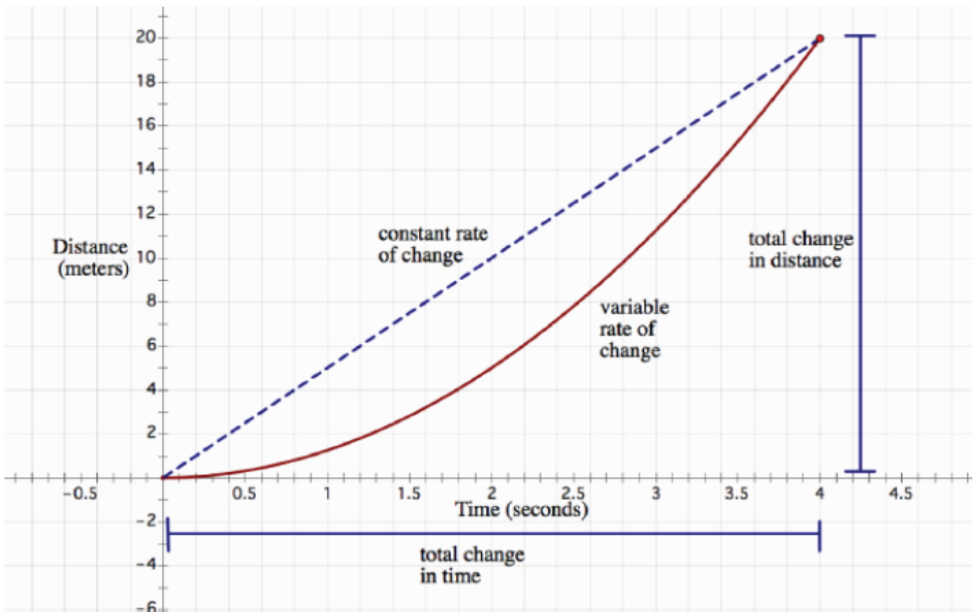
To understand this concept properly, students first need to have a solid grasp of the notion of constant rate of change. Saying that one quantity changes at a constant rate with respect to another means that any fraction of the total change of one quantity corresponds with the same fraction of the total change of the other quantity. For example, if an object is moving at a constant speed during a time interval, then in $\frac{5}{6}$ of the total time, the object will travel $\frac{5}{6}$ of its total distance, and to travel $\frac{2}{7}$ of the total distance requires $\frac{2}{7}$ of the total time. The figure below, adapted from Thompson (1994a), p. 232, illustrates this principle:



If a particle is moving at a constant rate (speed), and it travels 140 feet in 4 seconds, then the time it takes to travel *any* interval of length $\frac{1}{4} \cdot 140 = 35$ feet is $\frac{1}{4} \cdot 4 = 1$ second. Similarly, the time it takes to travel any interval of length $\frac{1}{5} \cdot 35 = 7$ feet is $\frac{1}{5} \cdot 1 = \frac{1}{5}$ second.

A teaching suggestion is to use motion along parallel lines in a coordinated manner, being explicit about the effort to coordinate the motion. Physical motion, animations, and other computer-based tools can help facilitate this learning.

With this notion in place, the average rate of change of one quantity with respect to another during an interval can be introduced as the constant rate of change necessary to yield the same net change in the function values over the entire interval. For example, if a person travels 20 meters in 4 seconds at a variable rate, then the average rate of change is 5 meters per second, because a person traveling at a constant speed of 5 meters per second would travel the same 20 meters in 4 seconds (see the figure below). In this approach, the average rate of change is a way to compare a variable rate of change to a constant rate of change over an interval.



3.1.2 Instantaneous rate of change

The passage from average rate of change to instantaneous rate of change is perhaps the best motivation for introducing the notion of limit in calculus. The following example illustrates one way to lead students through the transition from average to instantaneous rate of change.

Problem: *A package is dropped from a helicopter. The package's distance from the ground in feet t seconds after it is dropped is given by the equation*

$$h(t) = 400 - 16t^2.$$

It is known that the package can withstand the fall if its speed at impact with the ground is less than 165 *feet per second.*

(a) *How far does the package fall during the last second before it hits the ground? What is the package's average speed during this last second?*

(b) *What is the package's average speed during the last half second before it hits the ground?*

(c) *What is the package's average speed during its last tenth of a second in the air?*

(d) *What is the package's speed at the exact moment it hits the ground?*

3.1.3 The derivative as a function

To ease the transition from the notion of derivative at a point to the derivative as a function, it is possible to use technology to illustrate that a differentiable function has a derivative at each point, and plotting the instantaneous rates of change yields a new function. A nice interactive graph showing the derivative as a function can be explored at `http://www.calculusapplets.com/derivfunc.html`.

