Entanglement and transport in two disordered quantum many-body "toy" systems

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Overview

- The Disordered XY Chain.
  - Dynamical entanglement.
  - The transport of energy and particles.

- The Disordered Harmonic Oscillators System.
  - Dynamical correlations in the eigenstates.
  - Correlations in initially product states.

$$H = -\sum_{j=1}^{n-1} \mu_j [(1+\gamma_j)\sigma_j^x \sigma_{j+1}^x + (1-\gamma_j)\sigma_j^y \sigma_{j+1}^y] - \sum_{j=1}^{n} \nu_j \sigma_j^z$$

- $\Lambda = [1, n]$, Λ_0 a block of spins (subinterval of Λ).
- The Hilbert space: $\mathcal{H} := \bigotimes_{x \in \Lambda} \mathcal{H}_x = (\mathbb{C}^2)^{\otimes n}, \quad \dim \mathcal{H} = 2^n.$
- μ_j, γ_j and ν_j are i.i.d.

$\downarrow$ **Jordan-Wigner** $\downarrow$

$$H = \mathcal{C}^* M \mathcal{C},\ \mathcal{C} := (c_1, c_1^*, c_2, c_2^*, \dots, c_n, c_n^*)^t.$$

M is the block Jacobi matrix

$$M := \begin{pmatrix} -\nu_1 \sigma^z & \mu_1 S(\gamma_1) & & \\ \mu_1 S(\gamma_1)^t & \ddots & \ddots & \\ & \ddots & \ddots & \mu_{n-1} S(\gamma_{n-1}) \\ & & \mu_{n-1} S(\gamma_{n-1})^t & -\nu_n \sigma^z \end{pmatrix},$$

$$S(\gamma) = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}, \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Assumptions:

- The XY chain H has almost sure simple spectrum.
- M satisfies eigencorrelator localization, i.e
  $\mathbb{E}\left(\sup_{|g|\leq 1}\|g(M)_{jk}\|\right) \leq C_0(1+|j-k|)^{-\beta}$, for some $\beta > 6$.

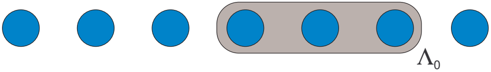
Applications:

$\mu_j = \mu$, $\gamma_j = \gamma$ for all $j \in \mathbb{N}$.
ν_j are i.i.d from an absolutely continuous, compactly supported distribution.

- Isotropic case ($\gamma = 0$): $M \longrightarrow$ Anderson Model.
- Anisotropic case ($\gamma \neq 0$):
  - Large disorder case. **Elgart/Shamis/Sodin (2012).**
  - Uniform spectral gap for M around zero. **Chapman /Stolz (2014).**

Dynamical Entanglement

The Entanglement Entropy and the Entanglement of Formation



Fix $\Lambda_0 \subseteq \Lambda$, consider the decomposition:

$$\mathcal{H} = \mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda\setminus\Lambda_0}, \text{ where } \mathcal{H}_{\Lambda_0} = \bigotimes_{x\in\Lambda_0} \mathcal{H}_x, \quad \mathcal{H}_{\Lambda\setminus\Lambda_0} = \bigotimes_{x\in\Lambda\setminus\Lambda_0} \mathcal{H}_x. \qquad (1)$$

Let ρ be a pure state in $\mathcal{B}(\mathcal{H})$, then

$$\boxed{\mathcal{E}(\rho) = -\operatorname{Tr}\left[\rho^1 \log \rho^1\right], \quad \text{where} \quad \rho^1 = \operatorname{Tr}_{\mathcal{H}_2} \rho.}$$

For any (mixed) state $\rho \in \mathcal{B}(\mathcal{H})$, then

$$\boxed{E_f(\rho) = \inf_{p_k,\psi_k} \sum_k p_k \mathcal{E}\left(|\psi_k\rangle\langle\psi_k|\right).}$$

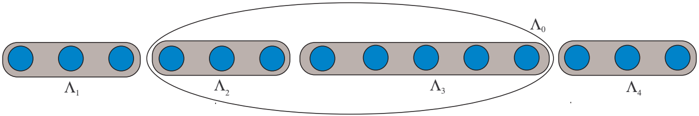
Dynamical Entanglement

Motivation Question

- For $1 \leq \ell \leq n$, let $H_{[1,\ell]}$ and $H_{[\ell+1,n]}$ be the restrictions of H to the corresponding interval.
- Let $\rho^{(1)}$ and $\rho^{(2)}$ be any eigenstates states of $H_{[1,\ell]}$ and $H_{[\ell+1,n]}$, respectively.
- We study $\rho_t := e^{-itH}\left(\rho^{(1)} \otimes \rho^{(2)}\right)e^{itH}$.
- ρ_t is an entangled state with respect to $\mathcal{H}_{[1,\ell]} \otimes \mathcal{H}_{[\ell+1,n]}$.

