

# ENTANGLEMENT OF A CLASS OF NON-GAUSSIAN STATES IN DISORDERED HARMONIC OSCILLATOR SYSTEMS.

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# The Harmonic Oscillators System

## THE HAMILTONIAN

$$H = \sum_{x \in \Lambda} (p_x^2 + k_x q_x^2) + \sum_{\substack{\{x, y\} \subset \Lambda \\ |x - y| = 1}} \lambda (q_x - q_y)^2$$

- $\Lambda := [-L, L]^d \cap \mathbb{Z}^d$  where  $L \geq 1$  and  $d \geq 1$ .
- $q_x$  and  $p_x = -i \frac{\partial}{\partial q_x}$  are the position and momentum operators.
- The Hilbert space  $\mathcal{H} = \bigotimes_{x \in \Lambda} \mathcal{L}^2(\mathbb{R}, dq_x)$ .
- $|\cdot|$  denotes the 1-norm.
- $\lambda \in (0, \infty)$  is the coupling parameter.
- $\{k_x\}_x$  are i.i.d. random variables with absolutely continuous distribution given by a bounded density supported in  $[0, k_{max}]$ .

# THE EFFECTIVE ONE-PARTICLE HAMILTONIAN

Let  $h_\Lambda$  be the finite volume Anderson model on  $\ell^2(\Lambda)$ , i.e.,

$$h_\Lambda = \lambda h_{0,\Lambda} + k,$$

where  $h_{0,\Lambda}$  is the negative discrete Laplacian over  $\ell^2(\Lambda)$  and  $k := \text{diag}\{k_x, x \in \Lambda\}$ .

Here are some facts about  $h_\Lambda$ :

- $\text{spec}(h_\Lambda)$  is almost surely simple.
- $\text{spec}(h_\Lambda) \subset \left[ \min_{x \in \Lambda} k_x, 4d\lambda + k_{\max} \right]$ .
  - ▶  $h_\Lambda$  is almost surely positive.
  - ▶  $h_\Lambda^{-1/2}$  is not uniformly bounded in the volume of the system and the disorder.
- $\exists C < \infty, \mu > 0$ , independent of the volume  $\Lambda$ , such that

$$\mathbb{E} \left( |\langle \delta_x, h_\Lambda^{-1/2} \delta_y \rangle| \right) \leq C e^{-\mu|x-y|}$$

for every  $x, y \in \Lambda$

# $H_\Lambda$ AS A FREE BOSON SYSTEM

- $h_\Lambda$  is diagonalizable in terms of an orthogonal matrix  $\mathcal{O}$  and a diagonal operator  $\gamma^2 = \text{diag}\{\gamma_x^2, x \in \Lambda\}$ , i.e.,  $h_\Lambda = \mathcal{O}\gamma^2\mathcal{O}^T$ .
- Define the *annihilation* and *creation* operators

$$a_x = \frac{1}{\sqrt{2}}(q_x + ip_x), \quad a_x^* = \frac{1}{\sqrt{2}}(q_x - ip_x), \quad \text{for } x \in \Lambda.$$

They satisfy the CCR  $[a_x, a_y] = 0$ , and  $[a_x, a_y^*] = \delta_{x,y}\mathbb{1}$ , for all  $x, y \in \Lambda$ .

- Define the bosons  $\{b_k\}_k$  using the Bogoliubov Transformation

$$b = U a + V a^*$$

where  $U = \frac{1}{2}(\gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}})\mathcal{O}^T$ ,  $V = \frac{1}{2}(\gamma^{\frac{1}{2}} - \gamma^{-\frac{1}{2}})\mathcal{O}^T$ .

