Quenched correlations in disordered harmonic oscillator systems

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The Harmonic Oscillators

The Hamiltonian

$$H_\Lambda = \sum_{x\in\Lambda} \left(p_x^2 + k_x q_x^2\right) + \sum_{\substack{\{x,y\}\subset\Lambda \\ |x-y|=1}} \lambda(q_x - q_y)^2$$

- $\Lambda := ([a_1, b_1] \times \ldots \times [a_d, b_d]) \cap \mathbb{Z}^d$ for integers $a_j < b_j$ for all j, and $d \geq 1$.
- For each $x \in \Lambda$, q_x and $p_x = -i\frac{\partial}{\partial q_x}$ are the position and momentum operators.
- The Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{L}^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R}^\Lambda)$.
- $\lambda \in (0, \infty)$ is the coupling parameter.
- $\{k_x\}_x$ are i.i.d. random variables with absolutely continuous distribution given by a bounded density ν supported in $[0, k_{\max}]$.

Known: Exponential decay of the position-momentum correlations at the:

- ground state
- thermal states.

Nachtergaele-Sims-Stolz (2012).

New Results:

1. Correlations at the energy eigenstates.
2. Correlations after a quantum quench.

Diagonalizing H_Λ

- Define the *annihilation* and *creation* operators

$$a_x = \frac{1}{\sqrt{2}}(q_x + ip_x), \quad a_x^* = \frac{1}{\sqrt{2}}(q_x - ip_x), \text{ for } x \in \Lambda.$$

They satisfy the *Canonical Commutations Relations* (CCR)

$$[a_x, a_y] = [a_x^*, a_y^*] = 0, \text{ and } [a_x, a_y^*] = \delta_{x,y}\mathbb{1}, \text{ for all } x, y \in \Lambda.$$

- The harmonic Hamiltonian can be writen as

$$H_\Lambda = \frac{1}{2}\begin{pmatrix} a^T & (a^*)^T \end{pmatrix}\begin{pmatrix} h_\Lambda - \mathbb{1} & h_\Lambda + \mathbb{1} \\ h_\Lambda + \mathbb{1} & h_\Lambda - \mathbb{1} \end{pmatrix}\begin{pmatrix} a \\ a^* \end{pmatrix}.$$

- h_Λ is the finite volume Anderson model on $\ell^2(\Lambda)$, i.e.,

$$h_\Lambda = \lambda h_{0,\Lambda} + k$$

where $h_{0,\Lambda}$ is the discrete Laplacian operator, and $k := \mathrm{diag}\{k_x,\, x \in \Lambda\}$.

Diagonalizing H_Λ

- $\sigma(h_\Lambda) \subset \left[\min_{x\in\Lambda} k_x, 4d\lambda + k_{\max}\right]$.
- h_Λ is diagonalizable in terms of an orthogonal matrix $\mathcal{O}$ and a diagonal operator $\gamma^2 = \operatorname{diag}\{\gamma_x^2, x \in \Lambda\}$, i.e., $h_\Lambda = \mathcal{O}\gamma^2\mathcal{O}^T$.
- Define the bosons $\{b_k, k = 1, \ldots, |\Lambda|\}$ using the Bogoliubov tranformation

$$b = U\, a + V\, a^*$$

where

$$U = \frac{1}{2}(\gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}})\mathcal{O}^T, \quad V = \frac{1}{2}(\gamma^{\frac{1}{2}} - \gamma^{-\frac{1}{2}})\mathcal{O}^T.$$

- H_Λ can be written as a free boson system

$$H_\Lambda = \sum_{k=1}^{|\Lambda|} \gamma_k (2b_k^* b_k + \mathbb{1})$$

A free boson system

$$H_\Lambda = \sum_{k=1}^{|\Lambda|} \gamma_k (2b_k^* b_k + \mathbb{1}) \longleftarrow \text{Free boson system.}$$

