An Area Law for the Entanglement entropy of Eigen-States in a Disordered XY-chain

Houssam Abdul-Rahman, Günter Stolz

University of Alabama at Birmingham

March 2015

AMS Sectional Meeting
Michigan State University
East Lansing, MI

Related Results

- Pastur and Slavin (2014): "*Area law scaling for the entropy of disordered quasifree fermions*".
  Result: An area law for the entanglement entropy of the ground state after disorder averaging.

- Hamza, Sims, and Stolz (2012): "*Dynamical localization in disordered quantum spin systems*".
  Result: Dynamical localization of the effective one-particle Hamiltonian implies zero-velocity Lieb-Robinson bound.

- Nachtergaele, Sims, and Stolz (2013): "*An area law for the bipartite entanglement of disordered oscillators systems*".
  Result: Dynamical localization of the effective one-particle Hamiltonian implies an area law for logarithmic negativity of thermal and ground states after disorder averaging.

Anisotropic XY Spin Chain

$$H_n = -\sum_{j=1}^{n-1} \mu_j [(1+\gamma_j)\sigma_j^X \sigma_{j+1}^X + (1-\gamma_j)\sigma_j^Y \sigma_{j+1}^Y] - \sum_{j=1}^{n} \nu_j \sigma_j^Z$$

- Hilbert space $\mathcal{H} = \bigotimes_{j\in\Lambda} \mathbb{C}^2$, $\Lambda = \{1, 2, \ldots, n\}$.
- $\gamma_j \in [0, 1]$, $\mu_j > 0$.
- ν_j's are i.i.d. random variables.

Remark:

1. $\gamma_j = 0 \longrightarrow$ Isotropic XY chain.
2. $\gamma_j = 1 \longrightarrow$ Ising chain.

Anisotropic XY Spin Chain

$$H_n = -\sum_{j=1}^{n-1} \mu_j[(1+\gamma_j)\sigma_j^X \sigma_{j+1}^X + (1-\gamma_j)\sigma_j^Y \sigma_{j+1}^Y] - \sum_{j=1}^{n} \nu_j \sigma_j^Z$$

- Hilbert space $\mathcal{H} = \bigotimes_{j\in\Lambda} \mathbb{C}^2$, $\Lambda = \{1, 2, \ldots, n\}$.
- $\gamma_j \in [0, 1]$, $\mu_j > 0$.
- ν_j's are i.i.d. random variables.

Remark:

1. $\gamma_j = 0 \longrightarrow$ Isotropic XY chain.
2. $\gamma_j = 1 \longrightarrow$ Ising chain.

The Bipartite Entanglement Entropy

- We consider the decomposition $\mathcal{H}_1 \otimes \mathcal{H}_2$, where

$$\mathcal{H}_1 = \bigotimes_{j\in\Lambda_0} \mathbb{C}^2, \quad \mathcal{H}_2 = \bigotimes_{j\in\Lambda\setminus\Lambda_0} \mathbb{C}^2$$

  where $\Lambda_0 \subset \Lambda$ is a connected (subinterval).

- The *von Neumann Entanglement Entropy* of a state ρ with respect to the subsystem Λ_0 is defined as follows:

$$\mathcal{E}(\rho) = -\operatorname{Tr}\rho_1 \log \rho_1, \quad \text{where} \quad \rho_1 = \operatorname{Tr}_{\mathcal{H}_2} \rho.$$

Jordan-Wigner Transform

- $a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j, \quad a^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_j.$
- $\sigma_j^X = a_j + a_j^*$, $\sigma_j^Y = i(a_j - a_j^*)$, and $\sigma_j^Z = 2a_j^* a_j - \mathbb{1}$.
- $c_1 := a_1, \quad c_j := \sigma_1^Z \sigma_2^Z \dots \sigma_{j-1}^Z a_j,\ j > 1.$
- $H_n = \widetilde{\mathcal{C}}^* M_n \widetilde{\mathcal{C}}$, where $\widetilde{\mathcal{C}} := (c_1, c_1^*, c_2, c_2^*, \dots, c_n, c_n^*)^t$.