- $H_\Lambda$  can be written as a free boson system

$$H_\Lambda = \sum_{k=1}^{|\Lambda|} \gamma_k (2b_k^* b_k + \mathbb{1})$$

# EIGENVALUES AND EIGENFUNCTIONS OF $H_\Lambda$

$$H = \sum_{k=1}^{|\Lambda|} \gamma_k (2b_k^* b_k + \mathbb{1}), \quad \gamma_k \text{'s are the eigenvalues of } h_\Lambda^{1/2}.$$

- $\{b_k\}_k$  are satisfying the CCR.
- There exists a unique vacuum  $\psi_0$  of the  $b$ 's, i.e.,  $b_k \psi_0 = 0$  for all  $k = 1, \dots, |\Lambda|$ .
- For every  $\alpha = (\alpha_1, \dots, \alpha_{|\Lambda|}) \in \mathbb{N}_0^{|\Lambda|}$ , The eigen-pair  $(\psi_\alpha, E_\alpha)$  is given as

$$\psi_\alpha = \prod_{k=1}^{|\Lambda|} \frac{1}{\sqrt{\alpha_k!}} (b_k^*)^{\alpha_k} \psi_0, \quad E_\alpha = \sum_{k=1}^{|\Lambda|} \gamma_k (2\alpha_k + 1).$$

- The g.s. energy is  $\sum_k \gamma_k = \text{Tr } h_\Lambda^{1/2}$ , and the gap above the g.s. is  $2 \min_k \gamma_k$

# ABOUT THE EIGENSTATES

- Recall: the eigenstates are

$$\psi_\alpha = \text{Const.} (b_1^*)^{\alpha_1} (b_2^*)^{\alpha_2} \dots (b_{|\Lambda|}^*)^{\alpha_{|\Lambda|}} \psi_0$$

- $b_k \psi_\alpha = \begin{cases} \sqrt{\alpha_k} \psi_{(\alpha_1, \dots, \alpha_k-1, \dots, \alpha_{|\Lambda|})}, & \alpha_k \in \mathbb{N} \\ 0, & \alpha_k = 0 \end{cases}$
- $b_k^* \psi_\alpha = \sqrt{\alpha_k + 1} \psi_{(\alpha_1, \dots, \alpha_k+1, \dots, \alpha_{|\Lambda|})}$
- $\alpha_k$  is called the  $k$ -th *mode* ( $\#$  of *quasi-particle* in the  $k$ -th site).
- $\|\alpha\|_1 = \sum_j \alpha_j$  is called the total number of modes (quasi-particles).
- $\alpha$  is called the *vector of occupation modes*.

# LOCALIZATION RESULTS OF THE EIGENSTATES

## Known:

- Area laws of the ground state and thermal states.

Nachtergaele/Sims/Stolz (2013), Vidal/Werner (2002)

- Exponential clustering results of arbitrary eigenstates.

AR/Sims/Stolz (2017).

## Open problem:

Proving area laws for the eigenstates of  $H_\Lambda$ .

- The ground state and the thermal states of free boson systems are Gaussian states (quasi-free).
- All eigenstates above the ground state are non-Gaussian.
- From Physics: There is a fundamental need to understand the entanglement beyond Gaussian states.
- There are (almost) **NO** rigorous results about the entanglement of non-Gaussian states.

# THE $N$ -MODES ENSEMBLE

- For each  $N \in \mathbb{N}_0$ , let  $\mathcal{J}_N$  be the set all vector of occupation modes that are describing a total of  $N$  modes,

$$\mathcal{J}_N = \{\alpha = (\alpha_1, \dots, \alpha_{|\Lambda|}) \in \mathbb{N}_0^{|\Lambda|}; \sum_j \alpha_j = N\}.$$

We define the “ $N$ -modes ensemble state”

$$\rho_N := \frac{1}{|\mathcal{J}_N|} \sum_{\alpha \in \mathcal{J}_N} |\psi_\alpha\rangle\langle\psi_\alpha|.$$

**Note:**  $\rho_N$  is the orthogonal projection onto the Fock space sector  $\text{span}\{\psi_\alpha; \sum_j \alpha_j = N\}$ .

