Coarsening by diffusion-annihilation in a bistable system driven by noise

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The stochastic Ginzburg-Landau equation in one dimension is the simplest continuum model describing the spatio-temporal evolution of a bistable system in the presence of thermal noise. Relaxation to equilibrium in this model proceeds by coarsening of the field during which regions in the two stables phases separated by localized kinks grow on the average. It is shown that coarsening in the presence of thermal noise effects, however small, is drastically different than in the deterministic situation. Coarsening by noise can be mapped onto the problem of diffusion-annihilation of independent particles on the line by identifying the particle locations with the kink positions. The diffusion-annihilation process displays universal self-similar features which are analyzed in detail.

The stochastic Ginzburg-Landau equation in one dimension,

$$u_t = \delta^2 u_{xx} + u - u^3 + \sqrt{\varepsilon}\,\xi(x,t), \qquad (1)$$

where $\xi(x,t)$ is a space-time white-noise process with

$$\langle \xi(x_1,t_1)\xi(x_2,t_2)\rangle = \delta(x_1 - x_2)\delta(t_1 - t_2), \qquad (2)$$

is the simplest continuum model describing the spatio-temporal evolution of a bistable system in the presence of noise. The model has received much attention in the literature in the deterministic setting ($\varepsilon = 0$) [1–3], but the noisy dynamics is still poorly understood, especially out of equilibrium [4–7]. This is regrettable since small thermal noise effects are ubiquitous and, as we will show, affect dramatically the coarsening process. We shall consider (1) with initial condition $u|_{t=0} = u_0$ and periodic boundary conditions on $x \in [-L, L]$ (though the boundary conditions are in fact irrelevant in what follows). (1) is written in appropriate dimensionless variables where the characteristic length-scale of the initial condition is of order one, and we shall be interested in the parameter range $L \gg 1$, $\varepsilon \ll \delta \ll 1$. This corresponds to a system very large compared with the characteristic length-scale of the initial condition, with diffusive effects acting only on scales much smaller than the characteristic length-scale of the initial condition, and where the effects of thermal noise are very weak. In this case, to leading order the initial data first relaxes by the ordinary differential equation $u_t = u - u^3$ towards the deterministic equilibria, $u = \pm 1$, except in the layers of size δ around the locations where $u_0 = 0$ where kinks form. The result is a multi-kink solution such as shown on Fig. 1 where the kink around $x = X$ has a hyperbolic tangent shape, $\pm\tanh((x - X)/\sqrt{2}\delta)$, and by assumption the inter-kink distance is of order one. This first stage of the dynamics is trivial and we shall focus on the subsequent evolution of the field, which is coarsening by motion of the kinks and annihilation at collision. The deterministic ($\varepsilon = 0$) coarsening has been extensively studied (see e.g. [1–3] and references therein); successive

kinks attract each other with a force exponentially small in δ, and coarsening is exponentially slow. The presence of noise, however small, drastically changes this picture and makes the kink motion diffusive. The relevant time-scale is slow (though much faster than the deterministic coarsening time-scale), of the order of $(\varepsilon\delta)^{-1}$, and the kinks move independently as

$$\dot{X}(s) = \eta(s), \qquad (3)$$

where we rescaled time as $s = \frac{3\sqrt{2}}{8}\varepsilon\delta t$ and $\eta(s)$ is a white-noise process with $\langle \eta(s)\eta(s')\rangle = \delta(s - s')$. In the coarsening stage the original dynamics in (1) can thus be mapped onto a diffusion-annihilation process of particles (the kinks) on the line.

The coarsening stage of the dynamics can be made arbitrary long for an arbitrary large domain [8] and displays universal self-similar features which we will discuss in detail through analysis of the diffusion-annihilation process. In particular, we shall show that the naive assumption of independence of the interval lengths between successive particles (the inter-kink distance in the original problem) is wrong. On the contrary, correlations between successive intervals are built by the dynamics. We will be able to account for these effects through a systematic expansion in correlation between neighbor intervals whose predictions are indistinguishable from the exact solutions obtained by numerical experiment.