- The operators b_k satisfy the CCR
- $\{\gamma_k > 0, k = 1, \ldots, |\Lambda|\}$ are the eigenvalues of $h_\Lambda^{\frac{1}{2}}$.
- The eigen-pair of H_Λ associated with $\alpha = (\alpha_1, \ldots, \alpha_{|\Lambda|}) \in \mathbb{N}_0^{|\Lambda|}$ is (ψ_α, E_α),

$$\psi_\alpha = \prod_{j=1}^{|\Lambda|} \frac{1}{\sqrt{\alpha_j!}} (b_j^*)^{\alpha_j} |\mathsf{vac}_b\rangle, \quad E_\alpha = \sum_{j=1}^{|\Lambda|} (2\alpha_j + 1)\gamma_j$$

Assumption: The eigencorrelator localization

There exist constants $C < \infty$ and $\eta > 0$ and $0 < s \leq 1$, independent of Λ, such that

$$\mathbb{E}\left(\sup_{|g|\leq 1} |\langle \delta_x, h_\Lambda^{-\frac{1}{2}} g(h) \delta_y \rangle|^s\right) < Ce^{-\eta|x-y|},$$

for all $x, y \in \Lambda$.

Satisfied for

- $d \geq 1$: large disordered case with $s = 1$.
- $d = 1$: any distribution density ν with $s = \frac{1}{2}$.

- For any observable A and state ρ, $\langle A \rangle_\rho := \mathrm{Tr}[A\rho]$.
- The position-position dynamical correlation at state ρ is

$$\langle \tau_t(p_x) p_y \rangle_\rho - \langle \tau_t(p_x) \rangle_\rho \langle p_y \rangle_\rho,\ x, y \in \Lambda.$$

- $\tau_t(q_x) = e^{itH_\Lambda} q_x e^{-itH_\Lambda}$.
- We will considers states ρ such that $\langle \tau_t(p_x) \rangle_\rho = \langle \tau_t(q_x) \rangle_\rho = 0$ for all $x \in \Lambda$ and $t \geq 0$.
- Define the positions-momenta correlation matrix

$$\Gamma_\rho(t) := \begin{pmatrix} \langle \tau_t(q) q^T \rangle_\rho & \langle \tau_t(q) p^T \rangle_\rho \\ \langle \tau_t(p) q^T \rangle_\rho & \langle \tau_t(p) p^T \rangle_\rho \end{pmatrix}$$

- Let

$$(\Gamma_\rho(t))_{xy} = \begin{pmatrix} \langle \tau_t(q_x) q_y \rangle_\rho & \langle \tau_t(q_x) p_y \rangle_\rho \\ \langle \tau_t(p_x) q_y \rangle_\rho & \langle \tau_t(p_x) p_y \rangle_\rho \end{pmatrix}$$

Eigenstates Correlations

Recall The eigencorrelator localization assumption: There exist constants $C < \infty$ and $\eta > 0$ and $0 < s \leq 1$, independent of Λ, such that

$$\mathbb{E}\left(\sup_{|g|\leq 1} |\langle \delta_x, h^{-\frac{1}{2}} g(h)\delta_y\rangle|^s\right) < Ce^{-\eta|x-y|}, \text{ for all } x, y \in \Lambda. \tag{1}$$

Theorem

Under the eigencorrelator localization assumption (above),

$$\mathbb{E}\left(\sup_{t} \|(\Gamma_{\rho_\alpha}(t))_{xy}\|^s\right) \leq CC'(1+\|\alpha\|_\infty)^{1+s} e^{-\eta|x-y|}$$

for all finite rectangular boxes $\Lambda \subset \mathbb{Z}^d$, $x, y \in \Lambda$ and $\alpha \in \mathbb{N}_0^{|\Lambda|}$. Here C, η and s are as in (1) and $C' < \infty$ depends on d, λ, s and $k_{\max}$, but is independent of Λ.

- Decompose Λ into M disjoint rectangular sub-boxes, $\Lambda = \biguplus_{\ell=1}^{M} \Lambda_\ell$.
- For each ℓ, let H_{Λ_ℓ} be the restriction of H_Λ to Λ_ℓ.
- Let $H_{0,\Lambda}$ be the Hamiltonian of the non-interacting system on $\mathcal{H}_\Lambda$,

$$H_{0,\Lambda} = \sum_{\ell=1}^{M} H_{\Lambda_\ell} \otimes \mathbb{1}_{\Lambda \setminus \Lambda_\ell}.$$

