Localization and Entanglement in Disordered Harmonic Oscillator Systems

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Many-body Localization (MBL)

- In single-body quantum systems, sufficiently strong disorder localizes wave functions in space. This is the essence of Anderson localization.
- By now, Anderson localization is well understood, both physically and mathematically.

Question:
What happens in a quantum system
when both disorder and interactions are present?

- This is the many-body setting where the situation is fundamentally different from the single particle case.
- Considerable analytical and numerical challenges persist even for the simplest many-body models.
- There are extensive efforts in the physics literature to understand the phenomenon of MBL as well as the many-body transition.

e.g. arXiv:1705.09103, arXiv:1804.11065

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Many-Body Localization indicators

- The many-body dynamics:
  - zero-velocity Lieb-Robinson bounds.
  - quasi-locality of observables.
- States localization:
  - decay of correlations.
  - small entanglement (area laws).

MBL Rigorous Results- Models

- XX/XY chain.
  Hamza/Sims/Stolz '12
  Pastur/Slavin '14
  AR/Stolz '15
  Sims/Warzel '16
  AR/Nachtergale/Sims/Stolz '16, '17
- Tonks-Girardeau gas.
  Seiringer/Warzel '16
- Oscillator systems.
  Nachtergaele/Sims/Stolz '12,'13
  AR/Sims/Stolz '17
  AR '18
  AR/Sims/Stolz arXiv:1810.12769
- XXZ spin chain in the Ising phase.
  Elgart/Klein/Stolz '18a, '18b
  Beaud/Warzel '17, '18
- Holstein model.
  Mavi/Schencker '18

Outline

I A disordered oscillator model.

II The regime of localized excitations.

- Zero velocity Lieb-Robinson bounds.
- Quasi-locality.
- Decay of correlations in eigenstates.

III Entanglement

- An area law for non-Gaussian states above the ground state.
- Dynamical Entanglement (work in progress).

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A Disordered Oscillator Model

The Hamiltonian

$$H_\Lambda = \sum_{x\in\Lambda} \left(p_x^2 + k_x q_x^2\right) + \sum_{\substack{\{x,y\}\subset\Lambda \\ |x-y|=1}} (q_x - q_y)^2$$

- $\Lambda := [-L, L]^\nu \cap \mathbb{Z}^\nu$ where $L \geq 1$ and $\nu \geq 1$.
- q_x and $p_x = -i\frac{\partial}{\partial q_x}$ are the position and momentum operators.
- The Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{L}^2(\mathbb{R}, dq_x)$.
- $\{k_x\}_x$ are i.i.d. random variables with absolutely continuous distribution given by a bounded density supported in $[0, k_{max}]$.

The Effective One-Particle Hamiltonian

$$H_\Lambda = \sum_{x\in\Lambda} p_x^2 + q^T h_\Lambda q, \qquad q := (q_x)_{x\in\Lambda}$$

- h_Λ is the Anderson model on $\ell^2(\Lambda)$, i.e., $h_\Lambda = h_{0,\Lambda} + k$, where
  - $h_{0,\Lambda}$ is the negative discrete Laplacian over $\ell^2(\Lambda)$.
  - $k := \mathrm{diag}\{k_x,\ x\in\Lambda\}$.
- Recall that:
  - spec(h_Λ) is almost surely simple.
  - spec$(h_\Lambda) \subset \left[\min_{x\in\Lambda} k_x, 4\nu + k_{\max}\right]$.
- h_Λ is almost surely positive, and $\|h_\Lambda\| \le 4\nu + k_{\max}$.
- $h_\Lambda^{-1/2}$ does not have a deterministic upper bound.