We first sketch the argument for existence of the coarsening stage and its equivalence with the diffusion-annihilation process (for details, see [9]). We establish two things. First, nucleations only arise on the much
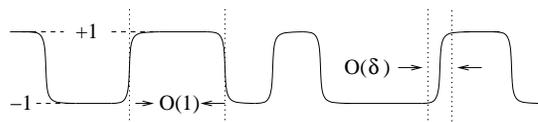

FIG. 1: Multi-kink solution to (1). Locally every layer has a shape of a simple kink and the inter-kink distance is of order one.

longer time-scale of the order $e^{\delta/\varepsilon}$. Second, kink structures are not destroyed by the noise, and their motion is governed by (3). Nucleation can be viewed as a space-time Poisson process which is adequately described by specifying the probability $\theta dx dt$ that a nucleation will occur within range dx and time interval dt. The nucleation rate θ can be estimated by a standard argument involving energy barriers. Because the deterministic part of the dynamics in (1) is the gradient flow for the energy

$$E[u] = \frac{1}{2}\int_{-L}^{L}(\delta^2 u_x^2 + 2V(u))dx$$

with potential $V(u) = \frac{1}{4}(1-u^2)^2$, it follows that the energy of a kink, $u = \pm\tanh((x-X)/\sqrt{2}\delta)$, is given by $E_k = \frac{8\sqrt{2}}{3}\delta$. Since the space-time scale of the nucleus, Δ_{nucl}, is only algebraic in δ, Kramers rule gives the estimate $\theta = \Delta_{nucl}^{-1}e^{-2E_k/\varepsilon} = O(e^{-\delta/\varepsilon})$, and the probability of a nucleation can be made arbitrary small on a space-time domain with area much smaller than $\theta^{-1} = O(e^{\delta/\varepsilon})$. Thus within a space-time domain with area much smaller than $e^{\delta/\varepsilon}$, coarsening without nucleation occurs and for small noise, $\varepsilon \ll \delta$, this domain can be made arbitrary large [10]. The argument also implies that kinks with hyperbolic tangent shape are preserved because deformations are tremendously expensive in energy.

Consider now the motion of the kink induced by the noise. Formally, the measure for the process defined by (1) on $0 \le t \le T$ can be written as $\exp(-S_T)$ with

$$S_T = \frac{1}{2\varepsilon}\int_0^T\int_{-L}^{L}(u_t - \delta^2 u_{xx} + V'(u))^2 dx dt.$$

Inserting the single kink ansatz solution, $u = \pm\tanh((x-X(t))/\sqrt{2}\delta)$, gives by space integration

$$S_T = \frac{2\sqrt{2}}{3\varepsilon\delta}\int_0^T\dot{X}^2 dt,$$

which is precisely the formal measure for the equation in (3) after rescaling of time. A similar argument for a multi-kink solution shows that the kinks move independently until annihilation at collision. A rigorous derivation of (3) for a single kink solution can be found in [7].

We shall now focus on the statistical description of the coarsening through analysis of the diffusion-annihilation process, a problem of interest in its own right. The statistical theory presented below is tested by comparison with the results of direct numerical simulations of N Brownian particles satisfying each (3) with an independent noise, moving on a ring of size L, and annihilating on collision. We made sure that the effects of the finiteness of the ring do not affect the dynamics by keeping the distance λ between the particles much less than the length of the ring L. We kept $\lambda/L < 10^{-5}$ and as a rule we started with $N = 10^6$ particles. Statistics were computed by averaging over the ensemble of particles on the ring.

The natural object for the statistical description of the bath of particles is the probability density function $n(l,t)$ of the length of an interval between two nearest particles:

$$\int_{l_1}^{l_2} n(l,t)dl = \text{Prob}\{\ell \in [l_1, l_2]\}, \qquad (4)$$

where ℓ is the interval lengths between any two successive particles along the line and for clarity of notation, we simply denote by t the rescaled time $s = \frac{3\sqrt{2}}{8}\varepsilon\delta t$ appearing in (3). n makes sense for a homogeneous bath in which the interval length between any two particles is on the average independent of the particle positions along the line. This property is conserved for all times if it is true initially, which we shall assume. Practically, n is obtained by bin-counting the interval lengths, which is the procedure we adopt in the direct numerical simulations. A complete statistical description of the bath requires not only $n \equiv n^{(1)}$, but also $n^{(k)}(l_1, \ldots, l_k, t)$, the probability density functions for k successive interval lengths, and we shall consider these objects as well.

