The Phase of Localized Excitations in Disordered Harmonic Oscillator Systems

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$$H_\Lambda = \sum_{x\in\Lambda} \left(p_x^2 + k_x q_x^2\right) + \sum_{\substack{\{x,y\}\subset\Lambda \\ |x-y|=1}} (q_x - q_y)^2$$

- $\Lambda := [-L, L]^d \cap \mathbb{Z}^d$ where $L \geq 1$ and $d \geq 1$.
- q_x and $p_x = -i\frac{\partial}{\partial q_x}$ are the position and momentum operators.
- The Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{L}^2(\mathbb{R}, dq_x)$.
- $|\cdot|$ denotes the 1-norm.
- $\{k_x\}_x$ are i.i.d. random variables with absolutely continuous distribution given by a bounded density supported in $[0, k_{max}]$.

The Effective One-Particle Hamiltonian

$$H_\Lambda = \sum_{x\in\Lambda} p_x^2 + q^T h_\Lambda q$$

- $q := (q_x)_{x\in\Lambda}$.
- $h_\Lambda = h_{0,\Lambda} + k$, where
  - $h_{0,\Lambda}$ is the negative discrete Laplacian over $\ell^2(\Lambda)$.
  - $k := \mathrm{diag}\{k_x,\ x \in \Lambda\}$.
- Some facts about h_Λ:
  - spec(h_Λ) is almost surely simple.
  - spec$(h_\Lambda) \subset \left[\min_{x\in\Lambda} k_x, 4\nu + k_{\max}\right]$.
- h_Λ is almost surely positive, and $\|h_\Lambda\| \le 4\nu + k_{\max}$.
- $h_\Lambda^{-1/2}$ is not uniformly bounded in the volume of the system and the disorder.

H_Λ AS A FREE BOSON SYSTEM

- $h_\Lambda = \sum_{j=1}^{|\Lambda|} \gamma_j^2 |\phi_j\rangle\langle\phi_j|$.
- For $j = 1, \ldots, |\Lambda|$, define

$$b_j := \frac{1}{\sqrt{2}}(\gamma_j^{\frac{1}{2}} \phi_j^T q + i\gamma_j^{-\frac{1}{2}} \phi_j^T p)$$

- The operators $\{b_j\}_j$ satisfy the CCR.
- H_Λ can be written as a free boson system

$$H_\Lambda = \sum_{j=1}^{|\Lambda|} \gamma_j (2b_j^* b_j + \mathbb{1})$$

Eigenvalues and eigenfunctions of H_Λ

$$H_\Lambda = \sum_{j=1}^{|\Lambda|} \gamma_j (2b_j^* b_j + \mathbb{1}), \quad \gamma_j\text{'s are the eigenvalues of } h_\Lambda^{1/2}.$$

- There exists a unique vacuum ψ_0 of the b's, i.e., $b_j \psi_0 = 0$ for all $j = 1, \dots, |\Lambda|$.
- For every $\alpha = (\alpha_1, \dots, \alpha_{|\Lambda|}) \in \mathbb{N}_0^{|\Lambda|}$, The eigen-pair (ψ_α, E_α) is given as

$$\psi_\alpha = \prod_{j=1}^{|\Lambda|} \frac{1}{\sqrt{\alpha_j!}} (b_j^*)^{\alpha_j} \psi_0, \qquad E_\alpha = \sum_{j=1}^{|\Lambda|} \gamma_j (2\alpha_j + 1).$$

- The g.s. energy is $\sum_j \gamma_j = \operatorname{Tr} h_\Lambda^{1/2}$, and the gap above the g.s. is $2 \min_j \gamma_j$

Previous Localization Results

Previous localization results of H_Λ required that h_Λ is *fully localized at all energies*. i.e.,

$$\mathbb{E}\left(\sup_{|u|\leq 1} |\langle \delta_x, h_\Lambda^{-1/2} u(h_\Lambda)\delta_y\rangle|\right) \leq Ce^{-\mu|x-y|}$$

- Known: for $\nu = 1$, or k_x with sufficiently large disorder.
- Expected to be wrong for $\nu \geq 3$ and small disorder.

