

LOCALIZATION AND ENTANGLEMENT IN DISORDERED HARMONIC OSCILLATOR SYSTEMS

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Joint work with R. Sims (UA) and G. Stolz (UAB).

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I A disordered oscillator model.

II The regime of localized excitations.

- ▶ Zero velocity Lieb-Robinson bounds.
- ▶ Quasi-locality.
- ▶ Decay of correlations in eigenstates.

III Entanglement

- ▶ Dynamical Entanglement (work in progress).

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A Disordered Oscillator Model

THE HAMILTONIAN

$$H_{\Lambda} = \sum_{x \in \Lambda} (p_x^2 + k_x q_x^2) + \sum_{\substack{\{x, y\} \subset \Lambda \\ |x - y| = 1}} (q_x - q_y)^2$$

- $\Lambda := [-L, L]^{\nu} \cap \mathbb{Z}^{\nu}$ where $L \geq 1$ and $\nu \geq 1$.
- q_x and $p_x = -i \frac{\partial}{\partial q_x}$ are the position and momentum operators.
- The Hilbert space $\mathcal{H}_{\Lambda} = \bigotimes_{x \in \Lambda} \mathcal{L}^2(\mathbb{R}, dq_x)$.
- $\{k_x\}_x$ are i.i.d. random variables with absolutely continuous distribution given by a bounded density supported in $[0, k_{max}]$.

THE EFFECTIVE ONE-PARTICLE HAMILTONIAN

$$H_\Lambda = \sum_{x \in \Lambda} p_x^2 + q^T h_\Lambda q, \quad q := (q_x)_{x \in \Lambda}$$

- h_Λ is the Anderson model on $\ell^2(\Lambda)$, i.e., $h_\Lambda = h_{0,\Lambda} + k$, where
 - ▶ $h_{0,\Lambda}$ is the negative discrete Laplacian over $\ell^2(\Lambda)$.
 - ▶ $k := \text{diag}\{k_x, x \in \Lambda\}$.
- Recall that:
 - ▶ $\text{spec}(h_\Lambda)$ is almost surely simple.
 - ▶ $\text{spec}(h_\Lambda) \subset \left[\min_{x \in \Lambda} k_x, 4\nu + k_{\max} \right]$.
- h_Λ is almost surely positive, and $\|h_\Lambda\| \leq 4\nu + k_{\max}$.
- $h_\Lambda^{-1/2}$ does not have a deterministic upper bound.

H_Λ AS A FREE BOSON SYSTEM

- Since h_Λ is positive with simple spectrum (almost surely), it can be diagonalized with eigenvalues $\{\gamma_j^2\}_{j=1}^{|\Lambda|}$ and unique (up to a phase)

orthogonal eigenvectors $\{\phi_j\}_{j=1}^{|\Lambda|}$, $h_\Lambda = \sum_{j=1}^{|\Lambda|} \gamma_j^2 |\phi_j\rangle\langle\phi_j|$.

- For $j = 1, \dots, |\Lambda|$, define

$$b_j := \frac{1}{\sqrt{2}} (\gamma_j^{\frac{1}{2}} \phi_j^T q + i \gamma_j^{-\frac{1}{2}} \phi_j^T p), \quad q := (q_x)_{x \in \Lambda}, \quad p := (p_x)_{x \in \Lambda}.$$

- Note that each operators b_j is fully determined by (γ_j, ϕ_j) .
- $\{b_j\}_j$ satisfy the CCR: $[b_j, b_k] = 0$ and $[b_j, b_k^*] = \delta_{j,k} \mathbb{1}$.
- H_Λ can be written as a free boson system

$$H_\Lambda = \sum_{j=1}^{|\Lambda|} \gamma_j (2b_j^* b_j + \mathbb{1})$$

EIGENVALUES AND EIGENFUNCTIONS OF H_Λ

$$H_\Lambda = \sum_{j=1}^{|\Lambda|} \gamma_j (2b_j^* b_j + \mathbb{1}), \quad \gamma_j \text{'s are the eigenvalues of } h_\Lambda^{1/2}.$$