Effective One-Particle Hamiltonian

M_n is the block Jacobi matrix

$$M_n := \begin{pmatrix} -\nu_1\sigma^Z & \mu_1 S(\gamma_1) & & \\ \mu_1 S(\gamma_1)^t & \ddots & \ddots & \\ & \ddots & \ddots & \mu_{n-1}S(\gamma_{n-1}) \\ & & \mu_{n-1}S(\gamma_{n-1})^t & -\nu_n\sigma^Z \end{pmatrix},$$

where $S(\gamma) = \sigma^Z + i\gamma\sigma^Y = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}$.

An Area Law

Theorem (Main result)

Suppose that H_n has simple spectrum almost surely, M_n is bounded uniformly in n, and dynamically localized in the sense that there exists $C < \infty$ and $\beta > 2$ such that

$$\mathbb{E}\left(\sup_{|g|\leq 1} \left\| [g(M_n)]_{\mathbf{jk}} \right\| \right) \leq \frac{C}{1 + |\mathbf{j} - \mathbf{k}|^{\beta}},$$

for all $n \in \mathbb{N}$ and $1 \leq \mathbf{j}, \mathbf{k} \leq n$. Then there exists $0 < C_\beta < \infty$ independent of $|\Lambda|$, $|\Lambda_0|$, such that

$$\mathbb{E}\left(\sup_{\phi} \mathcal{E}(|\phi\rangle\langle\phi|) \right) < C_\beta,$$

where the sup is taken over all normalized eigenfunctions ϕ of H_n.

Applications

- The simplicity of $\sigma(H_n)$ holds generally if the μ_j's and the γ_j's are independent of ν_j's, and the ν_j's are i.i.d with absolutely continuous distribution.
- Isotropic XY chain ($\gamma = 0$), $M_n \longleftrightarrow$ Anderson Model.
- Strong Magnetic field:
  - Large disorder: $\nu_j \to \lambda\nu_j$, λ large. [Elgart, Shamis and Sodin 2012]
  - M_n gapped: $\nu_j \geq c > 2$ or $\nu_j \leq c < -2$. [Chapman and Stolz 2013]

What we mean by. . . ?

- **Fermionic operators:** $\{b_j, j = 1, 2 \ldots, n\}$ are called Fermionic operators in $\mathcal{B}(\mathcal{H})$, if they satisfy the "*Canonical Anti-Commutation Relations*",

$$\{b_j, b_k\} = \{b_j^*, b_k^*\} = 0 \text{ and } \{b_j, b_k^*\} = \delta_{j,k}\,\mathbb{1} \quad \text{for all } 1 \le j, k \le n.$$

- **Fermionic system:** (of b_j's) $\mathcal{B} := (b_1, b_2, \ldots, b_n, b_1^*, \ldots, b_n^*)^t$.

Some Tools

- The *Correlation matrix* of the Fermionic system $\mathcal{B}$ with respect to a state ρ is defined to be the $2n \times 2n$ matrix

$$\Gamma_\rho^{\mathcal{B}} := \langle \mathcal{B}\mathcal{B}^* \rangle_\rho, \qquad \longrightarrow = \left(\operatorname{Tr} b_j^{\#} b_k^{\#} \rho\right)_{j,k=1,2,\ldots,n, \# \in \{\emptyset, *\}}$$

- *Bogoliubov matrices* map Fermionic system to another Fermionic system.

- If state ρ *satisfies Wick's rule with respect to* $\mathcal{B}$ (quasi-free / Gaussian) then

$$\operatorname{Tr} \rho \log \rho = \operatorname{tr} \Gamma_\rho^{\mathcal{B}} \log \Gamma_\rho^{\mathcal{B}}.$$

- Wick's rule is invariant under Bogoliubov mappings.

