

TRANSPORT AND ENTANGLEMENT IN DISORDERED XY CHAINS

Houssam Abdul-Rahman
University of Arizona

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B. Nachtergaele, R. Sims, and G. Stolz.

QMath13, Georgia Tech

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- The XY chain.
- Dynamical entanglement.
- Particle number transport.
- Energy transport.

The XY Chain

AN ANISOTROPIC XY CHAIN IN RANDOM TRANSVERSAL MAGNETIC FIELD

$$H = - \sum_{j=1}^{n-1} \mu_j [(1 + \gamma_j) \sigma_j^x \sigma_{j+1}^x + (1 - \gamma_j) \sigma_j^y \sigma_{j+1}^y] - \sum_{j=1}^n \nu_j \sigma_j^z$$

- $\Lambda = [1, n]$, Λ_0 a block of spins (subinterval of Λ).
- The Hilbert space: $\mathcal{H} := \bigotimes_{x \in \Lambda} \mathcal{H}_x = (\mathbb{C}^2)^{\otimes n}$, $\dim \mathcal{H} = 2^n$.
- μ_j , γ_j and ν_j are i.i.d.

The XY Chain

JORDAN-WIGNER TRANSFORM

↓ **Jordan-Wigner** ↓

$$H = \mathcal{C}^* M \mathcal{C}, \quad \mathcal{C} := (c_1, c_1^*, c_2, c_2^*, \dots, c_n, c_n^*)^t.$$

M is the block Jacobi matrix

$$M := \begin{pmatrix} -\nu_1 \sigma^z & \mu_1 S(\gamma_1) & & \\ \mu_1 S(\gamma_1)^t & \ddots & \ddots & \\ & \ddots & \ddots & \mu_{n-1} S(\gamma_{n-1}) \\ & & \mu_{n-1} S(\gamma_{n-1})^t & -\nu_n \sigma^z \end{pmatrix},$$

$$S(\gamma) = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The XY Chain

ASSUMPTIONS

Assumptions:

- The XY chain H has almost sure simple spectrum.
- M satisfies eigencorrelator localization, i.e.
$$\mathbb{E} \left(\sup_{|g| \leq 1} \|g(M)_{jk}\| \right) \leq C_0(1 + |j - k|)^{-\beta}, \text{ for some } \beta > 6.$$

Applications:

$\mu_j = \mu, \gamma_j = \gamma$ for all $j \in \mathbb{N}$.

ν_j are i.i.d from an absolutely continuous, compactly supported distribution.

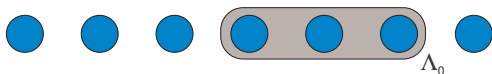
- Isotropic case ($\gamma = 0$): $M \rightarrow$ Anderson Model.
- Anisotropic case ($\gamma \neq 0$):
 - ▶ Large disorder case.
 - ▶ Uniform spectral gap for M around zero.

Elgart/Shamis/Sodin (2012).

Chapman /Stolz (2014).

Dynamical Entanglement

THE ENTANGLEMENT ENTROPY AND THE ENTANGLEMENT OF FORMATION



Fix $\Lambda_0 \subseteq \Lambda$, consider the decomposition:

$$\mathcal{H} = \mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda \setminus \Lambda_0}, \text{ where } \mathcal{H}_{\Lambda_0} = \bigotimes_{x \in \Lambda_0} \mathcal{H}_x, \quad \mathcal{H}_{\Lambda \setminus \Lambda_0} = \bigotimes_{x \in \Lambda \setminus \Lambda_0} \mathcal{H}_x. \quad (1)$$

Let ρ be a pure state in $\mathcal{B}(\mathcal{H})$, then

$$\mathcal{E}(\rho) = -\text{Tr} [\rho^1 \log \rho^1], \text{ where } \rho^1 = \text{Tr}_{\mathcal{H}_2} \rho.$$

For any (mixed) state $\rho \in \mathcal{B}(\mathcal{H})$, then

$$E_f(\rho) = \inf_{p_k, \psi_k} \sum_k p_k \mathcal{E}(|\psi_k\rangle\langle\psi_k|).$$

Dynamical Entanglement

MOTIVATION QUESTION



- For $1 \leq \ell \leq n$, let $H_{[1,\ell]}$ and $H_{[\ell+1,n]}$ be the restrictions of H to the corresponding interval.
- Let $\rho^{(1)}$ and $\rho^{(2)}$ be any eigenstates/thermal states of $H_{[1,\ell]}$ and $H_{[\ell+1,n]}$, respectively.
- We study $\rho_t := e^{-itH} \left(\rho^{(1)} \otimes \rho^{(2)} \right) e^{itH}$.
- ρ_t is an entangled state with respect to $\mathcal{H}_{[1,\ell]} \otimes \mathcal{H}_{[\ell+1,n]}$.

