1.11   Monthly Payments

Suppose you want to buy a car. You can put $1000 down and pay $250 per month. The interest rate is 5%, and the dealer wants the loan paid off in three years. What kind of car can you buy?

A recursive approach lets you experiment using your function modeling language. Can you afford a $10,000 car? Well, you'd borrow $9,000 and at the end of 36 months, you'd want the balance to be 0.

Begin with a simpler model, one that would hold in an ideal world where you didn't have to pay any interest.

- At the end of month 0, you owe $9,000 (that's how much you borrowed, and month 0 is when the loan starts).

- For any month after month 0, you own $250 less than the month before (because you paid $250).

If $b(n)$ is the balance in dollars at the end of the n-th month

$$b(n) = \begin{cases} 9,000 & \text{if } n = 0 \\ b(n-1) - 250 & \text{if } n > 0 \end{cases}$$

← Every now and then, car dealers offer a 0% interest rate.

For You to Try

How much do you owe after:

1. 1 month?

2. 2 months?

3. 6 months?

4. 1 year?

Can you pay off your car in 36 months? If not, how much do you still owe?

For Discussion

Find a closed form for the function b above.

What happens when we take the 5% interest into account?

- At the end of month 0, you owe \$9,000 (that's how much you borrowed, and month 0 is when the loan starts).

- For any month after month 0, you owe the balance last month, plus the interest on that balance, minus \$250.

The interest for one month is $\frac{1}{12}^{\text{th}}$ the interest for the whole year. The interest for a year is 5%, the interest for a month is $\frac{.05}{12}$ times the balance for that month.

This gives us a refinement of our description for b.

$$b(n) = \begin{cases} \$9,000 & \text{if } n = 0 \\ b(n-1) + \frac{.05}{12}b(n-1) - \$250 & \text{if } n > 0 \end{cases}$$

At the end of the first month you owe:

- \$9,000 (what you owed at the beginning of the month) *plus*

- $\frac{.05}{12} \cdot \$9,000$ (which is \$45.83) *minus*

- \$250 (your monthly payment).

$$\$9,000 + \frac{.05}{12} \cdot \$9,000 - \$250 = \$8787.50$$

The bank uses this rule: What you owe at the end of the month is what you owed at the start of the month, plus $\frac{1}{12}$ of the yearly interest on that amount, minus the monthly payment.

For You to Try

1. What would you owe at the end of month 2?

2. Month 3?

The Memory Error

If you model the new recursive definition for b above, you see that your system quickly runs out of memory. Why?

← Your calculator may return a message such as, "Recursion too deep."

How would a computer find $b(4)$ using this definition? You tell it that $n = 4$ and as it scans the definition it notes what should be done. It sees two places where it will need to compute $b(3)$.

$$b(4)=b(3)+\frac{.05}{12}\cdot b(3)-250$$

$$b(2)+\frac{.05}{12}\cdot b(2)-250 \qquad\qquad b(2)+\frac{.05}{12}\cdot b(2)-250$$
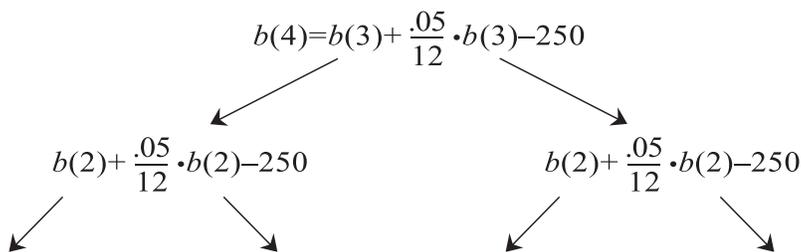
In each of these **2** computations of $b(3)$, the computer needs to compute $b(2)$ twice, for a total of **4** times. Continuing, it computes $b(1)$ **8** times, and $b(0)$ (the base case) **16** times. Then it feeds this information back through three layers of computation to do **1** last calculation to get $b(4)$.

Adding the boldfaced numbers of evaluations of b, we have

$$\mathbf{1 + 2 + 4 + 8 + 16} = 31 = 2^{(4+1)} - 1$$

That's how many computations are needed to compute $b(4)$. In general, calculating $b(n)$ requires tracking $2^{(n+1)} - 1$ evaluations of b. No wonder your calculator or computer runs into trouble quickly.

Habits: Estimating shows $2^{10} > 1,000$ and $2^{20} > 1,000,000$.