Main Steps

1. Diagonalize $H_n \to$ free Fermion Hamiltonian.
2. Find the eigenfunctions ψ_α's of H_n, and the eigen-states $\rho_\alpha := |\psi_\alpha\rangle\langle\psi_\alpha|$.
3. Introduce the "*Local Jordan-Wigner operators*" in $\mathcal{B}(\mathcal{H}_1)$, (system $\mathcal{C}_1$).
4. Prove that

$$\mathcal{E}(\rho_\alpha) = -\operatorname{Tr} \rho_{\alpha,1} \log \rho_{\alpha,1} = -\operatorname{tr} \Gamma^{\mathcal{C}_1}_{\rho_\alpha,1} \log \Gamma^{\mathcal{C}_1}_{\rho_\alpha,1},$$

   where $\rho_{\alpha,1} := \operatorname{Tr}_{\mathcal{H}_2} \rho_\alpha$, (Wick's rule).
5. Prove the uniform area law... (some steps are a generalized version of the proof given by Pastur and Slavin (2014)).

Diagonalizing H_n

- $H_n = \mathcal{C}^* \widetilde{M}_n \mathcal{C}$.
- $\mathcal{C} = (c_1, c_2, \ldots, c_n, c_1^*, \ldots, c_n^*)^t$ Fermionic system.
- $\widetilde{M}_n = \begin{pmatrix} A_n & B_n \\ -B_n & -A_n \end{pmatrix}$, where

$$A_n = \begin{pmatrix} -\nu_1 & \mu_1 & & \\ \mu_1 & \ddots & \ddots & \\ & \ddots & \ddots & \mu_{n-1} \\ & & \mu_{n-1} & -\nu_n \end{pmatrix}, B_n = \begin{pmatrix} 0 & \gamma_1\mu_1 & & \\ -\gamma_1\mu_1 & \ddots & \ddots & \\ & \ddots & \ddots & \gamma_{n-1}\mu_{n-1} \\ & & -\gamma_{n-1}\mu_{n-1} & 0 \end{pmatrix}.$$

$$H_n = \mathcal{C}^* \widetilde{M}_n \mathcal{C}.$$

- $\widetilde{M}_n$ is diagonalizable via a Bogoliubov matrix W.
- $\widetilde{M}_n = W^t \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} W$, $\Lambda := \mathrm{diag}\{\lambda_j \geq 0,\ j = 1, 2, \ldots, n\}$.
- $H_n = \mathcal{B}^* \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} \mathcal{B}$, where $\mathcal{B} := W\mathcal{C}$ is a Fermionic system.
- $H_n = 2 \sum_{j=1}^{n} \lambda_j b_j^* b_j - \sum_{j=1}^{n} \lambda_j \mathbb{1}$.

$$H_n = 2\sum_{j=1}^n \lambda_j b_j^* b_j - \sum_{j=1}^n \lambda_j \mathbb{1}.$$

- Let Ω be the vacuum state of the b_j's.
- $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \{0,1\}^n$.
- $\{\psi_\alpha := (b_1^*)^{\alpha_1} \dots (b_n^*)^{\alpha_n}\Omega,\ \alpha \in \{0,1\}^n\}$ are ONB of $\mathcal{H}$ and eigenfunctions of H_n.
- $\{\rho_\alpha := |\psi_\alpha\rangle\langle\psi_\alpha|\}_\alpha$ are the eigen-states of H_n.
- Let $\rho_{\alpha,1} := \mathrm{Tr}_{\mathcal{H}_2}\, \rho_\alpha$.

