Entanglement in Disordered Quantum XY Chains

Houssam Abdul-Rahman
Joint with B. Nachtergaele, R. Sims, and G. Stolz

Great Lakes Mathematical Physics Meeting, MSU

June 2016

Overview

- Basic concepts.
- Particle number transport.
- Dynamical entanglement.
- Open questions.

Basic Concepts

The Many-Body Hilbert Space

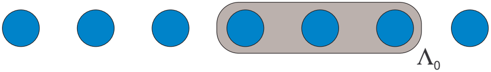
- $\Lambda = [1, n] := \{1, 2, \dots, n\}$.
- For each vertex $x \in \Lambda$ we associate the Hilbert space $\mathcal{H}_x := \mathbb{C}^2$.
- The Hilbert space associated with the system is

$$\mathcal{H} := \bigotimes_{x \in \Lambda} \mathcal{H}_x = (\mathbb{C}^2)^{\otimes n}, \quad \dim \mathcal{H} = 2^n.$$

- Notation: $e_\uparrow := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_\downarrow := \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

$$e_\alpha := e_{\alpha_1} \otimes e_{\alpha_2} \otimes \dots \otimes e_{\alpha_n}, \quad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \{\uparrow, \downarrow\}^n$$

is called the up-down configuration associated with α.



Fix $\Lambda_0 \subseteq \Lambda$, consider the decomposition:

$$\mathcal{H} = \mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda\setminus\Lambda_0}, \text{ where } \mathcal{H}_{\Lambda_0} = \bigotimes_{x\in\Lambda_0} \mathcal{H}_x, \quad \mathcal{H}_{\Lambda\setminus\Lambda_0} = \bigotimes_{x\in\Lambda\setminus\Lambda_0} \mathcal{H}_x. \quad (1)$$

Let ρ be a pure state in $\mathcal{B}(\mathcal{H})$ ($\rho \geq 0$, $\operatorname{Tr}\rho = 1$, and $\rho^2 = \rho$), then

- ρ is **separable:** if there exist pure states $\rho^{(1)} \in \mathcal{B}(\mathcal{H}_{\Lambda_0})$ and $\rho^{(2)} \in \mathcal{B}(\mathcal{H}_{\Lambda\setminus\Lambda_0})$, such that $\rho = \rho^{(1)} \otimes \rho^{(2)}$.

- ρ is **entangled:** if it is not separable.

The **entanglement entropy** of a pure state ρ with respect to the decomposition $\mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda \setminus \Lambda_0}$ is defined as follows:

$$\mathcal{E}(\rho) = -\operatorname{Tr}\left[\rho^1 \log \rho^1\right], \quad \text{where} \quad \rho^1 = \operatorname{Tr}_{\mathcal{H}_2} \rho.$$

For any pure state $\rho \in \mathcal{B}(\mathcal{H})$:

- $\mathcal{E}(\rho) \geq 0$.
- $\mathcal{E}(\rho) = 0$ if and only if ρ is a product state (Not entangled).
- $\mathcal{E}(\rho) \leq (\log 2)|\Lambda_0|$ (volume scaling).

Particle Number Transport

$$H_{\mathsf{iso}} = -\sum_{j=1}^{n-1} \mu_j [\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y] - \sum_{j=1}^{n} \nu_j \sigma_j^z$$

- $\Lambda = [1, n]$, Λ_0 a block of spins (subinterval of Λ).
- Hilbert space $\mathcal{H} = \left(\mathbb{C}^2\right)^{\otimes \Lambda}$.
- μ_j and ν_j are i.i.d.
- $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

A_j acts on the j^{th} component of the tensor product, i.e.

$$A_j = \mathbb{1}^{\otimes (j-1)} \otimes A \otimes \mathbb{1}^{\otimes (n-j)}$$

Particle Number Transport

Jordan-Wigner Transform

$\downarrow$ **Jordan-Wigner** $\downarrow$

$$H_{\mathsf{iso}} = c^* A c + \left(\sum_j \nu_j\right) \mathbb{1}, \text{ where } c := (c_1, c_2, \ldots, c_n)^t$$

$\{c_j\}_j$ satisfy the CAR algebra.