But where your calculator sees *two* calculations to be done separately, a person would notice that they are the same—both called $b(n-1)$. You could figure it out once, then plug it in both places.

$$b(4)=b(3)+\frac{.05}{12}\cdot b(3)-250$$

$$b(2)+\frac{.05}{12}\cdot b(2)-250$$
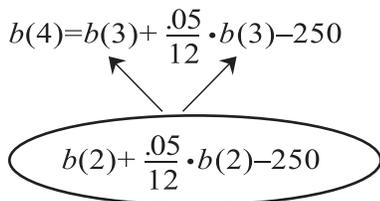
You can force the computer to do the calculation the smart way as well. If you take the expression $b(n-1) + \frac{.05}{12} \cdot b(n-1)$ and factor out the $b(n-1)$, you get:

$\leftarrow$ Smart calculation doesn't guarantee you'll never get the memory error, just that you won't get it as soon!

$$b(n-1) + \frac{.05}{12} \cdot b(n-1) = 1 \cdot b(n-1) + \frac{.05}{12} \cdot b(n-1)$$
$$= b(n-1)\left(1 + \frac{.05}{12}\right)$$

Now the computer only needs to track $n + 1$ evaluations of b in order to find $b(n)$. This change makes the computation manageable.

Habits: Algebra to the rescue.

For You to Try

Build a model in your function modeling language for the function b defined by

$$b(n) = \begin{cases} 9000 & \text{if } n = 0 \\ \left(1 + \frac{.05}{12}\right)b(n-1) - 250 & \text{if } n > 0 \end{cases}$$

How much do you owe after 6 months? How long does it take you to pay the loan down to \$8000?

← See the Technology Appendix in the back of this book to learn how you can add extra inputs that will let you change, on the fly, the monthly payment, the amount of the loan, or the interest rate.

Exercises

Try These

1. For a $9000 loan at 5% interest with a $250 monthly payment, how much would you owe at the end of a year?

2. For the loan in Exercise 1, can you pay off the loan in 36 months?

3. For the loan in Exercise 1, what monthly payment will let you pay off the car in
   (a) 36 months?
   (b) 39 months?
   (c) 48 months?

4. If you can really only afford $250 per month, and you must pay it off in 36 months at 5% interest, how much money can you borrow?

5. What would the interest rate have to be so that you could afford a $12,000 car with $1000 down and $310 per month payments for 36 months?

On Your Own

6. Pick an interest rate and keep it constant. Suppose you want to pay off a car in 36 months. Investigate how the monthly payment changes with the cost of the car:

   Make sure you say what interest rate you are using. Maybe you can get it from a newspaper car ad.

   (a) Make a table like this and complete it:

Cost of car (in thousands of dollars)	Monthly payment
10	
11	
12	
13	
14	
15	
16	
17	
18	
⋮	⋮

(b) Find either a closed form or a recursive definition for a function that agrees with your table. Build a model of your function.

(c) Use your model to find the monthly payment on a $26000 car.

← If this function works for all inputs, you can use it to calculate the monthly payment in terms of the cost of the car in thousands of dollars.

7. A local car dealer has an ad each week that offers two deals on its cars:

- You can get a $2000 rebate on the list price of the car. You then pay off the rest in 36 months at 5% interest.
- Instead of the rebate, you can get a low 0.9% interest rate and then pay off the full list price of the car in 36 months.

Cars on the lot sell for between $20,000 and $40,000. For which cars is it better to take the rebate? For which cars is it better to take the low rate? For what price car is the deal the same?

Habits: Make a table for costs between 20000 and 40000 in increments of 1000, and figure out the monthly payment for each. Your function modeling language and a spreadsheet can help here.

Patterns in Practice

8. Evaluate each sum.

(a) $1 + 2$
(b) $1 + 2 + 2^2$
(c) $1 + 2 + 2^2 + 2^3$
(d) $1 + 2 + 2^2 + 2^3 + 2^4$
(e) $1 + 2 + 2^2 + 2^3 + 2^4 + 2^5$
(f) $1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + \cdots + 2^9$

9. Suppose

$$s(n) = 1 + 2 + 2^2 + 2^3 + \cdots + 2^n$$

(a) Find a recursive definition for s.
(b) Find a closed form for s.

Project: More About Monthly Payments

There are severable variables at play when you buy a car:

- The cost of the car.

