Entanglement Dynamics for the Quantum Disordered XY Chain

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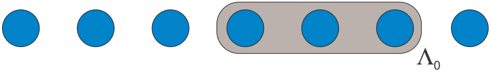
The Many-Body Hilbert Space

- $\Lambda = [1, n] := \{1, 2, \ldots, n\}$.
- For each vertex $x \in \Lambda$ we associate the Hilbert space $\mathcal{H}_x := \mathbb{C}^2$.
- The Hilbert space associated with the system is

$$\mathcal{H} := \bigotimes_{x \in \Lambda} \mathcal{H}_x = (\mathbb{C}^2)^{\otimes n}$$

- $\rho \in \mathcal{B}(\mathcal{H})$ is a **pure state** if $\rho \geq 0$, $\operatorname{Tr} \rho = 1$, and $\rho^2 = \rho$.
- There is a one to one correspondence between pure states and rank one projections.

The Bipartite Entanglements



Fix $\Lambda_0 \subseteq \Lambda$, consider the decomposition:

$$\mathcal{H} = \mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda\setminus\Lambda_0}, \text{ where } \mathcal{H}_{\Lambda_0} = \bigotimes_{x\in\Lambda_0} \mathcal{H}_x, \quad \mathcal{H}_{\Lambda\setminus\Lambda_0} = \bigotimes_{x\in\Lambda\setminus\Lambda_0} \mathcal{H}_x. \quad (1)$$

Let ρ be a pure state in $\mathcal{B}(\mathcal{H})$, then

- ρ **is separable:** if there exist pure states $\rho^{(1)} \in \mathcal{B}(\mathcal{H}_{\Lambda_0})$ and $\rho^{(2)} \in \mathcal{B}(\mathcal{H}_{\Lambda\setminus\Lambda_0})$, such that $\rho = \rho^{(1)} \otimes \rho^{(2)}$.
- ρ **is entangled:** if it is not separable.

Entanglement Entropy

The **Entanglement Entropy** of a pure state ρ with respect to the decomposition $\mathcal{H}_{\Lambda_0} \otimes \mathcal{H}_{\Lambda \setminus \Lambda_0}$ is defined as follows:

$$\mathcal{E}(\rho) = -\operatorname{Tr}\left[\rho_1 \log \rho_1\right], \quad \text{where} \quad \rho_1 = \operatorname{Tr}_{\mathcal{H}_2} \rho.$$

For any pure state $\rho \in \mathcal{B}(\mathcal{H})$:

- $\mathcal{E}(\rho) \geq 0$.
- $\mathcal{E}(\rho) = 0$ if and only if ρ is product state (Not Entangled).
- $\mathcal{E}(\rho) \leq (\log 2)|\Lambda_0|$ (volume scaling).

An XY Chain in Transversal Magnetic Field

$$H = -\sum_{j=1}^{n-1} \mu_j [(1+\gamma_j)\sigma_j^x \sigma_{j+1}^x + (1-\gamma_j)\sigma_j^y \sigma_{j+1}^y] - \sum_{j=1}^{n} \nu_j \sigma_j^z$$

- Hilbert space $\mathcal{H} = \left(\mathbb{C}^2\right)^{\otimes \Lambda}$.
- $\Lambda = [1, n]$, Λ_0 a block of spins (subinterval of Λ).
- μ_j, γ_j and ν_j are i.i.d.
- $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

A_j acts on the j^{th} component of the tensor product, i.e.

$$A_j = \mathbb{1}^{\otimes(j-1)} \otimes A \otimes \mathbb{1}^{\otimes(n-j)}$$

Jordan-Wigner Transform

$$H = -\sum_{j=1}^{n-1} \mu_j [(1+\gamma_j)\sigma_j^x \sigma_{j+1}^x + (1-\gamma_j)\sigma_j^y \sigma_{j+1}^y] - \sum_{j=1}^{n} \nu_j \sigma_j^z$$

- $a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$
- $\sigma_j^x = a_j + a_j^*,\ \sigma_j^y = i(a_j - a_j^*),$ and $\sigma_j^z = 2a_j^* a_j - \mathbb{1}.$
- $c_1 := a_1, \quad c_j := \sigma_1^z \sigma_2^z \ldots \sigma_{j-1}^z a_j,\ j > 1.$
- $H = \mathcal{C}^* M \mathcal{C}$, where $\mathcal{C} := (c_1, c_1^*, c_2, c_2^*, \ldots, c_n, c_n^*)^t$.

Effective One-Particle Hamiltonian

$$H = \mathcal{C}^* M \mathcal{C},\ \mathcal{C} := (c_1, c_1^*, c_2, c_2^*, \ldots, c_n, c_n^*)^t.$$

M is the block Jacobi matrix

$$M := \begin{pmatrix} -\nu_1 \sigma^z & \mu_1 S(\gamma_1) & & \\ \mu_1 S(\gamma_1)^t & \ddots & \ddots & \\ & \ddots & \ddots & \mu_{n-1} S(\gamma_{n-1}) \\ & & \mu_{n-1} S(\gamma_{n-1})^t & -\nu_n \sigma^z \end{pmatrix},$$

where $S(\gamma) = \sigma^z + i\gamma\sigma^y = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}$. Recall that: $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Motivation Question

- For $1 \leq \ell \leq n$, let $H_{[1,\ell]}$ and $H_{[\ell+1,n]}$ be the restrictions of H to the corresponding interval.
- Let $\rho^{(1)}$ and $\rho^{(2)}$ be any eigenstates of $H_{[1,\ell]}$ and $H_{[\ell+1,n]}$, respectively.
- We study the Schrödinger dynamics ρ_t of $\rho^{(1)} \otimes \rho^{(2)}$ with respect to the full chain:

$$\rho_t := e^{-itH} \left(\rho^{(1)} \otimes \rho^{(2)} \right) e^{itH}.$$

- Note that ρ_t is an Entangled state with respect to $\mathcal{H}_{[1,\ell]} \otimes \mathcal{H}_{[\ell+1,n]}$.