3.1.4 The role of time

Most problems dealing with rate of change involve time as the independent variable. Discussing examples where time is not the independent variable may be useful to solidify students' conceptual understanding.

The problem posed in the Introduction not only addresses the difference between average and instantaneous rate of change, but it also illustrates that the independent variable need not always be time. The following is an example where a quantity with the units of time is the dependent variable:

Example: *The period of a simple pendulum of length l is approximately*

$$T = 2\pi\sqrt{\frac{l}{g}},$$

where l is measured in meters and g is a constant with units meters per second squared. What is the rate of change of the period with respect to the length of the pendulum? Discuss the units in this calculation.

3.2 Emphasizing centrality

The following two examples illustrate how the theme of rate of change can be used to teach concepts that may seem unrelated or disjointed from the student's perspective.

3.2.1 Related rates

The following simple example shows that a good understanding of rates of change suffices to solve some related rates problems:

Example: *Suppose that plane tickets from Seattle to Tucson cost* 50% *more than tickets from New York. If the cost of tickets from New York is increasing by \$100 per year, how fast is the cost of tickets from Seattle increasing?*

The key idea here is that a relationship between quantities implies a relationship between their rates of change. If the ratio of the change in Y to the change in X is 2, then Y is changing twice as fast as X. In symbols,

$$\frac{dY}{dX} = 2 \Rightarrow \frac{dY}{dt} = 2 \cdot \frac{dX}{dt}.$$

As a generalization, provide tables of data showing the cost of Seattle tickets and the ratio of the cost of Seattle tickets to the cost of New York tickets. Use this data to estimate the rate of change of the cost of New York tickets.

3.2.2 The Chain Rule

Many students view the Chain Rule as a computational tool and do not realize that it is a principle about rates of change. The following example emphasizes the latter perspective:

Example: *Suppose the depth of snow, in inches, along a path is given by $h = 3x^2$, where x is the distance along the path, in feet, from the starting point.*

(a) *How fast is the depth of snow changing with respect to position two feet from the starting point?*

(b) *Now suppose that your position as you walk along the path is given by $x = 2t$, with t in seconds. How fast is the depth of the snow where you are walking changing with respect to time when you are two feet from the starting point?*

A good introduction to this problem would be to first regard it as a question about average rates of change, say over the second foot traveled, then do the problem as stated, which involves instantaneous rates of change.

The key idea here is that average rates of change can be treated as ratios, so that

$$\frac{\Delta h}{\Delta x} \cdot \frac{\Delta x}{\Delta t} = \frac{\Delta h}{\Delta t}.$$

This immediately suggests that a similar relationship should hold for derivatives, which it does:

$$\frac{dh}{dx} \cdot \frac{dx}{dt} = \frac{dh}{dt}.$$

The "cancellation" of the two "factors" of dx extends to the associated units; dimensional analysis can be quite helpful in checking chain rule computations.

4 What is known about student learning

4.1 Centrality of rate of change

Relating the rate of change of one quantity to another is fundamental to the study of both integral and differential calculus. Understanding derivatives requires understanding how changes in one quantity correspond to changes in another quantity. Meanwhile, understanding the Fundamental Theorem of Calculus relies on students' understanding of rates because a definite integral takes a function describing the rate of change over some interval and gives information about the total accumulation of change (Thompson, 1994a). The rate of change is the concept that unifies derivatives, integrals, related rates, and differential equations.

4.2 Rate is a quantity

Prior to calculus, most students learn to find the rate of change as a computational process. For example, algebra students may be given two points and asked to find the slope of a line passing through those points. In calculus, students are confronted with the new challenge of working with the rate of change of a function as an object rather than as a process (Tall, 1997). A typical calculus problem that requires treating the rate of change as an object is to write a differential equation for a population that grows at a rate proportional to itself. Previous research has found that students have a difficult time reasoning with rates as quantities rather than as the result of a computational process (Lobato & Thanheiser, 1999; Stump, 2001). This hurdle is rooted in the fact that a rate of change is an intensive quantity (Schwartz, 1988); that is, a single quantity that represents the relationship between two other quantities that co-vary.

Seeing and interpreting the rate of change of one quantity with respect to another is far more subtle than seeing and interpreting a single quantity like the height of a bar in a histogram, or the angle of steepness of a line. Teachers can use carefully designed instructional tools to facilitate this transition (Noble, Nemirovsky, Dimattia, & Wright, 2004; Schnepp & Nemirovsky, 2001). These tools rely on the theory that students develop advanced images of rates of change by coordinating how two quantities co-vary, and seeing the value of the rate as a way of quantifying the observed relationship (Thompson, 1994b).

5 References

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