Test 2 | **Linear Algebra** | **June 18, 2019**

MATH 413/513 | **Dr. Abdul-Rahman**

Name: ________________________ **Signature:** ________________________

Solve Seven out the following Eight questions.
In this test: V is a finite dimensional vector space over the field $\mathbb{F}$.

1. [15 points] Suppose $S, T \in \mathcal{L}(V)$. Prove that

$T \circ S$ is invertible **if and only if** both S and T are invertible.

2. [15 points] Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\text{null}(S)$ and $\text{range}(S)$ are T-invariant.

3. [15 points] Suppose that $T \in \mathcal{L}(V)$ has the property that every $v \in V$ is an eigenvector for T. Prove that there exists a constant $\alpha \in \mathbb{F}$ such that $T = \alpha \mathbb{1}$.

4. [15 points] Suppose that $P \in \mathcal{L}(V)$ has the properties that $P^2 = P$. Prove that $V = \operatorname{null}(P) \oplus \operatorname{range}(P)$.

5. [15 points]

(a) For each permutation $\pi \in \mathcal{S}_3$, compute the number of inversions in π, and classify π as being either an even or an odd permutation.

(b) Use part (a) to construct a formula for the determinant of a 3×3 matrix.

6. [15 points]

(a) Give an example of a map $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(\alpha v) = \alpha f(v)$, for all $\alpha \in \mathbb{F}$ and $v \in \mathbb{V}$, but $f \notin \mathcal{L}(\mathbb{R}^2, \mathbb{R})$.

(b) Give an example of a map $f : \mathbb{C} \to \mathbb{C}$ such that $f(u + v) = f(u) + f(v)$ for all $u, v \in V$, but $f \notin \mathcal{L}(\mathbb{C})$.

7. [15 points] Prove that

$$W := \{T \in \mathcal{L}(\mathbb{R}^4, \mathbb{R}^3);\ \dim\ \text{null}(T) > 2\}$$

is not a subspace of $\mathcal{L}(\mathbb{R}^4, \mathbb{R}^3)$.

8. [15 points] Suppose V is vector space with $\dim V = n \geq 2$. Show that for any $\phi \in \mathcal{L}(V, \mathbb{R}^2)$, then

$$\dim(\text{null}(\phi)) \geq n - 2.$$