Known localization results:

Nachtergaele/Sims/Stolz '12,'13
AR/Sims/Stolz '17

- Zero-velocity Lieb-Robinson bound.
- Exponential clustering of the ground/thermal states.
- Area laws for the entanglement of ground/thermal states.
- Exponential clustering of eigenstates and after a quantum quench.

Given any dimension $\nu \geq 1$, $\exists \lambda_0 > 0$ and $C < \infty, \mu > 0$ (independent of L) such that

$$\mathbb{E}\left(\sup_{|u|\leq 1} |\langle \delta_x, h_\Lambda^{-1/2} u(h_\Lambda) \chi_{[0,\lambda_0]}(h_\Lambda) \delta_y \rangle|\right) \leq C e^{-\mu|x-y|}$$

for all $x, y \in \Lambda$.

Nachtergaele/Sims/Stolz '12

What is the corresponding localization regime of H_Λ?

Low-energy localization of h_Λ

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$$\mathbb{E}\left(\sup_{|u|\leq 1} |\langle \delta_x, h_\Lambda^{-1/2} u(h_\Lambda) \chi_{[0,\lambda_0]}(h_\Lambda) \delta_y \rangle| \right) \leq C e^{-\mu|x-y|}$$

for all $x, y \in \Lambda$.

Nachtergaele/Sims/Stolz '12

What is the corresponding localization regime of H_Λ?

The phase of localizaed excitations

- For fixed λ_0, let

$$S_{\lambda_0} := \{j \in \{1, \ldots, |\Lambda|\};\ \gamma_j^2 \in [0, \lambda_0]\}$$
$$\mathcal{I} := \left\{\alpha = (\alpha_1, \ldots, \alpha_{|\Lambda|}) \in \mathbb{N}_0^{|\Lambda|};\ \operatorname{supp}\alpha \subseteq S_{\lambda_0}\right\}$$

- The phase of localized excitations:

$$P_{\mathcal{I}} := P_{\mathcal{I}}(H_\Lambda) := \sum_{\alpha \in \mathcal{I}} |\psi_\alpha\rangle\langle\psi_\alpha|$$

Note: $P_{\mathcal{I}}$ is the spectral projection of H_Λ associated with the energies

$$E_\alpha = E_0 + \sum_{j;\ \gamma_j^2 \in [0,\lambda_0]} 2\gamma_j \alpha_j.$$

- **Weyl operators:**
  - For $f : \Lambda \to \mathbb{C}$, the associated Weyl operator is defines as

$$\mathcal{W}(f) := \bigotimes_{x\in\Lambda} \exp\left(i(\mathrm{Re}[f_x]q_x + \mathrm{Im}[f_x]p_x)\right).$$

  - Note that $\mathrm{supp}(f) = \textbf{supp}(\mathcal{W}(f))$.

- **The Heisenberg dynamics:**
  - $\tau_t(\mathcal{W}(f)) = \exp(itH_\Lambda)\mathcal{W}(f)\exp(-itH_\Lambda)$.
  - Note that $\tau_t(\mathcal{W}(f))_{\mathcal{I}} = \tau_t(\mathcal{W}(f)_{\mathcal{I}})$, where $\mathcal{W}(f)_{\mathcal{I}} := P_{\mathcal{I}}\mathcal{W}(f)P_{\mathcal{I}}$.