- There exists a unique vacuum ψ_0 of the b 's, i.e., $b_j \psi_0 = 0$ for all j .
- For every $\alpha = (\alpha_1, \dots, \alpha_{|\Lambda|}) \in \mathbb{N}_0^{|\Lambda|}$, the eigen-pair (ψ_α, E_α) is given as

$$\psi_\alpha = \prod_{j=1}^{|\Lambda|} \frac{1}{\sqrt{\alpha_j!}} (b_j^*)^{\alpha_j} \psi_0, \quad E_\alpha = \sum_{j=1}^{|\Lambda|} \gamma_j (2\alpha_j + 1).$$

- $\{b_j\}_j$ are the modes (or quasi-particles).
- $\alpha \in \mathbb{N}^{|\Lambda|}$ describes the occupations of modes.
- The ground state energy is $\sum_j \gamma_j = \text{Tr } h_\Lambda^{1/2}$.
- The gap above the ground state is $2 \min_j \gamma_j$

Known: h_Λ IS FULLY LOCALIZED $\Rightarrow H_\Lambda$ IS LOCALIZED

h_Λ being *fully localized at all energies*. i.e.,

$$\mathbb{E} \left(\sup_{|u| \leq 1} |\langle \delta_x, h_\Lambda^{-1/2} u(h_\Lambda) \delta_y \rangle| \right) \leq C e^{-\mu|x-y|} \quad (1)$$

implies the following localization results

- Zero-velocity Lieb-Robinson bound.
- Exponential clustering of the ground/thermal states.
- Area laws for the entanglement of ground/thermal states.
- Exponential clustering of eigenstates and after a quantum quench.

Nachtergaele/Sims/Stolz '12,'13
AR/Sims/Stolz '17

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LOW-ENERGY LOCALIZATION OF h_Λ

Given any dimension $\nu \geq 1$, $\exists \lambda_0 > 0$ and $C < \infty, \mu > 0$ (independent of L) such that

$$\mathbb{E} \left(\sup_{|u| \leq 1} |\langle \delta_x, h_\Lambda^{-1/2} u(h_\Lambda) \chi_{[0, \lambda_0]}(h_\Lambda) \delta_y \rangle| \right) \leq C e^{-\mu|x-y|}$$

for all $x, y \in \Lambda$.

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What is the corresponding localization regime of H_Λ ?

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THE REGIME OF LOCALIZED EXCITATIONS

- For fixed λ_0 , let

$$S_{\lambda_0} := \{j \in \{1, \dots, |\Lambda|\}; \gamma_j^2 \in [0, \lambda_0]\}$$

$$\mathcal{I} := \left\{ \alpha = (\alpha_1, \dots, \alpha_{|\Lambda|}) \in \mathbb{N}_0^{|\Lambda|}; \text{supp } \alpha \subseteq S_{\lambda_0} \right\}$$

- The regime of localized excitations:

$$P_{\mathcal{I}} := P_{\mathcal{I}}(H_{\Lambda}) := \sum_{\alpha \in \mathcal{I}} |\psi_{\alpha}\rangle \langle \psi_{\alpha}|$$

- ψ_{α} for $\alpha \in \mathcal{I}$ is the eigenfunction of H_{Λ} that results from modes associated with the bottom of the spectrum of h_{Λ} .
- $P_{\mathcal{I}}$ is the spectral projection of H_{Λ} associated with the energies

$$E_{\alpha} = E_0 + \sum_{j; \gamma_j^2 \in [0, \lambda_0]} 2\gamma_j \alpha_j.$$

- If we choose $\lambda_0 \geq 4\nu + k_{\max}$, then $P_{\mathcal{I}} = \mathbb{1}$.

WEYL OPERATORS

To quantify localization for the oscillator system, it will be useful to identify a convenient class of observables.

- Let $f : \Lambda \rightarrow \mathbb{C}$, the associated Weyl operator is defined as

$$\mathcal{W}(f) := \bigotimes_{x \in \Lambda} \exp(i(\operatorname{Re}[f_x]q_x + \operatorname{Im}[f_x]p_x)).$$

- Note that $\operatorname{supp}(f) = \mathbf{supp}(\mathcal{W}(f))$.
- The Heisenberg dynamics: $\tau_t(\mathcal{W}(f)) = e^{itH_\Lambda} \mathcal{W}(f) e^{-itH_\Lambda}$.
- $\mathcal{W}(f)_I := P_I \mathcal{W}(f) P_I = C_f \mathcal{W}(\chi(h_\Lambda)_{[0, \lambda_0]} f) P_I$, where $0 < C_f \leq 1$.
- Note that $\tau_t(\mathcal{W}(f))_I = \tau_t(\mathcal{W}(f)_I)$.

