MATH 413/513 (LINEAR ALGEBRA)
HOMEWORK 5

DR. ABDUL-RAHMAN

SUMMER 2019

- **Submit questions 1 and 8 for grading.**
- **Due on: Tuesday June 11, 2019.**
- **You will have a quiz from the following questions on Tuesday June 11, 2019.**

In the following, U, V, and W are finite dimensional vector spaces over a field $\mathbb{F}$.

(1) Suppose U is 3-dimensional subspace of $\mathbb{R}^8$ and T is a linear map from $\mathbb{R}^8$ to $\mathbb{R}^5$ such that null $T = U$. Prove that T is surjective.

(2) Suppose that $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$ are both injective. Prove that $T \circ S$ is injective.

(3) Suppose $S, T \in \mathcal{L}(V)$. Prove that

$T \circ S$ is invertible **if and only if** both S and T are invertible.

(4) Let $S, T \in \mathcal{L}(V)$. Prove that $S \circ T = \mathbb{1}$ if and only if $T \circ S = \mathbb{1}$.

(5) Let $R, S, T \in \mathcal{L}(V)$. Prove that if RST is surjective then S is injective.

(6) For any positive integers m, n, $\mathbb{F}^{m,n}$ is used to denote the set of all $m \times n$ matrices with entries from the field $\mathbb{F}$. It is direct to check that $\mathbb{F}^{m,n}$ is a vector space under the obvious matrix addition and scalar multiplication. Find a basis for $\mathbb{F}^{2,2}$. What is the dimension of $\mathbb{F}^{m,n}$?

(7) Suppose $v_1, \ldots, v_n$ is a basis for V. Prove that the map $T : V \to \mathbb{F}^{n,1}$ defined by

$$Tv = \mathcal{M}(v)$$

is an isomorphism of V onto $\mathbb{F}^{n,1}$ where $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis $v_1, \ldots, v_m$.

(8) Suppose $\phi \in \mathcal{L}(V, \mathbb{F})$. Suppose $u \in V$ is not in null(ϕ). Prove that

$$V = \text{null}(\phi) \oplus \text{span}\{u\}.$$

(9) Let $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$. Prove that

$$\dim(\text{null}(T \circ S)) \leq \dim(\text{null}(T)) + \dim(\text{null}(S)).$$