$$\mathcal{E}(\rho_\alpha) = -\mathrm{Tr}\, \rho_{\alpha,1} \log \rho_{\alpha,1} = -\mathrm{tr}\, \Gamma^{?}_{\rho_{\alpha,1}} \log \Gamma^{?}_{\rho_{\alpha,1}}.$$

$$H_n = 2\sum_{j=1}^n \lambda_j b_j^* b_j - \sum_{j=1}^n \lambda_j \mathbb{1}.$$

- Let Ω be the vacuum state of the b_j's.
- $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \{0,1\}^n$.
- $\{\psi_\alpha := (b_1^*)^{\alpha_1} \dots (b_n^*)^{\alpha_n}\Omega,\ \alpha \in \{0,1\}^n\}$ are ONB of $\mathcal{H}$ and eigenfunctions of H_n.
- $\{\rho_\alpha := |\psi_\alpha\rangle\langle\psi_\alpha|\}_\alpha$ are the eigen-states of H_n.
- Let $\rho_{\alpha,1} := \mathrm{Tr}_{\mathcal{H}_2}\,\rho_\alpha$.

$$\mathcal{E}(\rho_\alpha) = -\,\mathrm{Tr}\,\rho_{\alpha,1}\log\rho_{\alpha,1} = -\,\mathrm{tr}\,\Gamma^{?}_{\rho_{\alpha,1}}\log\Gamma^{?}_{\rho_{\alpha,1}}.$$

Local Jordan-Wigner Operators

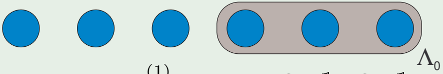
Suppose that $\Lambda_0 := \{r, r+1, \ldots, r+\ell-1\} \subset \Lambda$, $r \geq 1$.

- Define the local Jordan-Wigner operators $\{c_j^{(1)},\ j \in \Lambda_0\}$ on $\mathcal{H}_1$ as follows

$$c_r^{(1)} = a \otimes \mathbb{1}^{\otimes(\ell-1)}, \quad c_j^{(1)} := \left(\sigma^Z\right)^{\otimes(j-r)} \otimes a \otimes \mathbb{1}^{\otimes(\ell-j+r-1)}, \ j > r.$$

- Let $\mathcal{C}_1 := (c_r^{(1)}, c_{r+1}^{(1)}, \ldots, c_{r+\ell-1}^{(1)}, c_r^{(1)*}, \ldots, c_{r+\ell-1}^{(1)*})^t$ (The local Jordan-Wigner fermionic system).

Example ($n = 7,\ \Lambda_0 = \{4, 5, 6\}$)



$$\begin{aligned} c_4^{(1)} &= a \otimes \mathbb{1} \otimes \mathbb{1} \\ c_5^{(1)} &= \sigma^Z \otimes a \otimes \mathbb{1} \\ c_6^{(1)} &= \sigma^Z \otimes \sigma^Z \otimes a. \end{aligned}$$

The Entanglement Entropy

Theorem

The von Neumann Entanglement Entropies of the eigen-states $\{\rho_\alpha, \alpha \in \{0,1\}^n\}$ *with respect to the connected subsystem* $\Lambda_0 \subset \Lambda$ *are given by the formulas:*

$$\mathcal{E}(\rho_\alpha) = -\operatorname{tr} \Gamma^{\mathcal{C}_1}_{\rho_\alpha,1} \log \Gamma^{\mathcal{C}_1}_{\rho_\alpha,1}. \tag{1}$$

Note that $\mathcal{E}(\rho_\alpha)$ is

$$\mathcal{E}(\rho_\alpha) = -\sum_{j=1}^{|\Lambda_0|} \left(\xi_j^{(\alpha)} \log \xi_j^{(\alpha)} + (1 - \xi_j^{(\alpha)}) \log(1 - \xi_j^{(\alpha)}) \right),$$

where $\xi_j^{(\alpha)}$ are the eigenvalues of $\Gamma^{\mathcal{C}_1}_{\rho_\alpha,1}$.

The Correlation Matrix

How accessible is $\Gamma^{\mathcal{C}_1}_{\rho_{\alpha,1}}$??