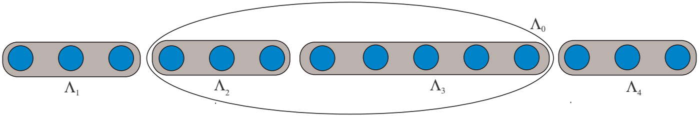
Question:

What can we say about the entanglement of ρ_t?



In general

- Decompose Λ into disjoint intervals $\Lambda_1, \Lambda_2, \ldots, \Lambda_m$.
- H_{Λ_k} is the restriction of H to Λ_k.
- ψ_k is an eigenfunction of H_{Λ_k}, and $\rho_k = |\psi_k\rangle\langle\psi_k|$.
- Define $\rho = \bigotimes_{k=1}^{m} \rho_k$, and its dynamics $\rho_t = e^{-itH}\rho e^{itH}$.



Theorem

There exists $C < \infty$ *such that*

$$\mathbb{E}\left(\sup_{t,\{\psi_k\}_{k=1,2,\ldots,m}} \mathcal{E}(\rho_t)\right) \leq C$$

for all n, m, *any choice of the interval* $\Lambda_0 \subset \Lambda$ *and all decompositions* $\Lambda_1, \ldots, \Lambda_m$ *of* $\Lambda = [1, n]$.

Dynamical Entanglement: Corollaries

1. Let ρ_β be the tensor product of local thermal states, then

$$\mathbb{E}\left(\sup_{t,\beta} E_f\left((\rho_\beta)_t\right)\right) \leq C.$$

2. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \{\uparrow, \downarrow\}^n$, the up-down configuration associated with α is given by: $e_\alpha = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \dots \otimes e_{\alpha_n}$,

$$\mathbb{E}\left(\sup_{\alpha} \mathcal{E}(e^{-itH}|e_\alpha\rangle\langle e_\alpha|e^{itH})\right) < C.$$

3. Let ψ be an eigenfunction of the full XY chain H.

$$\mathbb{E}\left(\sup_{\psi} \mathcal{E}(|\psi\rangle\langle\psi|)\right) < C.$$

Pastur/Slavin (2014). **AR/Stolz** (2015).

4. Let ρ_β be a thermal state of the full XY chain H.

$$\mathbb{E}\left(\sup_{\beta} E_f(\rho_\beta)\right) < C.$$

An Isotropic XY Chain in Random Transversal Magnetic Field

$$H_{\mathsf{iso}} = -\sum_{j=1}^{n-1} [\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y] - \sum_{j=1}^{n} \nu_j \sigma_j^z$$

$\downarrow$ **Jordan-Wigner** $\downarrow$

$H_{\mathsf{iso}} = c^* A c + \left(\sum_j \nu_j\right) \mathbb{1}$, where $c := (c_1, c_2, \ldots, c_n)^t$.

$$A := \begin{pmatrix} -\nu_1 & \mu & & \\ \mu & \ddots & \ddots & \\ & \ddots & \ddots & \mu \\ & & \mu & -\nu_n \end{pmatrix}, \quad \mathbb{E}\left(\sup_{|g|\leq 1} |\langle e_j, g(A) e_k \rangle| \right) \leq C e^{-\eta|j-k|}.$$

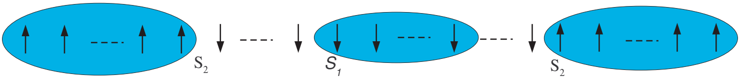
Particle Number Operator

$$\mathcal{N} := \sum_{j\in\Lambda} |e_\uparrow\rangle\langle e_\uparrow|_j \text{ and } \mathcal{N}_S := \sum_{j\in S} |e_\uparrow\rangle\langle e_\uparrow|_j.$$

- $\mathcal{N} e_\alpha = k e_\alpha$, where $k = |\{j \ : \alpha_j = \uparrow\}|$.
- Let $\rho = |e_\alpha\rangle\langle e_\alpha|$ then $\langle \mathcal{N} \rangle_\rho := \operatorname{Tr} \mathcal{N} \rho = k$ is the expected number of up-spins.
- $[H, \mathcal{N}] = 0 \Rightarrow$ The number of up-spins is conserved in time.
- $\rho_t = e^{-itH_{\text{iso}}} \rho e^{itH_{\text{iso}}}$ is the time evolution of ρ.
- $\langle \mathcal{N}_S \rangle_{\rho_t}$ is the expected number of up-spins in S at time t.