$$A := \begin{pmatrix} -\nu_1 & \mu_1 & & \\ \mu_1 & \ddots & \ddots & \\ & \ddots & \ddots & \mu_{n-1} \\ & & \mu_{n-1} & -\nu_n \end{pmatrix}.$$

Assumption: A satisfies eigencorrelator localization, i.e

$$\mathbb{E}\left(\sup_{|g|\leq 1} |\langle e_j, g(A) e_k \rangle|\right) \leq C e^{-\eta|j-k|}.$$

Particle Number Transport

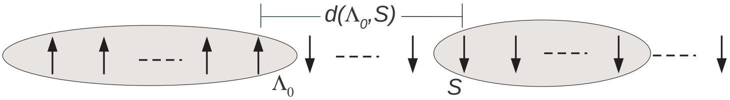
The Particle Number Operator

$$\mathcal{N} := \sum_{j\in\Lambda} |e_\uparrow\rangle\langle e_\uparrow|_j \text{ and } \mathcal{N}_S := \sum_{j\in S} |e_\uparrow\rangle\langle e_\uparrow|_j.$$

- $\mathcal{N} e_\alpha = k e_\alpha$, where $k = |\{j \ : \alpha_j = \uparrow\}|$.
- Let $\rho = |e_\alpha\rangle\langle e_\alpha|$ then $\langle \mathcal{N}\rangle_\rho := \operatorname{Tr} \mathcal{N}\rho = k$ is the expected number of up-spins.
- $[H, \mathcal{N}] = 0 \Rightarrow$ The number of up-spins is conserved in time.
- $\rho_t = e^{-itH_{\text{iso}}} \rho e^{itH_{\text{iso}}}$ is the time evolution of ρ.
- $\langle \mathcal{N}_S\rangle_{\rho_t}$ is the expected number of up-spins in S at time t.

Particle Number Transport

Results



- Fix $\Lambda_0 \subset \Lambda$ and $S \subset \Lambda \setminus \Lambda_0$.
- Initial state: $\rho = |\phi\rangle\langle\phi|$, where $\phi = (e_\uparrow)^{\otimes \Lambda_0} \otimes (e_\downarrow)^{\otimes \Lambda \setminus \Lambda_0}$

$$\mathbb{E}\left(\sup_t \langle \mathcal{N}_S \rangle_{\rho_t}\right) \leq C e^{-\frac{\eta}{2} dist(\Lambda_0, S)}$$

$$\mathbb{E}\left(\sup_t \left|\langle \mathcal{N}_S \rangle_{\rho_t} - \langle \mathcal{N}_S \rangle_{\rho}\right|\right) \leq \tilde{C}$$

Similar results for disordered Tonks-Girardeau gas, **Seiringer/Warzel** (2016).

$$H = -\sum_{j=1}^{n-1} \mu_j [(1+\gamma_j)\sigma_j^x \sigma_{j+1}^x + (1-\gamma_j)\sigma_j^y \sigma_{j+1}^y] - \sum_{j=1}^{n} \nu_j \sigma_j^z$$

- $\Lambda = [1, n]$, Λ_0 a block of spins (subinterval of Λ).
- Hilbert space $\mathcal{H} = \left(\mathbb{C}^2\right)^{\otimes\Lambda}$.
- μ_j, γ_j and ν_j are i.i.d.

$\downarrow$ **Jordan-Wigner** $\downarrow$

$$H = \mathcal{C}^* M \mathcal{C}, \; \mathcal{C} := (c_1, c_1^*, c_2, c_2^*, \ldots, c_n, c_n^*)^t.$$

M is the block Jacobi matrix

$$M := \begin{pmatrix} -\nu_1 \sigma^z & \mu_1 S(\gamma_1) & & \\ \mu_1 S(\gamma_1)^t & \ddots & \ddots & \\ & \ddots & \ddots & \mu_{n-1} S(\gamma_{n-1}) \\ & & \mu_{n-1} S(\gamma_{n-1})^t & -\nu_n \sigma^z \end{pmatrix},$$

$$S(\gamma) = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}, \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- For $1 \le \ell \le n$, let $H_{[1,\ell]}$ and $H_{[\ell+1,n]}$ be the restrictions of H to the corresponding interval.
- Let $\rho^{(1)}$ and $\rho^{(2)}$ be any eigenstates of $H_{[1,\ell]}$ and $H_{[\ell+1,n]}$, respectively.
- We study $\rho_t := e^{-itH}\left(\rho^{(1)} \otimes \rho^{(2)}\right)e^{itH}$.
- ρ_t is an entangled state with respect to $\mathcal{H}_{[1,\ell]} \otimes \mathcal{H}_{[\ell+1,n]}$.