- The interest rate.

- The monthly payment.

- The number of months (the "term") that you take to pay off the car.

These variables are related by a general function b that gives the balance at the end of n months on a loan amount of c dollars at an interest rate of $i\%$ (represented by a decimal) with a monthly payment of m dollars:

$$b(n, c, q, m) = \begin{cases} c & \text{if } n = 0 \\ (1 + q)b(n - 1, c, q, m) - m & \text{if } n > 0 \end{cases}$$

$\leftarrow$ Here, $q = 1 + \frac{1}{12}$. See the Technology Appendix at the end of this book for help modeling this function.

In this project, you'll look at the effects of two of the variables on the rest:

In Exercise 6 on page 100, you kept i and n constant and saw how m depended on c.

- If i and n are constant, how does c depend on m?

- If c and n are constant, how does m depend on i?

- If i and c are constant, how does m depend on n?

1. Pick an interest rate and keep it constant. Suppose you want to pay off a car in 36 months. Investigate how much you can spend on a car for a given monthly payment:

$\leftarrow$ Make sure you say what interest rate you are using. Maybe you can get it from the newspaper car ads.

   (a) Make a table like this:

Monthly payment	Cost of car
200	
210	
220	
230	
240	
250	
260	
270	
280	
⋮	⋮

(b) Find either a closed form or a recursive definition that lets you calculate the cost of the car in terms of the monthly payment. Build a model for your function.

← Include your model in your write-up.

(c) Use your model from part 1b to find the cost of a car you could get for a \$360 monthly payment. Check your result with the original model for b.

2. Investigate the effect of interest on the monthly payment. Pick a car that you'd like to buy and keep its price constant. Suppose you want to pay off a car in 36 months, and the interest rates might range from 0% to 15%.

← Make sure you say what your dream car costs. Of course, you want to get the lowest interest rate you can, but the point here is to see how interest affects monthly payments.

(a) Make a table like this:

Interest rate (in percent)	Monthly payment
0	
1	
2	
3	
4	
5	
6	
7	
8	
⋮	⋮

(b) Find the monthly payment on your car if the interest rate is 13%.

← Would the interest rate ever be 13%? Has it ever been 13% in the past?

(c) **Take It Further.** Find either a closed form or a recursive definition that lets you calculate the monthly payment in terms of the interest rate (expressed either as a decimal or a percent). Build a model for your function.

(d) **Take It Further.** Use the model you wrote in part 2c above to find the monthly payment on your car if the interest rate is 13%. Check your result with the original model for b.

3. Investigate the effect of the term on the monthly payment. Pick a car that you'd like to buy and keep its price constant. What is the monthly payment on the car as the number of months varies?.

← The **term** is the number of months you take to pay off the car.

(a) Make a table like this:

← Make sure you say what your dream car costs.

Term (in months)	Monthly payment
24	
28	
32	
36	
40	
44	
48	
52	
56	
⋮	⋮

(b) Find the monthly payment on your car if the term of the loan is five years.

(c) **Take It Further.** Find either a closed form or a recursive definition that lets you calculate the monthly payment in terms of the term of the loan. Build a model for your function.

(d) **Take It Further.** Use the model you wrote in 3c above to find the monthly payment on your car if the term of the loan is five years . Check your result with the original model for b.

n	$t(n)$	$\sum$
0	3	3
1	6	9
2	12	21
3	24	45
4	48	93
5	96	

1. Find both closed and recursive forms for t.

2. Find both closed and recursive forms for the associated series $T(n) = \sum_{k=0}^{n} t(k)$.

3. Show that $T(n) = 2T(n-1) + 3 = \mathfrak{M}_{2,3}(T(n-1))$

Habits: Can you use part 3 to find a closed form for T?

4. What is $\sum_{k=0}^{18} t(k)$?

Monthly Payments

In Lesson 11 of Chapter 1, you experimented with the monthly payment on a loan, modeled by the "balance function"

$$b(n, m) = \begin{cases} 10000 & \text{if } n = 0 \\ \left(1 + \frac{.05}{12}\right) b(n-1, m) - m & \text{if } n > 0 \end{cases}$$

Here, the parameters are

1. The amount borrowed is \$10000.

2. The interest rate is 5% .

3. Payments are made at the end of each month.

Then $b(n, m)$ gives you the balance on the loan at the end of n months, with a monthly payment of m dollars.