H_Λ AS A FREE BOSON SYSTEM

- Since h_Λ is positive with simple spectrum (almost surely), it can be diagonalized with eigenvalues $\{\gamma_j^2\}_{j=1}^{|\Lambda|}$ and unique (up to a phase) orthogonal eigenvectors $\{\phi_j\}_{j=1}^{|\Lambda|}$, $h_\Lambda = \sum_{j=1}^{|\Lambda|} \gamma_j^2 |\phi_j\rangle\langle\phi_j|$.
- For $j = 1, \dots, |\Lambda|$, define

$$b_j := \frac{1}{\sqrt{2}}(\gamma_j^{\frac{1}{2}} \phi_j^T q + i\gamma_j^{-\frac{1}{2}} \phi_j^T p), \;\; q := (q_x)_{x\in\Lambda}, \;\; p := (p_x)_{x\in\Lambda}.$$

- Note that each operators b_j is fully determined by (γ_j, ϕ_j).
- $\{b_j\}_j$ satisfy the CCR: $[b_j, b_k] = 0$ and $[b_j, b_k^*] = \delta_{j,k}\mathbb{1}$.
- H_Λ can be written as a free boson system

$$H_\Lambda = \sum_{j=1}^{|\Lambda|} \gamma_j (2b_j^* b_j + \mathbb{1})$$

Eigenvalues and eigenfunctions of H_Λ

$$H_\Lambda = \sum_{j=1}^{|\Lambda|} \gamma_j (2b_j^* b_j + \mathbb{1}), \quad \gamma_j\text{'s are the eigenvalues of } h_\Lambda^{1/2}.$$

- There exists a unique vacuum ψ_0 of the b's, i.e., $b_j \psi_0 = 0$ for all j.
- For every $\alpha = (\alpha_1, \ldots, \alpha_{|\Lambda|}) \in \mathbb{N}_0^{|\Lambda|}$, the eigen-pair (ψ_α, E_α) is given as

$$\psi_\alpha = \prod_{j=1}^{|\Lambda|} \frac{1}{\sqrt{\alpha_j!}} (b_j^*)^{\alpha_j} \psi_0, \qquad E_\alpha = \sum_{j=1}^{|\Lambda|} \gamma_j (2\alpha_j + 1).$$

- $\{b_j\}_j$ are the modes (or quasi-particles).
- $\alpha \in \mathbb{N}^{|\Lambda|}$ describes the occupations of modes.
- The ground state energy is $\sum_j \gamma_j = \operatorname{Tr} h_\Lambda^{1/2}$.
- The gap above the ground state is $2 \min_j \gamma_j$

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h_Λ is fully localized $\Rightarrow$ H_Λ is localized

h_Λ being *fully localized at all energies*. i.e.,

$$\mathbb{E}\left(\sup_{|u|\leq 1} |\langle \delta_x, h_\Lambda^{-1/2} u(h_\Lambda)\delta_y\rangle|\right) \leq Ce^{-\mu|x-y|} \quad (1)$$

implies the following localization results

- Zero-velocity Lieb-Robinson bound.
- Exponential clustering of the ground/thermal states.
- Area laws for the entanglement of ground/thermal states.
- Exponential clustering of eigenstates and after a quantum quench.

Nachtergaele/Sims/Stolz '12,'13
AR/Sims/Stolz '17

The singular eigencorrelator localization (1) is

- known for k_x with sufficiently large disorder.

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Low-energy localization of h_Λ

Given any dimension $\nu \geq 1$, $\exists \lambda_0 > 0$ and $C < \infty, \mu > 0$ (independent of L) such that

$$\mathbb{E}\left(\sup_{|u|\leq 1} |\langle \delta_x, h_\Lambda^{-1/2} u(h_\Lambda) \chi_{[0,\lambda_0]}(h_\Lambda) \delta_y \rangle|\right) \leq C e^{-\mu|x-y|}$$

for all $x, y \in \Lambda$.

Nachtergaele/Sims/Stolz '12

What is the corresponding localization regime of H_Λ?

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Nachtergaele/Sims/Stolz '12

What is the corresponding localization regime of H_Λ?