The most naive way to derive an equation for $n(l,t)$ will turn out to be wrong, but is instructive anyway. It is based on the assumption of statistical independence of successive interval lengths, in the sense that

$$n^{(2)}(l_1, l_2) = n(l_1)n(l_2), \qquad (5)$$

and similarly for higher order density functions [11]. Based on statistical independence it is easy to derive an evolution equation for n, which we shall further refer to as a mean-field (MF) approximation:

$$n_t = n_{ll} + n(\downarrow)\int_0^l n(l')n(l-l')dl', \qquad (6)$$

where here and below the down arrow indicates derivative over the corresponding argument evaluated at zero, $n(\downarrow) = n_l(0)$. The first term at the right hand-side accounts for diffusive motion of the intervals. The convolution term accounts for creation of an interval of length l by collision of two intervals of length l' and $l - l'$ (with $0 \le l' \le l$); the probability rate of these collision events is $n(\downarrow)$, the probability flux at $l = 0$. Since particles annihilate on collision, intervals of zero length disappear, and (6) must be solved with the absorbing boundary condition $n|_{l=0} = 0$. (6) admits a self-similar solution of the form $n(l,t) = (4t)^{-1/2}m(l/\sqrt{4t})$ with m satisfying

$$m'' + 2(\xi m)' + \int_0^\xi m(\zeta)m(\xi-\zeta)d\zeta = 0, \qquad (7)$$

where we used $m'(0) = 1$ which is easily derived by taking the first moment of (7) and using $\int_0^\infty md\xi = 1$.

The mean field equation in (6) is unsatisfactory because the independence assumption (5) is not true (see Fig. 2). In particular, the probability density of colliding
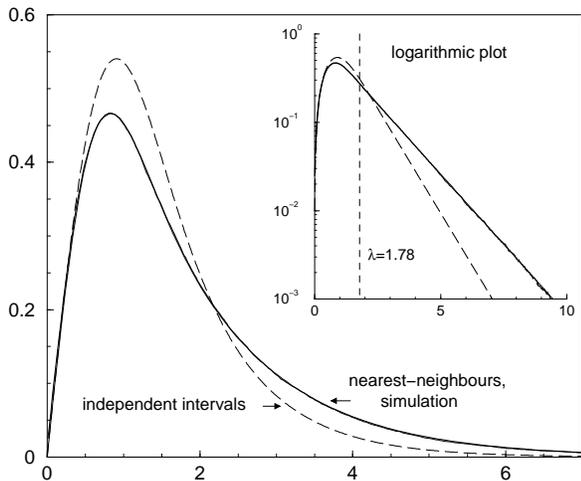
FIG. 2: The self-similar density functions m and m_c. Dashed line — mean field theory: Dotted line and dash-dotted — NN and SNN models, solid line — direct simulation. The average interval length is $\lambda \approx 1.78$.

FIG. 3: Relaxation of $n(l,t)$ from some initial condition $n_0(l)$ The dotted line corresponds to the NN approximation, the dash-dotted line to the SNN approximation, and the solid line to direct simulations.

intervals, i.e. the conditional probability density of an interval if its neighbor is about to disappear,

$$n_c(l) = \lim_{l' \to 0} \frac{n^{(2)}(l,l')}{n(l')} = n^{(2)}(l,\downarrow)/n(\downarrow), \qquad (8)$$

is not equal to n (see Fig. 2). As a result, the convolution term in (6) does not factorize as it was assumed and is

$$\int_0^l n^{(3)}(l',\downarrow,l-l')dl', \qquad (9)$$

leaving the equation for n unclosed.

We can obtain a closed equation proceeding to the next order beyond the independence assumption in (5) in a systematic expansion in correlation between successive intervals. We start from the exact (but unclosed) equation for $n^{(2)}$ which is in fact the second equation in an infinite hierarchy of coupled equations for the $n^{(k)}$:

$$\begin{aligned}
n_t^{(2)} &= n_{l_1 l_1}^{(2)} - n_{l_1 l_2}^{(2)} + n_{l_2 l_2}^{(2)} + 2n(\downarrow)n^{(2)} \\
&+ \int_0^{l_1} n^{(4)}(l,\downarrow,l_1-l,l_2)dl - n^{(3)}(\downarrow,l_1,l_2) \qquad (10) \\
&+ \int_0^{l_2} n^{(4)}(l_1,l,\downarrow,l_2-l)dl - n^{(3)}(l_1,l_2,\downarrow).
\end{aligned}$$