For Discussion

Review the construction of this function and why it does what we say it does. Model it in your function modeling language and experiment with it. You might want to introduce two more inputs for the initial amount of the loan and the interest rate so that you can control these things on the fly.

$\leftarrow$ See the Technology Appendix in the back of this book for advice on how to model b.

Exercise 6 from Lesson 1.11

Pick an interest rate and keep it constant. Suppose you want to pay off a car in 36 months. Investigate how the monthly payment changes with the cost of the car:

1. Make a table like this and complete it:

Cost of car (in thousands of dollars)	Monthly payment
10	
11	
12	
13	
14	
15	
16	
17	
18	
$\vdots$	$\vdots$

2. Find either a closed form or a recursive definition for a function that agrees with your table. Build a model of your function.

3. Use your model to find the monthly payment on a $26000 car.

← If this function works for all inputs, you can use it to calculate the monthly payment in terms of the cost of the car in thousands of dollars.

It appeared at the time that that Δ column for this table was *constant*. That is, it seemed that the monthly payment was a *linear function* of the cost of the car. We can use geometric series to see why.

Suppose you borrow C dollars at an interest rate of $i\%$, where i is converted to a decimal, so that we can calculate with it. Then the more general function b is defined recursively by:

$$b(n.m) = \begin{cases} C & \text{if } n = 0 \\ \left(1 + \frac{i}{12}\right) b(n-1, m) - m & \text{if } n > 0 \end{cases}$$

Notice that $\left(1 + \frac{i}{12}\right)$ is a constant. You could cover it up with your hand and deal with it as if it were one chunk q:

$$\text{Let } q = \left(1 + \frac{i}{12}\right)$$

So our function definition looks like this:

$$b(n.m) = \begin{cases} C & \text{if } n = 0 \\ qb(n-1, m) - m & \text{if } n > 0 \end{cases}$$

← Hmmm . . .

$$b(n, m) = \mathfrak{M}_{q,-m}(b(n-1, m))$$

More about this in *CME Project Precalculus*.

Let's unstack this definition and look for some regularity in the calculation:

$$\begin{aligned} b(n, m) &= q \cdot b(n-1, m) - m \\ &= q \left(q \cdot b(n-2, m) - m\right) - m = q^2 \cdot b(n-2, m) - qm - m \\ &= q^2 \left(q \cdot b(n-3, m) - m\right) - qm - m \\ &= q^3 \cdot b(n-3, m) - q^2 m - qm - m \\ &\vdots \\ &= q^n \cdot b(0, m) - q^{n-1}m - q^{n-2}m - \cdots - q^2 m - qm - m \\ &= C \cdot q^n - m(q^{n-1} + q^{n-2} + \cdots + q^2 + q + 1) \end{aligned}$$

The last series is *geometric*. Summing it, we get

$$b(n, m) = C\,q^n - m\frac{q^n - 1}{q - 1}$$

Setting $b(n, m)$ equal to 0 gives an explicit relationship between m and the cost of the car:

$$m = C\,\frac{(q-1)q^n}{q^n - 1}$$

or, in general,

Theorem 7.2

If a loan for C dollars is taken out for n months at an interest rate of $\frac{i}{100}$, then

$$monthly\ payment = C \times \frac{(q-1)q^n}{q^n - 1}$$

where n is the term of the loan and

$$q = 1 + \frac{interest\ rate}{12}$$

For Discussion

Why does this theorem say that the monthly payment is a linear function of the amount borrowed? If that linear function is written as

$$m = aC + b$$

what is a? b?

Exercises

Try These

1. Find $g(7)$ for each geometric sequence below:

(a) $g(1) = 5$, $r = 2$

(b) $g(1) = 5$, $r = -2$

(c) $g(1) = 5$, $r = 1$

(d) $g(1) = 5$, $r = \frac{1}{2}$

(e) $g(5) = 18$, $g(6) = 9$

(f) $g(3) = 12$, $g(5) = 30$

(g) $g(6) = 6$, $g(8) = 54$

(h) $g(5) = 6$, $g(9) = 54$

2. Can a sequence be geometric and arithmetic at the same time? Explain.

3. Reflect and Write.

- Write at least 4 pairs of sequences. A pair will consist of an arithmetic sequence a and a geometric sequence g with the same first term and the common difference of a equal to the common ratio of g. For example:

$\leftarrow$ Make sure you have first terms, common differences and common ratios of various signs and values.