Lemma

- $\Gamma^{\mathcal{C}}_{\rho_\alpha} = \chi_{\Delta_\alpha}(\widetilde{M}_n)$, *almost surely, where* $\Delta_\alpha := \{\lambda_j : \alpha_j = 0\} \cup \{-\lambda_j : \alpha_j = 1\}$, *and* $\{\pm\lambda_j, j = 1, 2, \ldots, n\}$ *are the eigenvalues of* $\widetilde{M}_n$.
- $\Gamma^{\mathcal{C}_1}_{\rho_{\alpha,1}}$ *is the restriction of* $\Gamma^{\mathcal{C}}_{\rho_\alpha}$ *to* $\mathrm{span}\{e_j, e_{n+j}, j \in \Lambda_0\}$.

Theorem (Recall)

Suppose that M_n is bounded uniformly in n, and dynamically localized in the sense that there exists $C < \infty$ and $\beta > 2$ such that

$$\mathbb{E}\left(\sup_{|g|\leq 1}\left\|[g(M_n)]_{\mathbf{jk}}\right\|\right) \leq \frac{C}{1+|\mathbf{j}-\mathbf{k}|^{\beta}},$$

for all $n \in \mathbb{N}$ and $1 \leq \mathbf{j}, \mathbf{k} \leq n$. Then the Entanglement Entropy of the eigen-states satisfy an area law, i.e. there exists $0 < C_\beta < \infty$ independent of n, ℓ and α, such that

$$\mathbb{E}\left(\sup_{\alpha} \mathcal{E}(\rho_\alpha)\right) < C_\beta.$$

Sketch of Proof

- We use a generalized version of the proof given by Pastur and Slavin (2014).
- We get
$$\mathcal{E}(\rho_\alpha) \leq 2\log 2 \sum_{\mathbf{j}\in\Lambda_0} \sum_{\mathbf{k}\in\Lambda\setminus\Lambda_0} \| \left[\chi_{\Delta_\alpha}(M_n)\right]_{\mathbf{jk}} \|.$$
- By taking the $\sup$ over α then averaging, we get
$$\mathbb{E}\left(\sup_\alpha \mathcal{E}(\rho_\alpha)\right) \leq 2\log 2 \sum_{\mathbf{j}\in\Lambda_0} \sum_{\mathbf{k}\in\Lambda\setminus\Lambda_0} \mathbb{E}\left(\sup_\alpha \left\| \left[\chi_{\Delta_\alpha}(M_n)\right]_{\mathbf{jk}} \right\| \right).$$

Sketch of Proof (Cont.)

- Now

$$\mathbb{E}\left(\sup_{\alpha}\left\|[\chi_{\Delta_\alpha}(M_n)]_{\mathbf{jk}}\right\|\right) \leq \mathbb{E}\left(\sup_{|g|\leq 1}\left\|[g(M_n)]_{\mathbf{jk}}\right\|\right) \leq \frac{C}{1+|\mathbf{j}-\mathbf{k}|^\beta}.$$

- We use $1+(x+y)^\alpha \geq \sqrt{(1+2x^\alpha)(1+2y^\alpha)}$, valid for x, $y>0$, $\alpha>1$.

- $$\sum_{j\in\Lambda_0}\sum_{k\in\Lambda\setminus\Lambda_0} \longrightarrow \sum_{j\in\Lambda_0}\sum_{\substack{k\,\in\,\Lambda\setminus\Lambda_0\\ k\,>\,r+\ell-1}} + \sum_{j\in\Lambda_0}\sum_{\substack{k\,\in\,\Lambda\setminus\Lambda_0\\ k\,<\,r}}.$$

- $$\mathbb{E}\left(\sup_{\alpha}\mathcal{E}(\rho_\alpha)\right) \leq 4C\log 2\left(\sum_{j=1}^{\infty}\frac{1}{\sqrt{1+2j^\beta}}\right)^2.$$

Thank you.