The regime of localized excitations

- For fixed λ_0, let

$$S_{\lambda_0} := \{j \in \{1, \dots, |\Lambda|\};\ \gamma_j^2 \in [0, \lambda_0]\}$$
$$\mathcal{I} := \left\{\alpha = (\alpha_1, \dots, \alpha_{|\Lambda|}) \in \mathbb{N}_0^{|\Lambda|};\ \operatorname{supp}\alpha \subseteq S_{\lambda_0}\right\}$$

- The regime of localized excitations:

$$P_{\mathcal{I}} := P_{\mathcal{I}}(H_\Lambda) := \sum_{\alpha \in \mathcal{I}} |\psi_\alpha\rangle\langle\psi_\alpha|$$

- ψ_α for $\alpha \in \mathcal{I}$ is the eigenfunstion of H_Λ that results from modes associated with the bottom of the spectrum of h_Λ.

- $P_{\mathcal{I}}$ is the spectral projection of H_Λ associated with the energies

$$E_\alpha = E_0 + \sum_{j;\ \gamma_j^2 \in [0,\lambda_0]} 2\gamma_j \alpha_j.$$

- If we choose $\lambda_0 \geq 4\nu + k_{\max}$, then $P_{\mathcal{I}} = \mathbb{1}$.

Weyl operators

To quantify localization for the oscillator system, it will be useful to identify a convenient class of observables.

- Let $f : \Lambda \to \mathbb{C}$, the associated Weyl operator is defines as

$$\mathcal{W}(f) := \bigotimes_{x \in \Lambda} \exp\left(i(\mathrm{Re}[f_x] q_x + \mathrm{Im}[f_x] p_x)\right).$$

- Note that $\mathrm{supp}(f) = \textbf{supp}(\mathcal{W}(f))$.
- The Heisenberg dynamics: $\tau_t(\mathcal{W}(f)) = e^{itH_\Lambda} \mathcal{W}(f) e^{-itH_\Lambda}$.
- $\mathcal{W}(f)_{\mathcal{I}} := P_{\mathcal{I}} \mathcal{W}(f) P_{\mathcal{I}} = C_f \mathcal{W}\big(\chi(h_\Lambda)_{[0,\lambda_0]} f\big) P_{\mathcal{I}}$, where $0 < C_f \leq 1$.
- Note that $\tau_t(\mathcal{W}(f))_{\mathcal{I}} = \tau_t(\mathcal{W}(f)_{\mathcal{I}})$.

Restricted Lieb-Robinson Bounds

Theorem (AR/Sims/Stolz arXiv:1810.12769)

For any $f, g : \Lambda \to \mathbb{C}$,

$$\mathbb{E}\left(\sup_t \|[\tau_t(\mathcal{W}(f)_{\mathcal{I}}), \mathcal{W}(g)_{\mathcal{I}}]\|\right) \leq C(1+\lambda_0^{1/2})^2 \sum_{x,y\in\Lambda} |f(x)||g(x)|e^{-\mu|x-y|}$$

where C and μ are the constants in the eigencorrelator localization bound.

Note: in the case where f and g have disjoint supports,

$$\sum_{x,y\in\Lambda} |f(x)||g(x)|e^{-\mu|x-y|} \leq \text{Const. } e^{-\mu' d(\text{supp}(f),\text{supp}(g))}.$$

Note: Similar restricted LR version was established in **Elgart/Klein/Stolz '18**

QUASI-LOCALITY

- For $X \subset \Lambda$, let $f : \Lambda \to \mathbb{C}$, such that $\operatorname{supp} f \subseteq X$.
- **supp**$(\mathcal{W}(f)) \subseteq X$ but **supp**$\tau_t(\mathcal{W}(f)) = \Lambda$.

Can $\tau_t(\mathcal{W}(f))$ be "approximated" by a local operator supported "near" X?

