

MATH 413/513 (LINEAR ALGEBRA)
HOMEWORK 6

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SUMMER 2019

- **Submit questions 2 and 9 for grading.**
- **Due on: Thursday June 13, 2019.**
- **You will have a quiz from the following questions on Friday June 14, 2019.**

In the following, U , V , and W are finite dimensional vector spaces over a field \mathbb{F} .

- (1) Let $T \in \mathcal{L}(V)$ and let U_1 and U_2 be two T -invariant subspaces of V .
 - (a) Prove that $U_1 + U_2$ is T -invariant.
 - (b) Prove that $U_1 \cap U_2$ is T -invariant.
- (2) Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\text{null } S$ and $\text{range } S$ are T -invariant.
- (3) Suppose that V is a finite dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that T has invariant subspace of dimension k for each $k = 1, \dots, \dim V$.
- (4) Let $n \in \mathbb{Z}_+$ be a positive integer and $T \in \mathcal{L}(\mathbb{F}^n)$ be defined by
$$T(x_1, \dots, x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n)$$
for every $x_1, x_2, \dots, x_n \in \mathbb{F}$. Compute the eigenvalues and associated eigenvectors for T .
- (5) Suppose that $V = U \oplus W$, where U and W are subspaces of V . Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for every $u \in U$ and $w \in W$. Find all eigenvalues and eigenvectors of P .
- (6) Suppose $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(V)$ invertible. Show that T and $S^{-1}TS$ have the same eigenvalues.
- (7) Suppose that $T \in \mathcal{L}(V)$ is invertible. Prove that
$$\text{null}(T - \lambda \mathbb{1}) = \text{null}(T^{-1} - \frac{1}{\lambda} \mathbb{1})$$
for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.
- (8) Suppose that $T \in \mathcal{L}(V)$ has the property that every $v \in V$ is an eigenvector for T . Prove that T must then be a scalar multiple of the identity function on V .
- (9) Let V be a finite dimensional over \mathbb{C} , $T \in \mathcal{L}(V)$, and $p(z) \in \mathbb{C}[z]$ be a polynomial. Prove that $\lambda \in \mathbb{C}$ is an eigenvalue of the linear operator $p(T) \in \mathcal{L}(V)$ if and only if T has an eigenvalue $\mu \in \mathbb{C}$ such that $p(\mu) = \lambda$.