An equation for $n \equiv n^{(1)}$ is obtained by integration of (10) over l_1 or l_2. The various terms in (10) can be interpreted similarly as in (6). In particular the integrals involving $n^{(4)}$ represent probability rates to form the given state (l_1, l_2) by annihilation of some intervals, while the terms with $n^{(3)}$ destroy the given state because of a collision with a particle from the outside. The term $2n(\downarrow)n^{(2)}$ preserves normalization and arises because the number of intervals is not preserved due to annihilation.
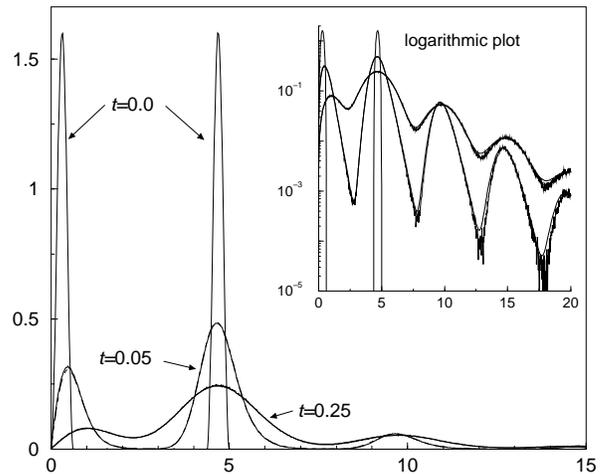
The simplest closure next to the mean field approximation is based on the assumption that the conditional probability density to observe an interval of length l_1 with consequent neighbors of respective lengths l_2, l_3, ... depends on l_2 only, i.e.

$$n(l_1|l_2,l_3,\ldots) = n(l_1|l_2). \qquad (11)$$

This immediately implies that

$$\begin{aligned}
n^{(3)}(l_1,l_2,l_3) &= n^{(2)}(l_1,l_2)n^{(2)}(l_2,l_3)/n(l_2), \\
n^{(4)}&(l_1,l_2,l_3,l_4) \\
&= n^{(2)}(l_1,l_2)n^{(2)}(l_2,l_3)n^{(2)}(l_3,l_4)/n(l_2)n(l_3),
\end{aligned} \qquad (12)$$

and so on for higher order density functions. Inserting (12) in (10) yields a closed equation for $n^{(2)}$ which we shall refer to as nearest-neighbor (NN) approximation. The NN approximation has a self-similar solution of the form $n^{(2)}(l_1,l_2,t) = (4t)^{-1}m^{(2)}(l_1/\sqrt{4t},l_2/\sqrt{4t})$ with $m^{(2)}$ satisfying

$$\begin{aligned}
m_{\xi\xi}^{(2)} &- m_{\xi\eta}^{(2)} + m_{\eta\eta}^{(2)} + 2\xi m_\xi^{(2)} + 2\eta m_\eta^{(2)} + 6m^{(2)} \\
&-m_c(\eta)m(\xi|\eta) - m_c(\xi)m(\eta|\xi) \\
&+ \int_0^\eta m(\xi|\zeta)m_c(\zeta)m_c(\eta-\zeta)d\zeta \\
&+ \int_0^\xi m(\eta|\zeta)m_c(\zeta)m_c(\xi-\zeta)d\zeta = 0.
\end{aligned} \qquad (13)$$

$m(\xi|\eta) = m^{(2)}(\xi,\eta)/m(\eta)$, $m_c(\xi) = m^{(2)}(\xi,\downarrow)/m(\downarrow)$ are the self-similar conditional probability densities.