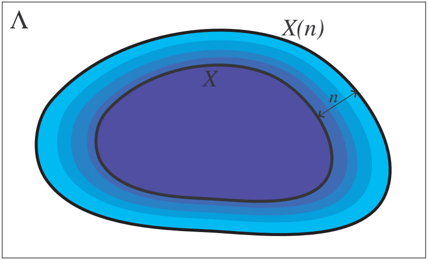

FIGURE: Define $X(n) := \{x \in \Lambda;\ d(x, X) \leq n\}$

Define the surface area of X

$$\partial X := \{x \in X, \text{there exists } y \in \Lambda \setminus X;\ |x-y| = 1\}.$$

Theorem (AR/Sims/Stolz arXiv:1810.12769)

Let $X \subset \Lambda$ and $f : \Lambda \to \mathbb{C}$ satisfy $\operatorname{supp}(f) \subset X$. For any $n \in \mathbb{N}_0$ and $t > 0$, there is an operator $\widehat{W}_{t,n} \in \mathcal{B}(\mathcal{H}_\Lambda)$ supported on $X(n)$, such that for any $\kappa > 0$ we have

$$\mathbb{E}\left(\sup_{\alpha \in \mathcal{I}; \|\alpha\|_\infty \le \kappa} \sup_{t \in \mathbb{R}} \left| \left\langle \psi_\alpha, (\tau_t(\mathcal{W}(f)) - \widehat{W}_{t,n}) \psi_\alpha \right\rangle \right| \right) \le \tilde{C} |\partial X| \|f\|_\infty^{2/3} e^{-\mu n/3}.$$

Note: Here

$$\tilde{C} = 2^{5/3} C \left(\sum_{z \in \mathbb{Z}^\nu} e^{-\mu|z|/6} \right)^4 (1+\kappa)^{1/3} (1+\lambda_0^{1/2})^2. \tag{2}$$

Exponential clustering of the eigenstates

The (restricted) dynamical correlations of local Weyl operators is given by

$$C_\alpha^{\mathcal{I}}(f,g;t) := \langle \tau_t(\mathcal{W}(f))_{\mathcal{I}} \mathcal{W}(g)_{\mathcal{I}} \rangle_{\psi_\alpha} - \langle \mathcal{W}(f)_{\mathcal{I}} \rangle_{\psi_\alpha} \langle \mathcal{W}(g)_{\mathcal{I}} \rangle_{\psi_\alpha}$$

where for any observable A, $\langle A \rangle_{\psi_\alpha} := \langle \psi_\alpha, A\psi_\alpha \rangle$.

Theorem (AR/Sims/Stolz arXiv:1810.12769)

For $\kappa \in \mathbb{N}_0$, and any functions $f, g : \Lambda \to \mathbb{C}$,

$$\mathbb{E}\left(\sup_{\alpha;\ \|\alpha\|_\infty \le \kappa} \sup_{t\in\mathbb{R}} |C_\alpha^{\mathcal{I}}(f,g,t)| \right) \le 8C(1+\lambda_0^{1/2})^2 \Big(\sum_{x,y\in\Lambda} |f(x)g(y)| e^{-\mu|x-y|} \Big)^{\frac{1}{\kappa+1}}$$

where C and μ are the constants in the eigencorrelator localization.

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- Fix a subregion $\Lambda_0 \subset \Lambda$ and decompose the Hilbert space $\mathcal{H}_\Lambda = \mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda \setminus \Lambda_0}$, where

$$\mathcal{H}_{\Lambda_0} = \bigotimes_{x \in \Lambda_0} \mathcal{L}^2(\mathbb{R}), \qquad \mathcal{H}_{\Lambda \setminus \Lambda_0} = \bigotimes_{x \in \Lambda \setminus \Lambda_0} \mathcal{L}^2(\mathbb{R})$$

- For any state $\rho \in \mathcal{B}(\mathcal{H}_\Lambda)$, the logarithmic negativity $\mathcal{N}(\rho)$ is defined as

$$\mathcal{N}(\rho) = \log \|\rho^{T_1}\|_1 \qquad (= \log \|\rho^{T_2}\|_1)$$

  where ρ^{T_1} is the *partial transpose* of ρ with respect to Λ_0 (the first component).

- Some properties:
  - If ρ is a separable state then $\mathcal{N}(\rho) = 0$.
  - If ρ is a pure state then $\mathcal{N}(\rho)$ is an upper bound for the von Neumann entanglement entropy.

