ENTANGLEMENT BOUNDS IN THE XXZ QUANTUM SPIN CHAIN

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THE ISING MODEL

- **Chain:** $\Lambda = \{1, 2, \dots, L\}$.
- **Hilbert Space:** $\mathcal{H}_\Lambda := \bigotimes_{j\in\Lambda} \mathbb{C}^2 = (\mathbb{C}^2)^{\otimes n}$.
- **Notations:** $|\uparrow\rangle := \begin{pmatrix}1\\0\end{pmatrix}$ and $|\downarrow\rangle := \begin{pmatrix}0\\1\end{pmatrix}$, and $|\uparrow\downarrow\rangle := |\uparrow\rangle \otimes |\downarrow\rangle$.
- $\sigma^Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is the Z-Pauli matrix, and $\mathcal{N} = |\downarrow\rangle\langle\downarrow|$ is the **Particle Number** operator.
- $A_j \in \mathcal{B}(\mathcal{H}_\Lambda)$ acts non-trivially only on the j-th component of the tensor product, i.e., $A_j = \mathbb{1}^{\otimes(j-1)} \otimes A \otimes \mathbb{1}^{\otimes(L-j)}$.

$$\boxed{H_\Lambda^{\text{Ising}} = \frac{1}{4}\sum_{j=1}^{L-1}(\mathbb{1} - \sigma_j^Z\sigma_{j+1}^Z) + \frac{1}{2}(\mathcal{N}_1 + \mathcal{N}_L)}$$

THE ISING MODEL

CLUSTERS OF PARTICLES

$$\boxed{H_\Lambda^{\mathsf{Ising}} = \frac{1}{4}\sum_{j=1}^{L-1}(\mathbb{1} - \sigma_j^Z \sigma_{j+1}^Z) + \frac{1}{2}(\mathcal{N}_1 + \mathcal{N}_L)}$$

- For $X \subseteq \Lambda$, let $\phi_X = \prod_{j\in X} S_j^- |\ldots \uparrow\uparrow\uparrow \ldots\rangle$, where $S^- = |\downarrow\rangle\langle\uparrow|$, i.e., ϕ_X has a down-spin (a particle) in the j-th component for every $j \in X$.

  Example: If $\Lambda := [1, 10]$ and $X := \{4, 5, 9\}$, then $\phi_X = |\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow\rangle$.

- **Clusters:** For $X \subseteq \Lambda$, let $\mathsf{cl}(X) = \#$connected components of X (# of clusters of down spins in ϕ_X), e.g., $\mathsf{cl}\big(\{4, 5, 9\}\big) = 2$

$$\boxed{H_\Lambda^{\mathsf{Ising}}\phi_X = \mathsf{cl}(X)\phi_X,\ \mathsf{cl}(X) \in \{0, 1, \ldots, \lceil\frac{L}{2}\rceil\}}$$

THE XXZ CHAIN IN THE ISING PHASE

$$H_\Lambda^{\mathsf{XXZ}} := \sum_{j=1}^{L-1} h_{j,j+1} + \frac{1}{2}\big(1 - \frac{1}{\Delta}\big)(\mathcal{N}_1 + \mathcal{N}_L), \quad \Delta > 1 \text{ where}$$

$$h_{j,j+1} := \frac{1}{4}(\mathbb{1} - \sigma_j^Z \sigma_{j+1}^Z) - \frac{1}{4\Delta}(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y)$$

- $\sigma^X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$
- Remark: H_Λ^{XXZ} is frustration free:

$$h_{j,j+1}|\uparrow\uparrow\rangle = h_{j,j+1}|\downarrow\downarrow\rangle = 0$$

$$h_{j,j+1}\big(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle\big) = \frac{1}{2}\big(1 \mp 1/\Delta\big)\big(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle\big).$$

- $H_\Lambda^{\mathsf{XXZ}}|\uparrow\ldots\uparrow\rangle = 0$ and $H_\Lambda^{\mathsf{XXZ}}|\downarrow\ldots\downarrow\rangle = (1 - 1/\Delta)|\downarrow\ldots\downarrow\rangle.$

PARTICLE NUMBER CONSERVATION

$$\boxed{H_\Lambda^{\mathsf{XXZ}} \text{ commutes with the particle number operator.}}$$

- Particle Number Operator: $\mathcal{N}_\Lambda = \sum_{j\in\Lambda} \mathcal{N}_j$.
- $\left[H_\Lambda^{\mathsf{XXZ}}, \mathcal{N}_\Lambda\right] = 0$.
- The eigenspaces of $\mathcal{N}_\Lambda$ are invariant under H_Λ^{XXZ}.