Though assumption (11) is not true and higher order closures are possible (which provides a systematic expansion in correlation), the NN approximation is in excellent agreement with the direct numerical experiments. Fig. 2 shows the self-similar density m and the collisional density m_c obtained from direct numerical simulations and
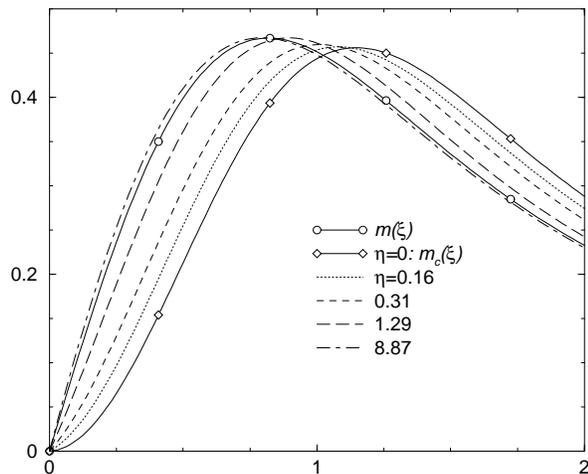
FIG. 4: Self-similar conditional probability densities $m^{(2)}(\xi|\eta)$ for different values of η in the NN model. When $\eta \to 0$ we obtain the collisional density m_c while $m^{(2)}(\xi|\eta) \approx m(\xi)$ for $\eta \gg \lambda$.

those predicted by the mean-field equation (7) and the nearest neighbor model (13). We can see that the NN approximation captures correctly both m and m_c for all values of ξ, including the large values in the tails of the density functions where the correct exponential decay is obtained. In contrast the mean-field equation (which does not distinguish m_c from m) fails for all ξ.

As a further test of the NN approximation, we studied the relaxation from an initial condition chosen so that $n_0(l)$ has two well defined peaks. Fig. 3 shows snapshots in time of $n(l,t)$ obtained from direct simulation and those predicted by the NN approximation. As one can see there is virtually no difference between the curves. The initial condition was chosen to illustrate an interesting phenomenon. It is easy to notice that the solution has pronounced humps at lengths equal to the multiples of the length of the second peak. This happens because as the short intervals from the first peak collide the large ones merge and create intervals of double length.

In the last numerical test presented in Fig. 4, we computed the conditional probability density $m(\xi|\eta) = m^{(2)}(\xi, \eta)/m(\eta)$ predicted by the NN approximation in the self-similar regime. As $\eta \to 0$, $m(\xi|\eta)$ converges to m_c as it should. On the other hand, as $\eta \to \infty$, $m(\xi|\eta)$ converges to m, which implies that correlations are not important for the intervals with lengths bigger than the mean length, $\lambda = \int_0^\infty \xi m d\xi \approx 1.78$. This in fact suggests an even simpler approximation than NN. Since the correlations are only important if they involve short intervals which are pretty rare due to annihilation, we can use the decoupling condition (11) when the corresponding term is responsible for the configuration undergoing a collision and otherwise decouple it completely using (5). The simplified nearest neighbor (SNN) approximation amounts to substituting all conditional probabilities

$m(\xi|\eta)$ by $m(\xi)$ in (13). All of the above numerical tests were also performed for SNN approximation: the results are presented on the same figures and they can barely be distinguished from those of the NN approximation.

In summary we have shown that coarsening in the stochastic Ginzburg-Landau dynamics is very different than in the deterministic case. Coarsening by noise can be mapped onto the diffusion-annihilation process by identifying the particles positions with the kinks locations in the original problem. The diffusion-annihilation displays universal self-similar features which were obtained going to the second order in a systematic expansion in correlation between successive intervals. To the best of our knowledge such analysis of relaxation in a space-time continuous system in the presence of noise is new.

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[8] Eventually the field relaxes to thermal equilibrium on a much longer time-scale of order $e^{\delta/\varepsilon}$. In this last stage of evolution the field (which for the parameter range we consider was left in one phase by the coarsening dynamics) is switched at random times by a single nucleation event propagating diffusively over the entire domain. This flip-flop process can be described by a Markov chain on the two states $\{+1, -1\}$; see W. E, W. Ren, and E. Vanden-Eijnden, submitted to: *Phys. Rev.Lett.*

[9] I. Fatkullin and E. Vanden-Eijnden, in preparation.

[10] If $L \geq e^{\delta/\varepsilon}$ there is no coarsening and the dynamics goes directly to a stage where both nucleation and diffusive propagation interplay in a non-trivial way. This process has been studied in equilibrium by M. Büttiker and collaborators [4].

[11] The independence assumption in (5) is in fact true in the problem where particles diffuse on the line not annihilating but crossing each other at collision. In this case the equilibrium density is exponential, $n(l) = e^{-l/\lambda}/\lambda$, where λ is the average interval length in the initial bath.