Question:

What can we say about the entanglement of ρ_t ?

Dynamical Entanglement

MOTIVATION QUESTION



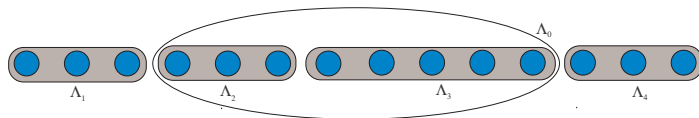
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Dynamical Entanglement

PROBLEM SETTING



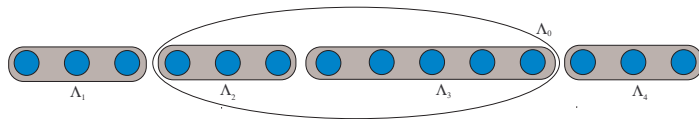
In general

- Decompose Λ into disjoint intervals $\Lambda_1, \Lambda_2, \dots, \Lambda_m$.
- H_{Λ_k} is the restriction of H to Λ_k .
- ψ_k is an eigenfunction of H_{Λ_k} , and $\rho_k = |\psi_k\rangle\langle\psi_k|$.
- Define $\rho = \bigotimes_{k=1}^m \rho_k$, and its dynamics $\rho_t = e^{-itH} \rho e^{itH}$.



Dynamical Entanglement: MAIN THEOREM

DYNAMICS OF PRODUCTS OF EIGENSTATES



THEOREM

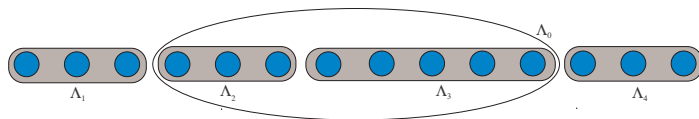
There exists $C < \infty$ such that

$$\mathbb{E} \left(\sup_{t, \{\psi_k\}_{k=1,2,\dots,m}} \mathcal{E}(\rho_t) \right) \leq C$$

for all n, m , any choice of the interval $\Lambda_0 \subset \Lambda$ and all decompositions $\Lambda_1, \dots, \Lambda_m$ of $\Lambda = [1, n]$.

Dynamical Entanglement: COROLLARIES

DYNAMICS OF PRODUCT OF THERMAL STATES



- ρ_{β_k} is a thermal state of H_{Λ_k} .
- Define $\rho_\beta = \bigotimes_{k=1}^m \rho_{\beta_k}$, and its dynamics $(\rho_\beta)_t = e^{-itH} \rho_\beta e^{itH}$.

Result:

$$\mathbb{E} \left(\sup_{t, \beta} E_f((\rho_\beta)_t) \right) \leq C$$

Dynamical Entanglement: COROLLARIES

DYNAMICS OF UP-DOWN SPINS

If $m = n$

- number of decompositions is n .
- eigenfunctions are up and down spins: $e_{\uparrow} := |\uparrow\rangle$ and $e_{\downarrow} := |\downarrow\rangle$.

For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \{\uparrow, \downarrow\}^n$, the up-down configuration associated with α is given by:

$$e_{\alpha} = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \dots \otimes e_{\alpha_n}$$

Result:

$$\mathbb{E} \left(\sup_{\alpha} \mathcal{E}(e^{-itH} | e_{\alpha} \rangle \langle e_{\alpha} | e^{itH}) \right) < C.$$

Barderson, Pollman, and Moore (2012).

Dynamical Entanglement: COROLLARIES

ENTANGLEMENT OF EIGENSTATES

For $m = 1$ (No Decomposition)

Let ψ be an eigenfunction of the full XY chain H .

Result: $\mathbb{E} \left(\sup_{\psi} \mathcal{E}(|\psi\rangle\langle\psi|) \right) < C.$ Pastur/Slavin (2014). AR/Stolz (2015).

Let ρ_{β} be a thermal state of the full XY chain H .

