MATH 413/513 (LINEAR ALGEBRA)
HOMEWORK 9

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SUMMER 2019

- **Submit questions 3 and 8 for grading.**
- **<u>Due on:</u> Wednesday June 26, 2019.**

(1) Suppose $n \in \mathbb{Z}_+$, define $T \in \mathcal{L}(\mathbb{F}^n)$ by
$$T(z_1, \ldots, z_n) := (0, z_1, \ldots, z_{n-1}).$$
Find a formula for $T^*(z_1, \ldots, z_n)$.

(2) Suppose $T \in \mathcal{L}(V)$, prove that $\operatorname{null} T^* = \big(\operatorname{range} T\big)^\perp$.

(3) Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*.

(4) Suppose that $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is T-invariant if and only if $U^\perp$ is T^*-invariant.

(5) Let $T \in \mathcal{L}(V)$ where V is a complex inner product space. Prove that
$$\begin{aligned}\langle Tu, v\rangle &= \frac{1}{4}\left[\langle T(u+v), u+v\rangle - \langle T(u-v), u-v\rangle + \right.\\ &\qquad \left. + \langle T(u+iv), u+iv\rangle - \langle T(u-iv), u-iv\rangle\right].\end{aligned}$$

(6) Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Prove that
$$T \text{ is self-adjoint if and only if } \langle Tv, v\rangle \in \mathbb{R} \text{ for every } v \in V.$$

(7) Let V be a finite dimensional inner product space over $\mathbb{C}$, and suppose that $T \in \mathcal{L}(V)$ has the property $T^* = -T$. (We call T a **skew Hermitian** operator of V.)
   (a) Prove that the operator $(iT) \in \mathcal{L}(V)$ is Hermitian.
   (b) Prove that T has purely imaginary eigenvalues.

(8) Consider the vector space $\mathbb{R}_2[x]$ of polynomials with degree ≤ 2. Define the following inner product on $\mathbb{R}_2[x]$,
$$\langle p, q\rangle = \int_0^1 p(x)q(x)\, dx.$$
Define $T \in \mathcal{L}(\mathbb{R}_2[x])$ by $T(a_0 + a_1x + a_2x^2) = a_1x$. Show that T is not self-adjoint.

(9) Suppose that T is a normal operator on V and that 3 and 4 are eigenvalues of T. Prove that there exists a vector $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.