- Let $\{\rho_\ell, \ell = 1, \ldots, M\}$ be states acting on $\mathcal{L}^2(\mathbb{R}^{\Lambda_\ell})$, and let

$$\rho := \bigotimes_{\ell=1}^{M} \rho_\ell.$$

- We study the positions-momenta correlations at the state

$$\rho_t = e^{-itH_\Lambda} \rho e^{itH_\Lambda}.$$

- Recall that $\rho_t = e^{-itH_\Lambda} \left(\bigotimes_{\ell=1}^{M} \rho_\ell \right) e^{itH_\Lambda}$.
- For every $x, y \in \Lambda$, let

$$(\Gamma_{\rho_t})_{xy} := (\Gamma_{\rho_t}(0))_{xy} = \begin{pmatrix} \langle q_x q_y \rangle_{\rho_t} & \langle q_x p_y \rangle_{\rho_t} \\ \langle p_x q_y \rangle_{\rho_t} & \langle p_x p_y \rangle_{\rho_t} \end{pmatrix}$$

- For all $x, y \in \Lambda_\ell$

$$(\Gamma_{\rho_\ell})_{x,y} := (\Gamma_{\rho_\ell}(0))_{xy} = \begin{pmatrix} \langle q_x q_y \rangle_{\rho_\ell} & \langle q_x p_y \rangle_{\rho_\ell} \\ \langle p_x q_y \rangle_{\rho_\ell} & \langle p_x p_y \rangle_{\rho_\ell} \end{pmatrix}$$

Recall: The Eigencorrelator Localization Assumption,

$$\mathbb{E}\left(\sup_{|g|\le 1} |\langle \delta_x, h_\Lambda^{-\frac{1}{2}} g(h)\delta_y\rangle|^s\right) < Ce^{-\eta|x-y|}, \text{ for all } x,y\in\Lambda. \tag{2}$$

Theorem

Under the assumption given above. Suppose that, for some $C' < \infty$, and $\eta' > 0$,

$$\mathbb{E}\left(\|(\Gamma_{\rho_\ell})_{xy}\|^s\right) \le C' e^{-\eta'|x-y|} \tag{3}$$

for all ℓ and all $x,y\in\Lambda_\ell$, where $0 < s \le 1$ is as in (2).
Then, for η from (2), $\tilde\eta := \frac{1}{6}\min\{\eta,\eta'\}$ and $\rho = \bigotimes_\ell \rho_\ell$, there exists a constant $C'' < \infty$ such that

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|(\Gamma_{\rho_t})_{xy}\|^{\frac{s}{3}}\right) \le (C')^{1/3} C'' e^{-\tilde\eta|x-y|}$$

for all $x,y\in\Lambda$. Here C' is the constant from (3) and C'' depends on d, λ, s, $k_{\max}$ and $\tilde\eta$, but is independent of Λ and the number of subregions M.

- Thermal States:

$$\rho_\beta = \frac{e^{-\beta H_\Lambda}}{\mathrm{Tr}[e^{-\beta H_\Lambda}]}, \text{ for } \beta \in (0,\infty).$$

Theorem

For a rectangular box $\Lambda \subset \mathbb{Z}^d$ and $\beta \in (0,\infty)$, let $\Gamma_{\rho_\beta} = \Gamma_{\rho_\beta}(0)$ their static position-momentum correlation matrices.
There exist $C < \infty$ and $\mu > 0$, dependent on d, λ and the distribution of the random variables k_x, but independent of Λ and β, such that

$$\mathbb{E}\left(\|(\Gamma_{\rho_\beta})_{xy}\|^{\frac{1}{2}}\right) \leq C \max\left\{1, \frac{1}{\beta}\right\}\ e^{-\mu|x-y|}$$

for all $x, y \in \Lambda$.

- Consider the thermal states of H_{Λ_ℓ} with inverse temperatures β_ℓ, $\ell = 1, \dots, M$, i.e.,

$$\rho_{\ell,\beta_\ell} = \frac{e^{-\beta_\ell H_{\Lambda_\ell}}}{\mathrm{Tr}[e^{-\beta_\ell H_{\Lambda_\ell}}]}.$$

- The product state

$$\rho_{\beta_1,\dots,\beta_M} := \bigotimes_{\ell=1}^{M} \rho_{\ell,\beta_\ell}.$$

- The Schrödinger evolution

$$(\rho_{\beta_1,\dots,\beta_M})_t = e^{-itH_\Lambda}(\rho_{\beta_1,\dots,\beta_M})e^{itH_\Lambda}.$$

- We assume the eigencorrelator localization with $s = 1/2$.