The eigenspaces of $\mathcal{N}_\Lambda$:

- **Recall:** For any $X \subseteq \Lambda$, ϕ_X has a down-spin in the j-th component for every $j \in X$.
- $\mathcal{N}_\Lambda \phi_X = |X|\phi_X$, $|X| \in \{0, 1, \dots, L\}$.
- For $N \in \mathbb{N}_0$, let $\mathcal{V}_N := \{X \subseteq \Lambda; |X| = N\}$ (the set of N-particles), then
$$\text{null}(\mathcal{N}_\Lambda - N\mathbb{1}) = \text{span}\{\phi_X;\ X \in \mathcal{V}_N\} = \ell^2(\mathcal{V}_N).$$

H_Λ^{XXZ} AS A DIRECT SUM

HARDCORE PARTICLE FORMULATION

$$H_\Lambda^{\mathsf{XXZ}} = \bigoplus_{N=0}^{L} H_{\Lambda,N}^{\mathsf{XXZ}}, \text{ where } H_{\Lambda,N}^{\mathsf{XXZ}} := H_\Lambda^{\mathsf{XXZ}} \downharpoonright_{\ell^2(\mathcal{V}_N)} = ??$$

- Recall that $\mathcal{V}_N = \{X \subseteq \Lambda;\ |X| = N\}$.
- For any $X = \{x_1 < \ldots < x_N\}$ and $Y = \{y_1 < \ldots < y_N\}$ in $\mathcal{V}_N$, let

$$d_N(X,Y) := \sum_{j=1}^{N} |x_j - y_j| = \|X^\uparrow - Y^\uparrow\|_1.$$

Symmetric Product of graphs: For any $N \in \Lambda$, we define the N-th symmetric products $\mathcal{G}_N = (\mathcal{V}_N, \mathcal{E}_N)$ where

- Vertex set: $\mathcal{V}_N$.
- Edge set: $\mathcal{E}_N := \big\{\{X,Y\} :\ X, Y \in \mathcal{V}_N;\ d_N(X,Y) = 1\big\}$

$H_{\Lambda,N}^{\mathrm{XXZ}} := H_{\Lambda}^{\mathrm{XXZ}} \restriction_{\ell^2(\mathcal{V}_N)} = ??$

$$\boxed{H_{\Lambda,N}^{\textsf{XXZ}} := -\frac{1}{2\Delta}\mathcal{L}_N + (1 - 1/\Delta)C_N}$$

- $\mathcal{L}_N$ is the graph Laplacian on $\mathcal{G}_N = (\mathcal{V}_N, \mathcal{E}_N)$, i.e.,

$$(\mathcal{L}_N f)(X) = \sum_{Y:\{X,Y\}\in\mathcal{E}_N} \big(f(Y) - f(X)\big), \text{ for } X \in \mathcal{V}_N.$$

- $C_N \phi_X = \mathsf{cl}(X)\phi_X$, where $\mathsf{cl}(X) \in \{0, 1, \dots\}$ is the number of clusters (of particles) in $X \subseteq \Lambda$.

Recall: $H_{\Lambda}^{\textsf{XXZ}} | \dots \uparrow\uparrow\uparrow \dots \rangle = 0$.

Observe:
spec$(H_{\Lambda}^{\textsf{XXZ}}) \setminus \{0\} \geq (1 - 1/\Delta)$ and $H_{\Lambda}^{\textsf{XXZ}} | \downarrow \dots \downarrow \rangle = (1 - 1/\Delta) | \downarrow \dots \downarrow \rangle$

THE SPECTRUM OF H_Λ^{XXZ}

$$\boxed{H_{\Lambda,N}^{\mathsf{XXZ}} := -\frac{1}{2\Delta}\mathcal{L}_N + (1-1/\Delta)C_N}$$

For $k \in \mathbb{N}_0$, let $\mathcal{V}_{N,k}$ be the set N-particles that are grouped in up to k clusters, i.e.,

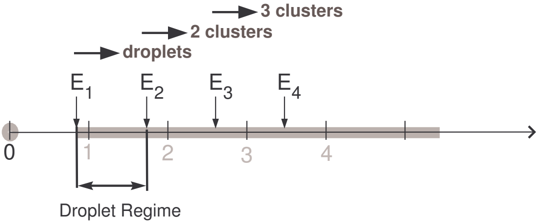
$$\mathcal{V}_{N,k} = \{X \in \mathcal{V}_N;\ \mathsf{cl}(X) \leq k\}, \text{ and } \overline{\mathcal{V}}_{N,k} = \mathcal{V}_N \setminus \mathcal{V}_{N,k}$$

Let $P_{\overline{\mathcal{V}}_{N,k}} : \ell^2(\mathcal{V}_N) \to \ell^2(\overline{\mathcal{V}}_{N,k})$ be the orthogonal projection onto $\ell^2(\overline{\mathcal{V}}_{N,k})$.