Particle Number/Energy Transport

Results



- Fix Fix $S_1 = [a, b] \subset \Lambda$ and $S_2 \subset \Lambda \setminus S_1$.
- Initial state: $\rho = \bigotimes_{j=1}^{n} \begin{pmatrix} \eta_j & 0 \\ 0 & 1-\eta_j \end{pmatrix}$, with $\eta_j = 0$ for all $j \notin S_2$.

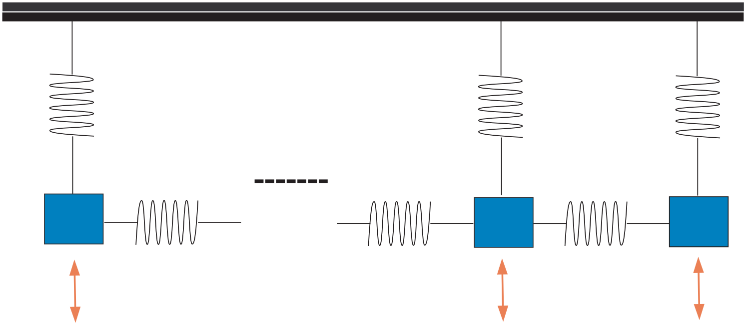
$$\mathbb{E}\left(\sup_t \langle \mathcal{N}_{S_1} \rangle_{\rho_t}\right) \leq \frac{4C}{(1+e^{-\eta})^2} e^{-\eta dist(S_1,S_2)}$$

Similar results for disordered Tonks-Girardeau gas, **Seiringer/Warzel** (2016).

$$\mathbb{E}\left(\sup_t \left|\langle H_{S_1} \rangle_{\rho_t} - \langle H_{S_1} \rangle_{\rho}\right|\right) \leq \frac{4CD}{(1+e^{-\eta})^2} e^{-\eta dist(S_1,S_2)},$$

where $D = \sup_n \|A_n\|$.

The Harmonic Oscillators



The Harmonic Oscillators

The Hamiltonian

$$H = \sum_{x \in \Lambda} \left(\frac{1}{2m} p_x^2 + \frac{k_x}{2} q_x^2 \right) + \sum_{\substack{\{x,y\} \in \Lambda \\ |x-y|=1}} \lambda (q_x - q_y)^2$$

- $\Lambda := [-L, L]^d \cap \mathbb{Z}^d$ where $L \geq 1$ and $d \geq 1$.
- q_x and $p_x = -i \frac{\partial}{\partial q_x}$ are the position and momentum operators.
- The Hilbert space $\mathcal{H} = \bigotimes_{x \in \Lambda} \mathcal{L}^2(\mathbb{R}, dq_x)$.
- $m, \lambda \in (0, \infty)$.
- $\{k_x\}_x$ are i.i.d. random variables with absolutely continuous distribution given by a bounded density ρ supported in $[0, k_{max}]$.

$$H = \sum_{k=1}^{|\Lambda|} \gamma_k (2B_k^* B_k + \mathbb{1}) \longleftarrow \text{Free boson system.}$$

- The operators B_k satisfy the CCR

$$[B_j, B_k] = [B_j^*, B_k^*] = 0, \quad [B_j, B_k^*] = \delta_{j,k}\mathbb{1} \quad \text{for all } j,k \in \{1, \dots, |\Lambda|\}.$$

- $\{\gamma_k\}_k$ are the eigenvalues of $h^{\frac{1}{2}}$ where

$$\langle \delta_x, h\delta_y \rangle = \begin{cases} \frac{k_x}{2} + 2d\lambda, & \text{if } x = y, \\ -\lambda, & \text{if } |x-y| = 1, \\ 0, & \text{else.} \end{cases}$$

Assumption:

There exist constants $C < \infty$ and $\eta > 0$ such that

$$\mathbb{E}\left(\sup_{|g|\leq 1} |\langle \delta_x, h^{\frac{\alpha}{2}} g(h)\delta_y \rangle|\right) < Ce^{-\eta|x-y|}, \text{ for } \alpha \in \{0, 1, -1\},$$

for all $x, y \in \Lambda$.