Entanglement of the Eigenstates of H_Λ

Known:

- Area laws of the ground state and thermal states.
  Nachtergaele/Sims/Stolz (2013), Vidal/Werner ('02)

- Exponential clustering results of arbitrary eigenstates.
  AR/Sims/Stolz ('17,'18).

Open problem:
Finding/Studying/Understanding the entanglement for the eigenstates of H_Λ.

Entanglement of (Non-)Gaussian States

- The ground state and the thermal states of free boson systems are Gaussian states (quasi-free).
- All eigenstates above the ground state are non-Gaussian. i.e.,

$$\langle \psi_\alpha, \mathcal{W}(f)\psi_\alpha \rangle = e^{-\frac{1}{4}\langle \tilde{f}, M\tilde{f}\rangle} \prod_{k=1}^{|\Lambda|} L_{\alpha_k}\left(\frac{\langle \tilde{f}, M\chi_k(M)\tilde{f}\rangle}{2} \right).$$

Here $\tilde{f} = \begin{pmatrix} \mathrm{Re}[f] \\ \mathrm{Im}[f] \end{pmatrix}$, $L_{\alpha_k}(\cdot)$ is the Laguerre polynomial of degree α_k, $M = h^{-1/2} \oplus h^{1/2}$, and $\chi_k(M) := \chi_{\{\gamma_k^{-1}\}}(h^{-1/2}) \oplus \chi_{\{\gamma_k\}}(h^{1/2})$.

Note: M is the correlation matrix $\langle RR^T \rangle_{\psi_\alpha}$, where $R = \begin{bmatrix} q \\ p \end{bmatrix}$.

- There are (almost) **NO** rigorous results about the entanglement of non-Gaussian states.

THE N-MODES ENSEMBLE

For each $N \in \mathbb{N}_0$, let $\mathcal{J}_N$ be the set all occupations α associated with a total of N modes,

$$\mathcal{J}_N = \{\alpha = (\alpha_1, \ldots, \alpha_{|\Lambda|}) \in \mathbb{N}_0^{|\Lambda|};\ \sum_j \alpha_j = N\}.$$

$$\rho_N := \frac{1}{|\mathcal{J}_N|} \sum_{\alpha \in \mathcal{J}_N} |\psi_\alpha\rangle\langle\psi_\alpha|.$$

- ρ_N is the orthogonal projection onto the Fock space sector span$\{\psi_\alpha;\ \sum_j \alpha_j = N\}$.
- $\mathrm{Tr}[H_\Lambda \rho_N] = \left(1 + \frac{2N}{|\Lambda|}\right) \sum_k \gamma_k \quad \longrightarrow_{|\Lambda| \to \infty} \quad$ g. s. energy $\sum_k \gamma_k$

The N-modes ensemble

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$$\rho_N := \frac{1}{|\mathcal{J}_N|} \sum_{\alpha \in \mathcal{J}_N} |\psi_\alpha\rangle\langle\psi_\alpha|.$$

- For all $N \in \mathbb{N}$, ρ_N is non-Gaussian, in fact

$$\mathrm{Tr}[\rho_N \mathcal{W}(f)] = e^{-\frac{1}{4}\langle \tilde{f}, M\tilde{f}\rangle} \mathcal{Q}_N \left(\frac{\langle \tilde{f}, M\tilde{f}\rangle}{2} \right).$$

  where $\mathcal{Q}_N$ is a polynomial of degree N.
- The exact logarithmic negativity can be found using correlation matrices.

An Area Law

- $\rho_N := \frac{1}{|\mathcal{J}_N|} \sum_{\alpha \in \mathcal{J}_N} |\psi_\alpha\rangle\langle\psi_\alpha|$, and we have $\mathbb{E}\left(|\langle \delta_x, h_\Lambda^{-1/2} \delta_y\rangle|\right) \leq C\, e^{-\mu|x-y|}$.
- Let $\partial\Lambda_0 := \{x \in \Lambda_0;\ \exists\ y \in \Lambda \setminus \Lambda_0 \text{ with } |x-y| = 1\}$.