Question:

What can we say about the Entanglement Entropy of ρ_t?

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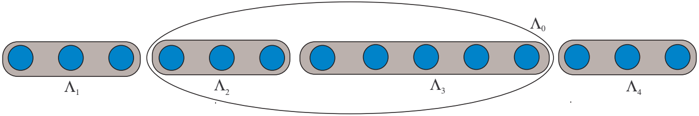
Problem Setting



In general

- Decompose Λ into disjoint intervals $\Lambda_1, \Lambda_2, \ldots, \Lambda_m$.
- H_{Λ_k} is the restriction of H to Λ_k.
- ψ_k is an eigenfunction of H_{Λ_k}, and $\rho_k = |\psi_k\rangle\langle\psi_k|$.
- Define $\rho = \bigotimes_{k=1}^{m} \rho_k$, and its dynamics $\rho_t = e^{-itH} \rho e^{itH}$.

Assumptions

Assumptions:

- The XY chain H has almost sure simple spectrum.
- M satisfies eigencorrelator localization, i.e $\mathbb{E}\left(\sup_{|g|\leq 1}\|g(M)_{jk}\|\right) \leq C_0(1+|j-k|)^{-\beta}$, for some $\beta > 6$.

Applications:

$\mu_j = \mu$, $\gamma_j = \gamma$ for all $j \in \mathbb{N}$.
ν_j are i.i.d from an absolutely continuous, compactly supported distribution.

- Isotropic case ($\gamma = 0$): $M \longrightarrow$ Anderson Model.
- Anisotropic case ($\gamma \neq 0$):
  - Large disorder case: Elgart, Shamis, and Sodin (2012).
  - Uniform Spectral gap for M around zero: Chapman and Stolz (2014).

An Area Law

Theorem

Then there exists $C < \infty$ *such that*

$$\mathbb{E}\left(\sup_{t,\{\psi_k\}_{k=1,2,\dots,m}} \mathcal{E}(\rho_t)\right) \leq C$$

for all n, m, *any choice of the interval* $\Lambda_0 \subset \Lambda$ *and all decompositions* $\Lambda_1, \dots, \Lambda_m$ *of* $\Lambda = [1, n]$.

Hamza/Sims/Stolz (2012).
Nachtergale/Sims/Stolz (2013).
Sims/Warzel (2016).

If $m = n$

- number of decompositions is n.
- eigenfunctions are up and down spins:

$$e_{\uparrow} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_{\downarrow} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{\uparrow, \downarrow\}^n$, define the up-down configuration associated with α:

$$e_{\alpha} = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \ldots \otimes e_{\alpha_n}$$

Result:

Eigencorrelator localization of $M \Rightarrow \mathbb{E}\left(\sup_{\alpha} \mathcal{E}(e^{-itH}|e_{\alpha}\rangle\langle e_{\alpha}|e^{itH})\right) \leq C$

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For $m = 1$ (No Decomposition)

Let ψ be an eigenfunction of the full XY chain H.

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Pasture/Slavin (2014).
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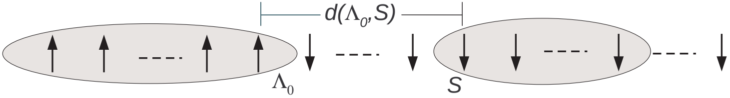
Particle Number Transport

$$H_0 = -\sum_{j=1}^{n-1}[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y] - \sum_{j=1}^{n}\nu_j\sigma_j^z$$

The particle number operator $\mathcal{N} := \sum_{j=1}^{n} |e_\uparrow\rangle\langle e_\uparrow|_j$.

- $\mathcal{N} e_\alpha = k e_\alpha$, where $k = \#\{j \ : \alpha_j = \uparrow\}$.
- Let $\rho = |e_\alpha\rangle\langle e_\alpha|$ then $\langle\mathcal{N}\rangle_\rho := \mathrm{Tr}\,\mathcal{N}\rho = k$.
- $[e^{itH_0}, \mathcal{N}] = 0 \Rightarrow$ the number of up-spins is conserved in time.
- $\mathcal{N}_S := \sum_{j\in S} |e_\uparrow\rangle\langle e_\uparrow|_j$ counts the number of up-spins in $S \subset \Lambda$.



- Fix $\Lambda_0 \subset \Lambda$ and $S \subset \Lambda \setminus \Lambda_0$.
- Initial state: $\rho = |\phi\rangle\langle\phi|$, where $\phi = (e_\uparrow)^{\otimes \Lambda_0} \otimes (e_\downarrow)^{\otimes \Lambda \setminus \Lambda_0}$

Theorem

For the isotropic XY chain, there exist constants $C, \eta < \infty$ such that

$$\mathbb{E}\left(\sup_t \langle \mathcal{N}_S \rangle_{\rho_t}\right) \leq C e^{-\eta d(\Lambda_0, S)}$$

Similar results for disordered Tonks-Girardeau Gas, **Seiringer and Warzel** (2016)

Thank you.