$a_1 = 2, 5, 8, 11, 14 \ldots$ (initial term $= 2$, common difference $= 3$)

$g_1 = 2, 6, 18, 54, 162 \ldots$ (initial term $= 2$, common ratio $= 3$)

The common difference for the arithmetic series a is 3.

The common ratio for the geometric series g is 3.

- Compare the behavior of sequences in each pair. Do both of them increase or decrease?

4. If $s(n) = 5 \cdot \frac{1}{2}^n$,

(a) Find a formula for $\displaystyle\sum_{i=0}^{n} s(i)$.

(b) Use the formula to find $\displaystyle\sum_{i=0}^{15} s(i)$.

5. Rosita took out a loan for a car costing $11,000. She paid a $1,000 down payment and got a 4% interest rate.

(a) If the term of the loan is 36 months, what is the monthly payment?

(b) If Rosita pays $500 per month (which is more than the minimum monthly payment), how long will it take her to pay off the car?

On Your Own

6. Write a closed form definition for a function that generates each sequence in Exercise 1.

7. Draw a square. Connect the midpoints of the sides of this square to form another square. Connect the midpoints of the sides of this second square to form the next one. You can continue to make more squares in this pattern.

    (a) If the sidelength of the first square you drew is 1, what is the sidelength of the second square? Of the third square?

    (b) Do these lengths form a geometric sequence? Why or why not? If yes, what is the first term and common ratio of this geometric sequence?

    (c) If the area of the first square you drew is 1, what is the area of the second square? Of the third square?

    (d) Do these areas form a geometric sequence? Why or why not? If yes, what is the first term and common ratio of this geometric sequence? If no, what kind of a sequence do they form?

    (e) **Take It Further.** Prove that connecting the midpoints of the sides of a square really gives you another square.

8. (a) Turn a blank piece of $8\frac{1}{2}''$ x $11''$ paper lengthwise and draw a 45°angle in the bottom left corner of the paper.

    (b) Draw a small square, $\frac{1}{2}''$ on each side, with one side on the horizontal side of the angle and a vertex on the other side of the angle.

    (c) Extend the right hand side of this square vertically until it intersects the sloping side of the 45°angle. Draw a second square with this segment as its left-hand side, as shown.

    (d) Continue, drawing two more squares.

    (e) Now you have a sequence of 4 squares. The length of the side of the first square is $\frac{1}{2}''$. What are the lengths of the sides of the other squares?

    (f) Find at least two geometric sequences in this picture.

9. An old tale tells of a King in an ancient Asian country who was tricked into paying for a favor as follows. The King was to pay by placing 1 grain of rice on the first square of
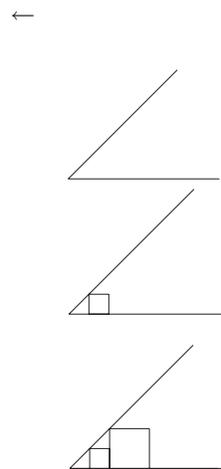
a chessboard, 2 on the next square, 4 on the next square, 8 on the next square and keep doubling like this until all 64 squares have rice. The King thought this might take several pounds of rice—perhaps even a ton—but surely not more than that!

(a) How many grains of rice would the King place on the last (64th) square?

(b) How many *total* grains of rice would the King place on the chessboard?

(c) If there are 8,000 grains of rice in a pound, how many pounds of rice did the King pay?

10. Suppose you make up a joke, and it is hysterical. That day (day 1), you tell your two best friends, and they can't stop laughing. The next day (day 2), each of them tells two people, who also love the joke. On day three, the four people who heard it the day before each tell two new people.

And so it continues. Each day, everyone who heard the joke yesterday tells two people (who have never heard the joke). How quickly does that joke spread?

← The spread of disease can also be modeled with geometric series, but the spread of jokes is nicer to think about.

(a) How many people hear the joke (for the first time) on the 8th day?

(b) How many people (total) have heard the joke by the end of the 8th day?

(c) How long would it take all the students in your school to hear the joke?

(d) How long would it take all the students in your town or city to hear the joke?

(e) How long would it take everyone in the world to hear the joke?

← Say there are about 6 billion people in the world.

11. Jaden wanted to get a loan for $9,000 over three years. He wanted to know what monthly payment to expect for different interest rates.