Result:

$$\boxed{\mathbb{E}\left(\sup_t \|(\Gamma_{(\rho_{\beta_1,\dots,\beta_M})_t})_{xy}\|^{\frac{1}{6}}\right) \leq C' \max\left\{1, \beta^{-1/3}\right\} e^{-\tilde{\eta}|x-y|}}$$

Correlations After a Quantum Quench

Corollaries: Energy Eigenstates

- For $\ell = 1, \dots, M$, let $\alpha_\ell \in \mathbb{N}_0^{|\Lambda_\ell|}$, and ρ_{α_ℓ} be the corresponding "local" eigenstate of H_{Λ_ℓ}.
- Let N be the highest mode, $\|\alpha_\ell\|_\infty \le N$ for all $\ell = 1, \dots, M$.
- The product state

$$\rho_\alpha = \bigotimes_{\ell=1}^{M} \rho_{\alpha_\ell}.$$

- The time evolution

$$(\rho_\alpha)_t := e^{-itH_\Lambda} \rho_\alpha e^{itH_\Lambda}.$$

- We assume the eigencorralator localization with $s = \frac{1}{2}$.

Result:

$$\mathbb{E}\left(\sup_t \|(\Gamma_{(\rho_\alpha)_t})_{xy}\|^{\frac{1}{6}} \right) \le \tilde{C}(1+N)^{\frac{1}{2}} e^{-\frac{\eta}{6}|x-y|}$$

for all $x, y \in \Lambda$.

- Fix $\beta > 0$ and $N < \infty$.
- Consider the local states ρ_ℓ, $\ell = 1, \dots, M$, where each ρ_ℓ is one of the following:
  - a thermal state with inverse temperature $\beta_\ell \in (\beta, \infty)$.
  - an eigenstate with occupation number vector α_ℓ with $\|\alpha_\ell\|_\infty \leq N$.
- Let $\rho = \bigotimes_{\ell=1}^{M} \rho_\ell$ and $\rho_t = e^{-itH_\Lambda} \left(\bigotimes_{\ell=1}^{M} \rho_\ell \right) e^{-itH_\Lambda}$
- We assume the eigencorrelator localization with $s = 1/2$.

Result:

$$\mathbb{E}\left(\sup_t \|(\Gamma_{\rho_t})_{xy}\|^{\frac{1}{6}} \right) \leq C \max\left\{ (1+N)^{\frac{3}{2}}, \frac{1}{\beta} \right\}^{\frac{1}{3}} e^{-\tilde{\eta}|x-y|}$$

for all $x, y \in \Lambda$.

- If the $M = |\Lambda|$.
- $\{H_{\{x\}}, x \in \Lambda\}$ with $H_{\{x\}} = p_x^2 + k_x q_x^2$.
- Let $\{n_x, x \in \Lambda\}$ be the set of occupation numbers in sites $x \in \Lambda$.
- Let $N = \max_x n_x$, i.e., the maximum occupation number.
- The eigenstates are

$$\phi_{n_x}(q_x) = \text{Const.}\, H_{n_x}(\sqrt[4]{k_x} q_x) e^{-\frac{\sqrt{k_x}}{2} q_x^2}, \text{ for } x \in \Lambda.$$

- Let $\rho = \bigotimes_{x \in \Lambda} \rho_x$ and $\rho_t = e^{-itH_\Lambda} \left(\bigotimes_{x \in \Lambda} \rho_x \right) e^{itH_\Lambda}$.
- We assume eigencorrelator localization with $s = 1/2$.

Result:

$$\mathbb{E}\left(\sup_t \|(\Gamma_{\rho_t})_{xy}\|^{\frac{1}{6}} \right) \le C(1 + 2N)^{\frac{1}{6}} e^{-\frac{\eta}{6}|x-y|}$$

for all $x, y \in \Lambda$.

Thank you.