RESTRICTED LIEB-ROBINSON BOUNDS

THEOREM

For any $f, g : \Lambda \rightarrow \mathbb{C}$,

$$\mathbb{E} \left(\sup_t \|[\tau_t(\mathcal{W}(f)_{\mathcal{I}}), \mathcal{W}(g)_{\mathcal{I}}]\| \right) \leq C(1 + \lambda_0^{1/2})^2 \sum_{x,y \in \Lambda} |f(x)||g(x)| e^{-\mu|x-y|}$$

where C and μ are the constants in the eigencorrelator localization bound.

Note: in the case where f and g have disjoint supports,

$$\sum_{x,y \in \Lambda} |f(x)||g(x)| e^{-\mu|x-y|} \leq \text{Const. } e^{-\mu' d(\text{supp}(f), \text{supp}(g))}.$$

Note: Similar restricted LR version was established in **Elgart/Klein/Stolz '18**

QUASI-LOCALITY

- For $X \subset \Lambda$, let $f : \Lambda \rightarrow \mathbb{C}$, such that $\text{supp } f \subseteq X$.
- $\text{supp}(\mathcal{W}(f)) \subseteq X$ but $\text{supp}\tau_t(\mathcal{W}(f)) = \Lambda$.

Can $\tau_t(\mathcal{W}(f))$ be “approximated” by a local operator supported “near” X ?

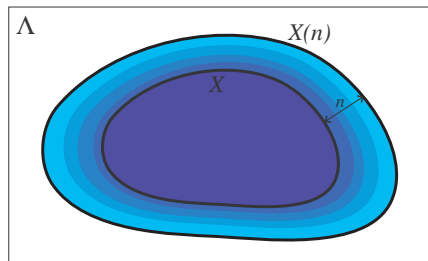


FIGURE: Define $X(n) := \{x \in \Lambda; d(x, X) \leq n\}$

QUASI-LOCALITY

Define the surface area of X

$$\partial X := \{x \in X, \text{there exists } y \in \Lambda \setminus X; |x - y| = 1\}.$$

THEOREM

Let $X \subset \Lambda$ and $f : \Lambda \rightarrow \mathbb{C}$ satisfy $\text{supp}(f) \subset X$. For any $n \in \mathbb{N}_0$ and $t > 0$, there is an operator $\widehat{W}_{t,n} \in \mathcal{B}(\mathcal{H}_\Lambda)$ supported on $X(n)$, such that for any $\kappa > 0$ we have

$$\mathbb{E} \left(\sup_{\alpha \in \mathcal{I}; \|\alpha\|_\infty \leq \kappa} \sup_{t \in \mathbb{R}} \left| \langle \psi_\alpha, (\tau_t(\mathcal{W}(f)) - \widehat{W}_{t,n})\psi_\alpha \rangle \right| \right) \leq \tilde{C} |\partial X| \|f\|_\infty^{2/3} e^{-\mu n/3}.$$

Note: Here

$$\tilde{C} = 2^{5/3} C \left(\sum_{z \in \mathbb{Z}^\nu} e^{-\mu|z|/6} \right)^4 (1 + \kappa)^{1/3} (1 + \lambda_0^{1/2})^2. \quad (2)$$

EXPONENTIAL CLUSTERING OF THE EIGENSTATES

The (restricted) dynamical correlations of local Weyl operators is given by

$$C_\alpha^{\mathcal{I}}(f, g; t) := \langle \tau_t(\mathcal{W}(f))_{\mathcal{I}} \mathcal{W}(g)_{\mathcal{I}} \rangle_{\psi_\alpha} - \langle \mathcal{W}(f)_{\mathcal{I}} \rangle_{\psi_\alpha} \langle \mathcal{W}(g)_{\mathcal{I}} \rangle_{\psi_\alpha}$$

where for any observable A , $\langle A \rangle_{\psi_\alpha} := \langle \psi_\alpha, A \psi_\alpha \rangle$.