(a) Complete this table showing the payments for different interest rates r.

r	m
0%	
1%	
2%	
3%	
4%	
5%	

   (b) If Jaden can afford to pay $300 a month, what is the largest interest rate (to one-tenth of a percent) that he can accept for his loan?

12. What price can Hector afford to pay for a car if he can put $1500 down and pay $500 per month? He can get a 5-year 4% loan.

13. To keep from getting in a rut, lots of athletes challenge themselves. Often, runners make a pact that they will never run a shorter distance than they did the week before, and they will try to increase their distance 10% each week.

Suppose Michelle runs 5 miles this week, and she has made this pact.

   (a) How far will Michelle run next week?
   (b) How far will she run the week after next?
   (c) Michelle plans to run in a marathon 33 weeks from now. How far will she run the week before the marathon?
   (d) How much *total* mileage will Michelle log between now and the marathon, if she keeps her pact?

Remember this exercise from Lesson 7.1?

Emile got a letter in the mail saying that a wealthy relative had died and had left an inheritance that would be made public at a meeting the following week. At the meeting, the lawyer read the following:

To Emile, because he likes to work on problems, I leave a choice. He can have one of two inheritances: ← Year 0 is the year in which the will is read.

   Option 1: *A starting amount of $10,000 with an annual payment at the end of each year that is $1000 more than the amount he received the previous year, for the next 26 years.*

Option 2: *A starting amount of one cent with an annual payment at the end of each year that is twice the amount he received in the previous year, for the next 26 years.*

To find the total Emile gets under Option 1, you can sum an arithmetic series. After 26 years, he will get:

$$10,000 + 11,000 + 12,000 + 13,000 + \cdots + (10,000 + 1,000n)$$

To find the total Emile gets under Option 2, you can sum a geometric series. After 26 years, he will get:

$$.01 + .02 + .04 + .08 + \cdots + (.01) \cdot 2^n$$

14. (a) What is n? That is, what is the last term you would add in each series?
   (b) Sum the arithmetic series to find the total Emile gets from Option 1.
   (c) Sum the geometric series to find the total Emile gets from Option 2.

Remember Thor and Pandora? They dropped a ball from a height of 2 meters and measured the height of each bounce. The following table shows the results of this experiment.

$\leftarrow$

Bounces	Height (m)
0	2.00
1	1.68
2	1.41
3	1.19
4	1.00
5	0.84
6	0.71
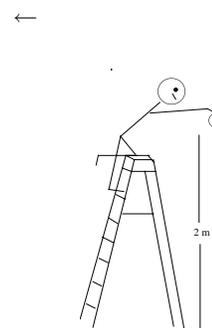
One question you might ask is: what total distance does the ball travel in, say, 100 bounces. The problem is more complicated than it might first look. The following exercises (15-19) will help you answer this question.

15. First, just think about the distance the ball *falls*. (Don't worry about the distance it travels bouncing up.) When

it is first dropped, it falls 2 meters. How far does it fall after the first bounce? Second bounce? Third bounce?

16. We saw that the bounce height was actually (pretty close to) a geometric sequence. So you can rewrite your answers above as: 2 meters, $2x$ meters, $2x^2$ meters, and $2x^3$ meters. What is the common ratio x?

17. So, to find the distance the ball falls in 100 bounces, you have to add up:

$$2 + 2x + 2x^2 + 2x^3 + \cdots + 2x^{100}$$

(where x is whatever you found in problem 16). Go ahead.

18. What about the distance the ball bounces up? Use any method you like to find out that total distance.

19. Find the total distance the ball travels (up and down). Do you think your answer would be different if you calculated the distance for 1,000 bounces instead? Why or why not?

Patterns in Practice

20. (*Zeno's journey*) Zeno started at one side of a room (24 feet wide) and jumped half way across. Then he jumped half way across the remaining distance to the wall. Then he jumped half of that . . . .

     ← This famous problem is known as "Zeno's paradox," named for a Greek mathematician who lived in the 5th century BC. See *A Tour of the Calculus* by David Berlinski for more about this problem.

    (a) How far has Zeno traveled after 3 jumps? How much further is he to the wall?

    (b) How far has Zeno traveled after 6 jumps? How much further is he to the wall?

    (c) How far has Zeno traveled after n jumps? How much further is he to the wall?

    (d) Will Zeno ever get to the wall if he keeps up this "jumping half way" scheme?