- ▶ For all  $N \in \mathbb{N}$ ,  $\rho_N$  is non-Gaussian.
- ▶  $\text{Tr}[H_\Lambda \rho_N] = \left(1 + \frac{2N}{|\Lambda|}\right) \sum_k \gamma_k$ .

# BIPARTITE ENTANGLEMENT

## THE LOGARITHMIC NEGATIVITY

- Fix a subregion  $\Lambda_0 \subset \Lambda$ .
- Decompose the Hilbert space  $\mathcal{H}_\Lambda = \mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda \setminus \Lambda_0}$ , where

$$\mathcal{H}_{\Lambda_0} = \bigotimes_{x \in \Lambda_0} \mathcal{L}^2(\mathbb{R}), \quad \mathcal{H}_{\Lambda \setminus \Lambda_0} = \bigotimes_{x \in \Lambda \setminus \Lambda_0} \mathcal{L}^2(\mathbb{R})$$

- For any state  $\rho \in \mathcal{B}(\mathcal{H}_\Lambda)$ , the logarithmic negativity  $\mathcal{N}(\rho)$  is defined as

$$\mathcal{N}(\rho) = \log \|\rho^{T_1}\|_1 \quad (= \log \|\rho^{T_2}\|_1)$$

where  $\rho^{T_1}$  is the *partial transpose* of  $\rho$  with respect to  $\Lambda_0$  (the first component).

- Some properties:
  - ▶ If  $\rho$  is a separable state then  $\mathcal{N}(\rho) = 0$ .
  - ▶ If  $\rho$  is a pure state then  $\rho$  is an upper bound for the von Neumann entanglement entropy.

# AN AREA LAW

- Recall:  $\rho_N := \frac{1}{|\mathcal{J}_N|} \sum_{\alpha \in \mathcal{J}_N} |\psi_\alpha\rangle\langle\psi_\alpha|$ ,  $\mathbb{E} \left( |\langle \delta_x, h_\Lambda^{-1/2} \delta_y \rangle| \right) \leq C e^{-\mu|x-y|}$ .
- Let  $\partial\Lambda_0 := \{x \in \Lambda_0; \exists y \in \Lambda \setminus \Lambda_0 \text{ with } |x - y| = 1\}$ .

## THEOREM

For any  $\Lambda_0 \subset \Lambda$ ,  $N \in \mathbb{N}_0$ , and the corresponding  $N$ -modes ensemble  $\rho_N$ , there exists  $\tilde{C} < \infty$  such that

$$\mathbb{E} (\mathcal{N}(\rho_N)) \leq \tilde{C}(2N + 1)|\partial\Lambda_0| \quad (1)$$

where the constant  $\tilde{C}$  is independent of  $N$ ,  $\Lambda_0$  and  $\Lambda$ .

**Note:**  $\tilde{C} = C(4d\lambda + k_{\max})^{1/2} \left( \sum_{x \in \mathbb{Z}^d} e^{-\mu|x|} \right)^2$ .

# KEY LEMMAS

## The Weyl operators:

- For  $z \in \mathbb{C}$ ,  $\mathcal{W}_z := \exp\left(\frac{i}{\sqrt{2}}(\bar{z}a_x + za_x^*)\right)$ .
- For  $f : \Lambda \rightarrow \mathbb{C}$ ,  $\mathcal{W}(f) := \bigotimes_{x \in \Lambda} \mathcal{W}_{f_x}$ .
- The first Lemma finds  $\langle \mathcal{W}(f) \rangle_{\rho_N} = \frac{1}{|\mathcal{J}_N|} \sum_{\alpha \in \mathcal{J}_N} \langle \psi_\alpha, \mathcal{W}(f) \psi_\alpha \rangle = ??$
- Let  $M := \begin{pmatrix} h_\Lambda^{-1/2} & 0 \\ 0 & h_\Lambda^{1/2} \end{pmatrix}$ , recall that  $h_\Lambda = \mathcal{O}\gamma^2\mathcal{O}^T$ .
- Let  $\chi_k(M)$  be the spectral projection associated with the eigenvalues  $\gamma_k$  and  $\gamma_k^{-1}$ .