$$\boxed{P_{\overline{\mathcal{V}}_{N,k}} H_{\Lambda,N}^{\mathsf{XXZ}} P_{\overline{\mathcal{V}}_{N,k}} \geq E_{k+1} P_{\overline{\mathcal{V}}_{N,k}} \text{ where } E_k = k\big(1-1/\Delta\big)}$$

THE SPECTRUM OF H_Λ^{XXZ}

- $H_\Lambda^{\mathsf{XXZ}} |\ldots\uparrow\uparrow\uparrow\ldots\rangle = 0.$
- $\mathsf{spec}(H_\Lambda^{\mathsf{XXZ}}) \setminus \{0\} \geq (1 - 1/\Delta)$
- $P_{\overline{\mathcal{V}}_{N,k}} H_{\Lambda,N}^{\mathsf{XXZ}} P_{\overline{\mathcal{V}}_{N,k}} \geq E_{k+1} P_{\overline{\mathcal{V}}_{N,k}}$ where $E_k = k\big(1 - 1/\Delta\big)$.



MBL IN THE DROPLET REGIME

In a nonnegative background field:

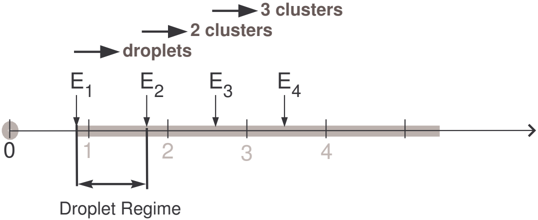
$$\boxed{H_\Lambda^{\mathsf{XXZ}}(v) = H_\Lambda^{\mathsf{XXZ}} + \sum_{j\in\Lambda} v_j \mathcal{N}_j,\ \ v_j \geq 0.}$$

When $\{v_j \geq 0\}_{j\in\Lambda}$ are i.i.d. random variables from an absolutely continuous distribution with bounded density, we have the following on the **droplet regime**:

- Zero-velocity Lieb-Robinson bound.
- Exponential decay of correlations.
- Area laws.

Elgart/Klein/Stolz '17,'18
Beaud/Warzel '17,'18

COMBES-THOMAS BOUND ABOVE THE DROPLET REGIME



THEOREM (AR/FISCHBACHER/STOLZ '19 - SIMPLIFIED)

For any non-negative background field v, *and any* $N \in \{1, \dots, N\}$, $\delta > 0$, $k \in \mathbb{N}$, *and* $\psi_N \in \ell^2(\mathcal{V}_N)$ *an eigenstate of* H_Λ^{XXZ} *in* $\mathit{Ran}\big(\chi_{[0,E_{k+1}-\delta)}(H_\Lambda^{\mathsf{XXZ}}(v))\big)$. *There exists* $C < \infty$ *and* $\mu > 0$ *such that*

$$\big|\langle \phi_X, \psi_N \rangle\big| \leq C e^{-\mu d_N(X, \mathcal{V}_{N,k})}$$

for any $X \subseteq \overline{\mathcal{V}}_{N,k}$.

ENTANGLEMENT

- Fix $\Lambda_0 := [1, \ell] \subset \Lambda$.
- Consider the corresponding bipartition of the Hilbert space $\mathcal{H}_\Lambda = \mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda \setminus \Lambda_0}$.
- The **Entanglement Entropy** of state $\psi \in \mathcal{H}_\Lambda$ with respect to the decomposition $\mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda \setminus \Lambda_0}$ is the entropy of the reduced state, i.e.,

$$\mathcal{E}(\psi) = -\operatorname{Tr}[\rho_1 \log \rho_1], \text{ where } \rho_1 := \operatorname{Tr}_{\mathcal{H}_{\Lambda \setminus \Lambda_0}} |\psi\rangle\langle\psi|.$$

AN ENTANGLEMENT BOUND

- Fix $\Lambda_0 = [1, \ell] \subset \Lambda$, and consider the corresponding bipartition of the Hilbert space $\mathcal{H}_\Lambda = \mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda \setminus \Lambda_0}$.

THEOREM (AR/FISCHBACHER/STOLZ '19)

For any non-negative background field v, any $k \in \mathbb{N}$, and $\delta > 0$

$$\limsup_{\ell \to \infty} \limsup_{L \to \infty} \frac{\sup_\psi \mathcal{E}(\psi)}{\log \ell} \leq 2k - 1.$$

The supremum is taken over all $\psi \in Ran\left(\chi_{[0, E_{k+1} - \delta)}(H_\Lambda^{XXZ}(v))\right)$.

Beaud/Warzel '18 for $k = 1$.

Conjecture (disorder case): $\limsup_{\ell \to \infty} \limsup_{L \to \infty} \frac{1}{\log \ell} \mathbb{E}\left(\sup_\psi \mathcal{E}(\rho_\psi)\right) \leq k - 1.$

Thank you.