Satisfied for

- $d = 1$.
- $d > 1$ in the large disorder case.

The Harmonic Oscillators

Eigenstates

$$H = \sum_{k=1}^{|\Lambda|} \gamma_k (2B_k^* B_k + \mathbb{1}).$$

- There is a unique vacuum Ω_b (the ground state of H).
- The eigen-pair of H associated with $\alpha = (\alpha_1, \ldots, \alpha_{|\Lambda|}) \in \mathbb{N}_0^{|\Lambda|}$ is (ψ_α, E_α),

$$\psi_\alpha = \prod_{j=1}^{|\Lambda|} \frac{1}{\sqrt{\alpha_j!}} (B_j^*)^{\alpha_j} \Omega_b, \quad E_\alpha = \sum_j (2\alpha_j + 1)\gamma_j$$

- For any α, the corresponding eigenstate is $\rho_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$.

Let $C_\alpha(A,B,t) := \langle \tau_t(A)B\rangle_{\rho_\alpha} - \langle A\rangle_{\rho_\alpha}\langle B\rangle_{\rho_\alpha}$, where $\tau_t(A) = e^{itH}Ae^{-itH}$.

In the following Theorem: $A \in \{q_x, p_x\}$, $B \in \{q_y, p_y\}$.

Theorem

For any $x, y \in \Lambda$ and $\alpha \in \ell^\infty(\mathbb{N}_0^{|\Lambda|})$, there exist constants $C < \infty$ and $\eta > 0$ such that

$$\mathbb{E}\left(\sup_t |C_\alpha(A,B,t)|\right) < C(1+\|\alpha\|_\infty)^2 e^{-\eta|x-y|}.$$

- Define $a_x = \frac{1}{\sqrt{2}}(q_x + ip_x)$, and $a_x^* = \frac{1}{\sqrt{2}}(q_x - ip_x)$.
- For $f : \Lambda \to \mathbb{C}$, the Weyl operator is defined as

$$W(f) = \exp\left(\frac{i}{\sqrt{2}}(a(f) + a^*(f))\right), \text{ where } a(f) = \sum_{x\in\Lambda} \overline{f(x)} a_x, \quad a^* = \sum_{x\in\Lambda} f(x) a_x^*.$$

- Let $C_\alpha(f, g, t) := \langle \tau_t(W(f))W(g)\rangle_{\rho_\alpha} - \langle W(f)\rangle_{\rho_\alpha} \langle W(g)\rangle_{\rho_\alpha}$.

THEOREM

For any excitation vector $\alpha \in \ell^\infty(\mathbb{N}_0^{|\Lambda|})$ with $\|\alpha\|_\infty = N$, and any vectors $f, g \in \ell^2(\Lambda)$, there exist constants $\eta > 0$ and $C_N < \infty$ such that

$$\mathbb{E}\left(\sup_t |C_\alpha(f, g, t)|\right) \le C_N \sum_{x,y} |f(x)|^{\frac{1}{2N}} |g(y)|^{\frac{1}{2N}} \; e^{-\frac{\eta}{2N}|x-y|}$$

The Harmonic Oscillators

Quenched Correlations

- Decompose $\Lambda = \Lambda_1 \uplus \Lambda_2$.
- Let H_{Λ_1} and H_{Λ_2} be the restrictions of H to Λ_1 and Λ_2, respectively.
- Let $\rho^{(1)}$ and $\rho^{(2)}$ be any eigenstate/thermal states of H_{Λ_1} and H_{Λ_2}, respectively.
- $\rho_t := e^{-itH}\left(\rho^{(1)} \otimes \rho^{(2)}\right) e^{itH}$.
- We study the correlations $C_{\rho_t}(A,B) := \langle AB\rangle_{\rho_t} - \langle A\rangle_{\rho_t}\langle B\rangle_{\rho_t}$ where $A \in \{q_x, p_x\}$, $B \in \{q_y, p_y\}$.

Theorem

For any $x, y \in \Lambda$ and $\alpha \in \ell^\infty(\mathbb{N}_0^{|\Lambda_1|})$, there exist constants $C < \infty$ and $\eta > 0$ such that

$$\mathbb{E}\left(\sup_t |C_{\rho_t}(A,B)|^{\frac{1}{3}}\right) < C(1 + \|\alpha\|_\infty)^{\frac{2}{3}} e^{-\eta|x-y|}.$$

Thank you.