# KEY LEMMAS

## The Weyl expectations at the eigenstates:

### LEMMA

Let  $\alpha = (\alpha_1, \dots, \alpha_{|\Lambda|}) \in \mathbb{N}_0^{|\Lambda|}$  be the vector of occupation modes and let  $\psi_\alpha$  be the corresponding eigenstate of  $H_\Lambda$ . Then for any  $f : \Lambda \rightarrow \mathbb{C}$ ,

$$\langle \psi_\alpha, \mathcal{W}(f) \psi_\alpha \rangle = e^{-\frac{1}{4} \langle \tilde{f}, M \tilde{f} \rangle} \prod_{k=1}^{|\Lambda|} L_{\alpha_k} \left( \frac{\langle \tilde{f}, M \chi_k(M) \tilde{f} \rangle}{2} \right).$$

Here  $\tilde{f} = \begin{pmatrix} \operatorname{Re}[f] \\ \operatorname{Im}[f] \end{pmatrix}$  and  $L_{\alpha_k}$  is the Laguerre polynomial of degree  $\alpha_k$ .

**note:**

$$L_k(x) = \sum_{n=0}^k \binom{k}{n} \frac{(-1)^n x^n}{n!}, \text{ for } k \in \mathbb{N}. \quad (L_0(x) = 1, L_1(x) = 1 - x, \dots)$$

# KEY LEMMAS

## AN OBSERVATION ABOUT THE LAGUERRE POLYNOMIALS

### LEMMA

Fix  $N \in \mathbb{N}_0$  and  $K \in \mathbb{N}$ , then for any  $x_1, \dots, x_K \in \mathbb{R}$ ,

$$\sum_{\substack{\alpha = (\alpha_1, \dots, \alpha_K) \in \mathbb{N}_0^K \\ \sum_j \alpha_j = N}} \prod_{k=1}^K L_{\alpha_k}(x_k) = \mathcal{Q}_N \left( \sum_{j=1}^K x_k \right),$$

where  $\mathcal{Q}_N$  is a polynomial of degree  $N$ .

This gives

$$\langle \mathcal{W}(f) \rangle_{\rho_N} = \frac{1}{|\mathcal{J}_N|} \mathcal{Q}_N \left( \frac{1}{2} \langle \tilde{f}, M \tilde{f} \rangle \right) e^{-\frac{1}{4} \langle \tilde{f}, M \tilde{f} \rangle}.$$

# KEY LEMMAS

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# KEY LEMMAS

## LEMMA (NACHTERGAELE/SIMS/STOLZ '13)

Fix  $a > 0$ , there exists a unique self adjoint trace class operator  $\rho_a$  such that

$$\langle \mathcal{W}_z \rangle_{\rho_a} = e^{-\frac{1}{4}a|z|^2}, \quad \forall z \in \mathbb{C}, \quad \text{where } \|\rho_a\|_1 = \begin{cases} 1 & a \geq 1 \\ \frac{1}{a} & a < 1 \end{cases}$$

## LEMMA (AR '17)

Fix  $a > 0$  and  $\ell \in \mathbb{N}_0$ , there exists a unique self adjoint trace class operator  $\rho_a^{(\ell)}$  such that

$$\langle \mathcal{W}_z \rangle_{\rho_a^{(\ell)}} = L_\ell \left( \frac{1}{2}a|z|^2 \right) e^{-\frac{1}{4}a|z|^2}, \quad \forall z \in \mathbb{C},$$

$$\text{where } \|\rho_a^{(\ell)}\|_1 \leq \begin{cases} a^\ell & a \geq 1 \\ \frac{1}{a^{\ell+1}} & a < 1 \end{cases}$$

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*Thank you.*