Question:

What can we say about the entanglement entropy of ρ_t?

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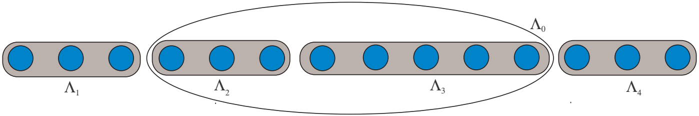


In general

- Decompose Λ into disjoint intervals $\Lambda_1, \Lambda_2, \ldots, \Lambda_m$.
- H_{Λ_k} is the restriction of H to Λ_k.
- ψ_k is an eigenfunction of H_{Λ_k}, and $\rho_k = |\psi_k\rangle\langle\psi_k|$.
- Define $\rho = \bigotimes_{k=1}^{m} \rho_k$, and its dynamics $\rho_t = e^{-itH} \rho e^{itH}$.

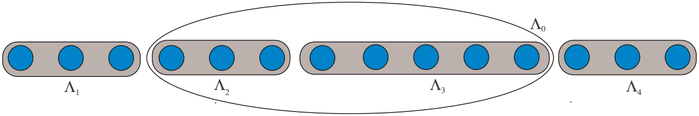
Assumptions:

- The XY chain H has almost sure simple spectrum.
- M satisfies eigencorrelator localization, i.e
  $\mathbb{E}\left(\sup_{|g|\leq 1} \|g(M)_{jk}\|\right) \leq C_0(1+|j-k|)^{-\beta}$, for some $\beta > 6$.

Applications:

$\mu_j = \mu$, $\gamma_j = \gamma$ for all $j \in \mathbb{N}$.
ν_j are i.i.d from an absolutely continuous, compactly supported distribution.

- Isotropic case ($\gamma = 0$): $M \longrightarrow$ Anderson Model.
- Anisotropic case ($\gamma \neq 0$):
  - Large disorder case. **Elgart/Shamis/Sodin (2012).**
  - Uniform spectral gap for M around zero. **Chapman /Stolz (2014).**



Theorem

There exists $C < \infty$ *such that*

$$\mathbb{E}\left(\sup_{t,\{\psi_k\}_{k=1,2,\ldots,m}} \mathcal{E}(\rho_t)\right) \leq C$$

for all n, m, *any choice of the interval* $\Lambda_0 \subset \Lambda$ *and all decompositions* $\Lambda_1, \ldots, \Lambda_m$ *of* $\Lambda = [1, n]$.

Dynamics of Up-Down Spins

If $m = n$

- number of decompositions is n.
- eigenfunctions are up and down spins: $e_\uparrow$ and $e_\downarrow$.

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{\uparrow, \downarrow\}^n$, recall that the up-down configuration associated with α is given by:

$$e_\alpha = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \ldots \otimes e_{\alpha_n}$$

Result:

$$\boxed{\mathbb{E}\left(\sup_\alpha \mathcal{E}(e^{-itH}|e_\alpha\rangle\langle e_\alpha|e^{itH})\right) < C.}$$

Barderson, Pollman, and Moore PRL (2012).

Entanglement of Eigenstates

For $m = 1$ (No Decomposition)

Let ψ be an eigenfunction of the full XY chain H.

Result:

$$\mathbb{E}\left(\sup_{\psi} \mathcal{E}(|\psi\rangle\langle\psi|)\right) < C.$$

Pastur/Slavin (2014).
Elgart/Pastur/Shcherbina (2015).

Open Questions

- Dynamical entanglement of the tensor product of more general pure states.
- Area law for the entanglement of thermal states.

  Sims/Warzel (2016)
- "Logarithmic negativity" in the XY chain.

  Eisler/Zimboras (2015)

Thank you.