Result: $\mathbb{E} \left(\sup_{\beta} E_f(\rho_{\beta}) \right) < C.$

Particle Number Transport

AN ISOTROPIC XY CHAIN IN RANDOM TRANSVERSAL MAGNETIC FIELD

$$H_{\text{iso}} = - \sum_{j=1}^{n-1} [\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y] - \sum_{j=1}^n \nu_j \sigma_j^z$$

↓ **Jordan-Wigner** ↓

$$H_{\text{iso}} = c^* A c + \left(\sum_j \nu_j \right) \mathbb{1}, \text{ where } c := (c_1, c_2, \dots, c_n)^t.$$

$$A := \begin{pmatrix} -\nu_1 & \mu & & \\ \mu & \ddots & \ddots & \\ & \ddots & \ddots & \mu \\ & & \mu & -\nu_n \end{pmatrix}, \quad \mathbb{E} \left(\sup_{|g| \leq 1} |\langle e_j, g(A) e_k \rangle| \right) \leq C e^{-\eta |j-k|}.$$

Particle Number Transport

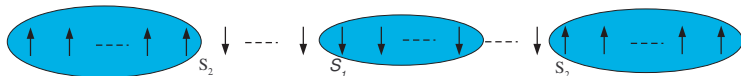
THE PARTICLE NUMBER OPERATOR

$$\mathcal{N} := \sum_{j \in \Lambda} |e_{\uparrow}\rangle\langle e_{\uparrow}|_j \text{ and } \mathcal{N}_S := \sum_{j \in S} |e_{\uparrow}\rangle\langle e_{\uparrow}|_j.$$

- $\mathcal{N}e_{\alpha} = ke_{\alpha}$, where $k = |\{j : \alpha_j = \uparrow\}|$.
- Let $\rho = |e_{\alpha}\rangle\langle e_{\alpha}|$ then $\langle \mathcal{N} \rangle_{\rho} := \text{Tr } \mathcal{N}\rho = k$ is the expected number of up-spins.
- $[H, \mathcal{N}] = 0 \Rightarrow$ The number of up-spins is conserved in time.
- $\rho_t = e^{-itH_{\text{iso}}} \rho e^{itH_{\text{iso}}}$ is the time evolution of ρ .
- $\langle \mathcal{N}_S \rangle_{\rho_t}$ is the expected number of up-spins in S at time t .

Particle Number Transport

RESULTS



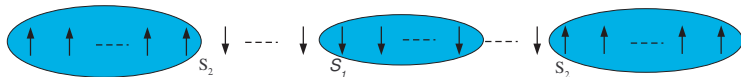
- Fix $S_1 \subset \Lambda$ and $S_2 \subset \Lambda \setminus [\min S_1, \max S_1]$.
- Initial state: $\rho = \bigotimes_{j=1}^n \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix}$, with $\eta_j = 0$ for all $j \notin S_2$.

$$\mathbb{E} \left(\sup_t \langle \mathcal{N}_{S_1} \rangle_{\rho_t} \right) \leq \frac{4C}{(1 + e^{-\eta})^2} e^{-\eta \text{dist}(S_1, S_2)}$$

Similar results for disordered Tonks-Girardeau gas, **Seiringer/Warzel** (2016).

Energy Transport

ISOTROPIC CASE



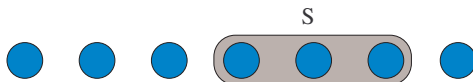
- Fix $S_1 = [a, b] \subset \Lambda$ and $S_2 \subset \Lambda \setminus S_1$.
- Initial state: $\rho = \bigotimes_{j=1}^n \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix}$, with $\eta_j = 0$ for all $j \notin S_2$.

$$\mathbb{E} \left(\sup_t |\langle H_{S_1} \rangle_{\rho_t} - \langle H_{S_1} \rangle_{\rho}| \right) \leq \frac{4CD}{(1 + e^{-\eta})^2} e^{-\eta \text{dist}(S_1, S_2)},$$

where $D = \sup_n \|A_n\|$.

Energy Transport

ANISOTROPIC CASE



- Fix $S = [a, b] \subset \Lambda$.
- H_S is the restriction of the XY chain to S .

- Initial state: $\rho = \bigotimes_{j=1}^n \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix}$.

$$\mathbb{E} \left(\sup_t |\langle H_S \rangle_{\rho_t} - \langle H_S \rangle_{\rho}| \right) \leq \tilde{C},$$

Thank you.