Theorem (AR '18)

For any $\Lambda_0 \subset \Lambda$, $N \in \mathbb{N}_0$, and the corresponding N-modes ensemble ρ_N, there exists $\tilde{C} < \infty$ such that

$$\mathbb{E}\left(\mathcal{N}(\rho_N)\right) \leq \tilde{C}(2N+1)|\partial\Lambda_0| \tag{3}$$

where the constant $\tilde{C}$ is independent of N, Λ_0 and Λ.

Note: $\tilde{C} = C(4d\lambda + k_{\max})^{1/2} \left(\sum_{x \in \mathbb{Z}^d} e^{-\mu|x|} \right)^2$.

Dynamical Entanglement

A simple case

- Let H_{Λ_0} and $H_{\Lambda\setminus\Lambda_0}$ be the restrictions of H_Λ to Λ_0 and $\Lambda \setminus \Lambda_0$, respectively.
- Let ρ_1 and ρ_2 be any thermal/ground states of H_{Λ_0} and $H_{\Lambda\setminus\Lambda_0}$, respectively.
- Note that (for example) if ρ_1 and ρ_2 are ground states then $\rho_1 \otimes \rho_2$ is the ground state of $H_{\Lambda_0} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\Lambda\setminus\Lambda_0}$
- We study $\rho_t := e^{-itH_\Lambda}(\rho_1 \otimes \rho_2)e^{itH_\Lambda}$.
- ρ_t is and entangled state with respect to $\mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda\setminus\Lambda_0}$.

Question: What can we say about the entanglement of ρ_t?

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Dynamical Entanglement

What about correlations of ρ_t?

Define the positions-momenta correlations

$$Cor_{\rho_t}(A_x, B_y) := \langle A_x B_y \rangle_{\rho_t} - \langle A_x \rangle_{\rho_t} \langle B_y \rangle_{\rho_t}, \ \ A, B \in \{p, q\}, \ \ x, y \in \Lambda.$$

Then the following are upper bounds for

$$\mathbb{E}\left(\sup_t |C_{\rho_t}(A_x, B_y)|^{1/3}\right), \text{ for all } A, B \in \{q, p\},$$

- $\rho_0 = \rho_{\alpha_1} \otimes \rho_{\alpha_2}$ (eigenstate-eigenstate):

$$\leq C_1 \big(1 + N\big)^{2/3} e^{-\eta_1 |x-y|}, \text{ where } \max\left\{\|\alpha_1\|_\infty, \|\alpha_2\|_\infty\right\} \leq N.$$

- $\rho_0 = \rho_\beta \otimes \rho_\beta$ (thermal-thermal):

$$\leq C_2 \max\left\{1, \beta^{-1/3}\right\} e^{-\eta_2 |x-y|}.$$

AR/Sims/Stolz '17

- In the gaped case with gap $\hat{\gamma}$:

$$\mathcal{N}\big(\rho_t\big) \leq C_{\hat{\gamma}}\ |\partial\Lambda_0| \text{ where } C_{\hat{\gamma}} \to \infty \text{ as } \hat{\gamma} \to 0.$$

- If $\rho_1^{\beta_1}$ and $\rho_2^{\beta_2}$ are thermal states (with inverse temperatures β_1 and β_2) and

$$(\rho^{\beta_1,\beta_2})_t = e^{-itH}\big(\rho_1^{\beta_1} \otimes \rho_2^{\beta_2}\big)e^{itH}$$

then

$$\mathbb{E}\left(\mathcal{N}\big((\rho^{\beta_1,\beta_2})_t\big)\right) \leq C(1+\max\{\beta_1,\beta_2\})\big(\max\{2+C_h^2, 2+8t^2\}\big)^{1/2}\ |\partial\Lambda_0|.$$

- If ρ_1 and ρ_2 are ground states then

$$\mathbb{E}\big(\mathcal{N}(\rho_t)^{1/2}\big) \leq C\big(\max\{2+C_h^2, 2+8t^2\}\big)^{1/4}\ |\partial\Lambda_0|.$$

Thank you.