THEOREM

For $\kappa \in \mathbb{N}_0$, and any functions $f, g : \Lambda \rightarrow \mathbb{C}$,

$$\mathbb{E} \left(\sup_{\alpha; \|\alpha\|_\infty \leq \kappa} \sup_{t \in \mathbb{R}} |C_\alpha^{\mathcal{I}}(f, g, t)| \right) \leq 8C(1 + \lambda_0^{1/2})^2 \left(\sum_{x, y \in \Lambda} |f(x)g(y)| e^{-\mu|x-y|} \right)^{\frac{1}{\kappa+1}}$$

where C and μ are the constants in the eigencorrelator localization.

BIPARTITE ENTANGLEMENT

THE LOGARITHMIC NEGATIVITY

- Fix a subregion $\Lambda_0 \subset \Lambda$ and decompose the Hilbert space $\mathcal{H}_\Lambda = \mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda \setminus \Lambda_0}$, where

$$\mathcal{H}_{\Lambda_0} = \bigotimes_{x \in \Lambda_0} \mathcal{L}^2(\mathbb{R}), \quad \mathcal{H}_{\Lambda \setminus \Lambda_0} = \bigotimes_{x \in \Lambda \setminus \Lambda_0} \mathcal{L}^2(\mathbb{R})$$

- For any state $\rho \in \mathcal{B}(\mathcal{H}_\Lambda)$, the logarithmic negativity $\mathcal{N}(\rho)$ is defined as

$$\mathcal{N}(\rho) = \log \|\rho^{T_1}\|_1 \quad (= \log \|\rho^{T_2}\|_1)$$

where ρ^{T_1} is the *partial transpose* of ρ with respect to Λ_0 (the first component).

- Some properties:
 - If ρ is a separable state then $\mathcal{N}(\rho) = 0$.
 - If ρ is a pure state then $\mathcal{N}(\rho)$ is an upper bound for the von Neumann entanglement entropy.

ENTANGLEMENT OF THE EIGENSTATES OF H_Λ

Known:

- Area laws of the ground state and thermal states.
Vidal/Werner ('02), Nachtergaele/Sims/Stolz ('13), Beaud/Sieber/Warzel ('18)
- Area laws for the uniform ensemble of equal-modes excitations (N-modes ensemble).

AR ('18).

Open problem:

Finding/Studying/Understanding the entanglement for the eigenstates of H_Λ .

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DYNAMICAL ENTANGLEMENT

A SIMPLE CASE

- Let H_{Λ_0} and $H_{\Lambda \setminus \Lambda_0}$ be the restrictions of H_Λ to Λ_0 and $\Lambda \setminus \Lambda_0$, respectively.
- Let ρ_1 and ρ_2 be any thermal/ground states of H_{Λ_0} and $H_{\Lambda \setminus \Lambda_0}$, respectively.
- Note that (for example) if ρ_1 and ρ_2 are ground states then $\rho_1 \otimes \rho_2$ is the ground state of $H_{\Lambda_0} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\Lambda \setminus \Lambda_0}$
- We study $\rho_t := e^{-itH_\Lambda}(\rho_1 \otimes \rho_2)e^{itH_\Lambda}$.
- ρ_t is an entangled state with respect to $\mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda \setminus \Lambda_0}$.

Question: What can we say about the entanglement of ρ_t ?

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DYNAMICAL ENTANGLEMENT

INITIAL RESULTS

- In the gaped case with gap $\hat{\gamma}$:

$$\mathcal{N}(\rho_t) \leq C_{\hat{\gamma}} |\partial\Lambda_0| \text{ where } C_{\hat{\gamma}} \rightarrow \infty \text{ as } \hat{\gamma} \rightarrow 0.$$

- If $\rho_1^{\beta_1}$ and $\rho_2^{\beta_2}$ are thermal states (with inverse temperatures β_1 and β_2) and

$$(\rho^{\beta_1, \beta_2})_t = e^{-itH} (\rho_1^{\beta_1} \otimes \rho_2^{\beta_2}) e^{itH}$$

then

$$\mathbb{E} \left(\sup_t [\mathcal{N}(\rho^{\beta_1, \beta_2})_t]^{1/2} \right) \leq C(1 + \max\{\beta_1, \beta_2\}) |\partial\Lambda_0|.$$

- If ρ_1 and ρ_2 are ground states then

$$\mathbb{E} \left(\sup_t [\mathcal{N}(\rho_t)]^{1/4} \right) \leq C |\partial\Lambda_0|.$$

Thank you.