Lieb-Robinson Bounds

Theorem (Restricted Lieb-Robinson bound)

For any $f, g : \Lambda \to \mathbb{C}$,

$$\mathbb{E}\left(\sup_t \|[\tau_t(\mathcal{W}(f)_\mathcal{I}), \mathcal{W}(g)_\mathcal{I}]\|\right) \leq C(1+\lambda_0^{1/2})^2 \sum_{x,y\in\Lambda} |f(x)||g(x)|e^{-\mu|x-y|}$$

where C *and* μ *are the constants in the eigencorrelator localization bound.*

Note: in the case where f and g have disjoint supports,

$$\sum_{x,y\in\Lambda} |f(x)||g(x)|e^{-\mu|x-y|} \leq \text{Const. } e^{-\mu' d(\text{supp}(f),\text{supp}(g))}.$$

Note: Similar restricted LR version was established in **Elgart/Klein/Stolz ’17**

Quasi-Locality

- For $X \subset \Lambda$.
- Let $f : \Lambda \to \mathbb{C}$, such that $\operatorname{supp} f \subseteq X$.
- **supp**$(\mathcal{W}(f)) \subseteq X$ but **supp**$\tau_t(\mathcal{W}(f)) = \Lambda$.

Can $\tau_t(\mathcal{W}(f))$ be "approximated" by a local operator supported "near" X?

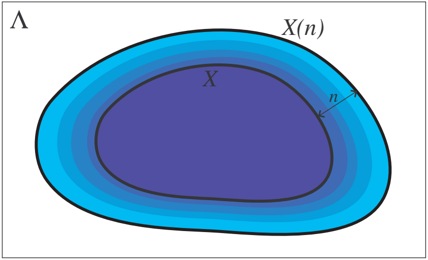

Figure: Define $X(n) := \{x \in \Lambda;\ d(x, X) \leq n\}$

Define the surface area of X

$$\partial X := \{x \in X, \text{there exists } y \in \Lambda \setminus X;\ |x-y| = 1\}.$$

Theorem

Let $X \subset \Lambda$ and $f : \Lambda \to \mathbb{C}$ satisfy $\mathrm{supp}(f) \subset X$. For any $n \in \mathbb{N}_0$ and $t > 0$, there is an operator $\widehat{W}_{t,n} \in \mathcal{B}(\mathcal{H}_\Lambda)$ supported on $X(n)$, such that for any $\kappa > 0$ we have

$$\mathbb{E}\left(\sup_{\alpha;\|\alpha\|_\infty \le \kappa} \sup_{t\in\mathbb{R}} \left\| \left(\tau_t(\mathcal{W}(f)) - \widehat{W}_{t,n}\right)_{\mathcal{I}} \psi_\alpha \right\|\right) \le \tilde{C}(1+\kappa)^{1/3}(1+\lambda_0^{1/2})^2|\partial X|\|f\|_\infty^{2/3} e^{-\mu n/3}.$$

Note: Here

$$\tilde{C} = 2^{5/3} C \left(\sum_{z\in\mathbb{Z}^\nu} e^{-\mu|z|/6}\right)^4. \qquad (1)$$

The (restricted) dynamical correlations of local Weyl operators is given by

$$C_\alpha^{\mathcal{I}}(f,g;t) := \langle \tau_t(\mathcal{W}(f))_{\mathcal{I}} \mathcal{W}(g)_{\mathcal{I}} \rangle_{\psi_\alpha} - \langle \mathcal{W}(f)_{\mathcal{I}} \rangle_{\psi_\alpha} \langle \mathcal{W}(g)_{\mathcal{I}} \rangle_{\psi_\alpha}$$

where for any observable A, $\langle A \rangle_{\psi_\alpha} := \langle \psi_\alpha, A\psi_\alpha \rangle$.

THEOREM

For $\kappa \in \mathbb{N}_0$, and any functions $f, g : \Lambda \to \mathbb{C}$,

$$\mathbb{E}\left(\sup_{\alpha;\ \|\alpha\|_\infty \le \kappa} \sup_{t\in\mathbb{R}} |C_\alpha^{\mathcal{I}}(f,g,t)| \right) \le 8C(1+\lambda_0^{1/2})^2 \Big(\sum_{x,y\in\Lambda} |f(x)g(y)| e^{-\mu|x-y|} \Big)^{\frac{1}{\kappa+1}}$$

where C and μ are the constants in the eigencorrelator localization.

Thank you.